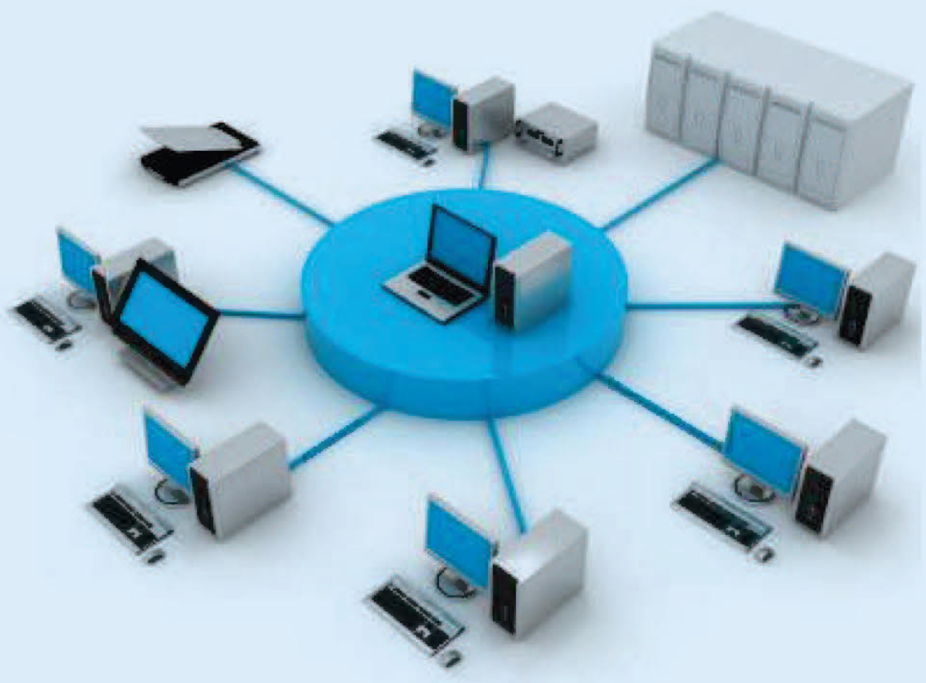

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Business Mathematics

Institute of Open and Distance Education

Faculty of Management

Business Mathematics

1BBA6

DR. C.V. RAMAN UNIVERSITY

Chhattisgarh, Bilaspur

A STATUTORY UNIVERSITY UNDER SECTION 2(F) OF THE UGC ACT

1BBA6

Business Mathematics

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1.1 PARTIAL DIFFERENTIATIONS

1.1.1 FUNCTIONS OF TWO VARIABLES : We know that the quantities like area, volume depend on two and three other quantities, respectively. For example :

Area of a triangle $= \frac{1}{2} \times$ base $\times$ corresponding attitude

Area of a rectangle = length $\times$ breadth

Volume of a parallelopiped = length $\times$ breadth $\times$ height

In terms of functions we can say that area is a function of two variables, where as volumes is a function of three variables.

Let A, B and C are three sets such that A represents base of a triangle and B represents attitude of the corresponding bases of the triangle and C represent the area of triangle, then we can define a function *f* from the set A × B to C. This function *f* is called function of two variables.

Definition 1.1: Let A, B and C are any three non-empty sets, then the function f: A × B→ C, is called a function of two variables, where

$$a \in A \text{ and } b \in B \text{ then } \exists\ c \in C, \text{ such that } f(a,b) = c$$

The set A × B is called domain of the function and C is called codomain of the function.

In general we write $u = f(x, y)$ for a function of two variables x and y. The variables x, and y, are independent where as u is dependent. The function u is called single valued.

Similarly the function of three or more variables can be expressed as

$$u = f(x, y, z)$$

$$\text{or } v = f(x_1, x_2, \ldots x_n)$$

1.1.2 **Limit of a function of two variables :** If function $f = (x, y)$ is said to be limit L, at (a, b) if for every $\in > 0$ however small there exists δ, such that

$$|f(x, y) - L| < \in \text{ where } |(x-a)| < \delta \text{ and } |(y-b)| < \delta$$

we also write $\lim f(x, y) = L$

1.1.3 **Continuity of function of two variables :** A function $f = (x, y)$ is said to be continuous at (a, b) if for every $\in > 0$ however small there exists $\delta > 0$, such that

$$|f(x, y) - f(a, b)| < \in$$

$$a - \delta < x < a + \delta \quad \text{and} \quad b - \delta < y < b + \delta$$

Alternatively, $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

$$\text{or} \quad \lim_{(h,h) \to (0+0)} f(a + h, b + k) = f(a, b)$$

1.1.4 PARTIAL DERIVATIVES (OR PARTIAL DIFFERENTIAL COEFFICIENT) :

Let $f(x, y)$ be a function of two independent variable x and y, then

$\lim_{h \to 0} \dfrac{f(x + h, y) - f(x, y)}{h}$, if exists, is called partial derivatives or partialdifferential coefficient of $f(x, y)$ with respect to x, when y is treated as constant. It is denoted by $\frac{\partial f}{\partial x}$ or f_x. similarly $\lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$, if exists is called partial derivative or differential coefficient of $f(x, y)$ with respect y, wheb x is treated as constant. It is denoted by $\frac{\partial f}{\partial y}$ *or* f_y.

The process of finding the partial derivative or partial differential coefficient, is known as partial differentiation.

It is clear that the partial derivative of a function of two variables keeping one variable as constant is same as the ordinary derivatives.

The other higher partial derivative of $f_x, f_y \left(or \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ with respect to x an denoted

by $$f_{xx}, \left(or \frac{\partial^2 f}{\partial x^2},\ fyx\ or\ \frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial^2 f}{\partial y^2} \right)$$

For $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2},\quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Remark : $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, It is not always possible that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

SOLVED EXAMPLES

Example 1 : If $u = \sin(x+y)$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial y \partial x}$, and show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial^2 u}{\partial x^2} + u = 0,\quad \frac{\partial 2u}{\partial y^2} + u = 0,\quad \frac{\partial^2 u}{\partial x \partial y} + u = 0$$

Solution : Given$u = \sin(x+y)$

Differentiating above partially w. r. t. x

$$\frac{\partial u}{\partial x} = \cos(x+y), \quad \text{Similarly} \quad \frac{\partial u}{\partial y} = \cos(x+y)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+y), \quad \frac{\partial^2 u}{\partial y^2} = -\sin(x+y)$$

$$= -u \qquad\qquad = -u$$

$$\frac{\partial u}{\partial y \partial x} = -\sin(x+y)\,(1), \quad \frac{\partial^2 u}{\partial x \partial y} = -\sin(x+y)$$

$$= -u \qquad\qquad = -u$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + u = 0, \qquad \therefore\ \frac{\partial^2 u}{\partial y^2} + u = 0$$

and $$\frac{\partial^2 u}{\partial y \partial x} + u = 0, \qquad \frac{\partial^2 u}{\partial x \partial y} + u = 0$$

From above it is clear that,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

Example 2 : If $u = \log\left(\frac{x-y}{x+y}\right)$. Find $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial y \partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial x \partial y}$

Solution : Given $u = \log\left(\frac{x-y}{x+y}\right)$

Differentiating above p. w. r. t. x

$$\frac{\partial u}{\partial x} = \frac{(x+y)}{(x-y)} \times \left\{\frac{1(x+y) - 1(x-y)}{(x+y)^2}\right\},$$

$$\frac{\partial u}{\partial x} = \frac{2y}{(x^2 - y^2)},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{0(x^2 - y^2) + 2x \times 2y}{(x^2 - y^2)^2},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-4xy}{(x^2 - y^2)^2},$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2(x^2 - y^2) + 2y \times 2y}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2x^2 - 2y^2 + 4y^2}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2x^2 + 2y^2}{(x^2 - y^2)^2}$$

Differentiating above p. w. r. t. y

$$\frac{\partial u}{\partial y} = \frac{(x+y)}{(x-y)} \times \left\{\frac{-1(x+y) - 1(x-y)}{(x+y)^2}\right\}$$

$$\frac{\partial u}{\partial y} = \frac{\{-x - y - x + y\}}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{0(x^2 + y^2) - 2y(-2x)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{4xy}{(x^2 + y^2)^2}$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-2(x^2 - y^2) - 2x(-2x)}{(x^2 - y^2)^2}$$

$$= \frac{2x^2 + 2y^2}{(x^2 - y^2)^2}$$

$$= \frac{2\left(x^2+y^2\right)}{\left(x^2-y^2\right)^2} \qquad\qquad = \frac{2\left(x^2+y^2\right)}{\left(x^2-y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Alternatively, $u = \log(x-y) - \log(x+y)$

(Find u_x, u_{xy}, u_y, u_{yx})

Example 3 : If $u = x\tan y + y\tan x$ then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution : We have $u = x\tan y + y\tan x$

Differentiating above p. w. r. t. x

$$\frac{\partial u}{\partial x} = \tan y + y\sec^2 x,$$

$$\frac{\partial^2 u}{\partial y \partial x} = \sec^2 y + \sec^2 x \qquad \ldots(1)$$

Similarly,

$$\frac{\partial u}{\partial y} = x\sec^2 y + \tan x$$

and $$\frac{\partial 2u}{\partial x \partial y} = \sec^2 y + \sec^2 x \qquad \ldots(2)$$

From equation (1) and (2)

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Example 4 : If $u = y^x$, then verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution : We have $u = y^x$,

$$\therefore \frac{\partial u}{\partial x} = y^x \log y \qquad\qquad \frac{\partial u}{\partial y} = xy^{x-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = xy^{x-1}\log y + y^x \times \frac{1}{y} \qquad\qquad \frac{\partial^2 u}{\partial y \partial x} = y^{x-1} + x \cdot y^{x-1}\log y$$

$$= xy^{x-1}\log y + y^{x-1} \qquad\qquad = (1 + x\log y)y^{x-1}$$

$$= (1 + x\log x)y^{x-1}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Example 5 : If $u = \log \frac{xy}{x^2 + y^2}$ then verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution : We have $u = \log \frac{xy}{x^2 + y^2}$

$$u = \log x + \log y - \log\left(x^2 + y^2\right)$$

Differentiating partially w. r. t. x,

Similarly,

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{y} - \frac{2y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^2 + y^2 - 2x^2}{x\left(x^2 + y^2\right)} \quad \text{or} \quad \frac{\partial u}{\partial y} = \frac{+x^2 + y^2 - 2y^2}{y\left(x^2 + y^2\right)}$$

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{x\left(x^2 + y^2\right)} \quad \text{or} \quad \frac{\partial u}{\partial y} = \frac{x^2 - y^2}{y\left(x^2 + y^2\right)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2y\left(x^3 + xy^2\right) - 2xy\left(x^2 - y^2\right)}{x^2\left(x^2 + y^2\right)^2} \text{ and } \frac{\partial^2 u}{\partial x \partial y} = \frac{2x\left(xy^2 + y^3\right) - 2xy\left(x^2 - y^2\right)}{y^2\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2yx^3 + 2xy^3 - 2xy^3 - 2x^3 y}{x^2\left(x^2 + y^2\right)^2} \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{2x^3 y + 2xy^3 - 2x^3 y + 2xy^3}{y^2\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{4x^3 y}{x^2\left(x^2 + y^2\right)^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{4xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{4x^3 y}{x^2\left(x^2 + y^2\right)^2} = \frac{4xy}{\left(x^2 + y^2\right)^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{4xy^3}{y^2\left(x^2 + y^2\right)^2}$$

$$= \frac{4xy}{\left(x^2 + y^2\right)^2}$$

$\therefore$ From above we have

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Example 6 : If $u = x^3 y + y^3 z + z^3 x$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial z} + z\dfrac{\partial u}{\partial z} = 4u$

Solution : We have $u = x^3 y + y^3 z + z^3 x$, ...(1)

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 y + z^3 \qquad ...(2)$$

$$\frac{\partial u}{\partial y} = x^3 + 3y^2 z \qquad ...(3)$$

$$\frac{\partial u}{\partial z} = y^3 + 3z^2 x \qquad ...(4)$$

Multiplying (2), (3), (4) respectively by x, y and z and adding we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3x^2 y + z^3 + x^3 + 3y^2 z + y^3 + 3z^2 x$$

$$= 4\left(x^3 y + y^3 z + z^3 x\right)$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4u \qquad \text{from (1)}$$

Example 7 : If $u = f(x + ay) + \phi(x - ay)$ then prove that $x\dfrac{\partial^2 u}{\partial y^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

Solution : $u = f(x + ay) + \phi(x - ay)$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(x + ay) + \phi'(x - ay)$$

and $$\frac{\partial^2 u}{\partial x^2} = f''(x + ay) + \phi''(x - ay) \qquad ...(1)$$

Now, $$\frac{\partial u}{\partial y} = af'(x + ay) + a\phi'(x - ay)$$

$$\frac{\partial^2 u}{\partial y^2} = a^2 f''(x + ay) + a^2 \phi''(x - ay) \qquad ...(2)$$

from equation (1) and (2) we have,

$$\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Example 8 : If $u = \sin^{-1}\left(\dfrac{y}{x}\right) + \tan^{-1}\left(\dfrac{x}{y}\right)$ then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Solution : Given $u = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{x}{y}\right)$

Differentiating above partially w. r. t. x

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-y^2/x^2}} \times \left(\frac{-y}{x^2}\right) + \frac{1}{\left(1+x^2/y^2\right)} \times \frac{1}{y}$$

or $$\frac{\partial u}{\partial x} = -\frac{\not{x}y}{x^2\sqrt{1-y^2/x^2}} + \frac{y^2}{x^2+y^2} \times \frac{1}{y}$$

$$\frac{\partial u}{\partial x} = -\frac{y}{x\sqrt{x^2-y^2}} + \frac{y}{x^2+y^2}$$

$$\therefore x\frac{\partial u}{\partial x} = -\frac{y}{\sqrt{x^2-y^2}} + \frac{xy}{x^2+y^2} \qquad ...(1)$$

We also have

$$\therefore \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2/x^2}} \times \frac{1}{x} + \frac{1}{1+x^2/y^2} \times \left(-\frac{x}{y^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\not{x}}{\sqrt{x^2-y^2}} \times \frac{1}{x} - \frac{x}{x^2+y^2}$$

$$\therefore y\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2-y^2}} - \frac{xy}{x^2+y^2} \qquad ...(2)$$

Adding (1) and (2) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Example 9 : If $u = \sin^{-1}\left(\frac{x-y}{x+y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution : Given $u = \sin^{-1}\left(\frac{x-y}{x+y}\right)$

Differentiating above partially w. r. t. x

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x-y}{x+y}\right)^2}} \times \left\{\frac{1(x+y)-1(x-y)}{(x+y)^2}\right\}$$

or $$\frac{\partial u}{\partial x}=\frac{(x+y)}{\sqrt{x^2+y^2+2xy-x^2-y^2+2xy}}\times\frac{x+y-x+y}{(x+y)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2y}{2\sqrt{xy}(x+y)}$$

or $$x\frac{\partial u}{\partial x}=\frac{xy}{(x+y)\sqrt{xy}}=\frac{\sqrt{xy}}{(x+y)} \qquad ...(1)$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x-y}{x+y}\right)^2}}\cdot\left\{\frac{-1(x+y)-1(x-y)}{(x+y)^2}\right\}$$

or $$=\frac{(x+y)}{\sqrt{x^2+y^2+2xy-x^2-y^2+2xy}}\times\frac{-x-y-x+y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{(x+y)}\frac{-2x}{2\sqrt{xy}}=\frac{-x}{\sqrt{xy}(x+y)}$$

$$\therefore y\frac{\partial u}{\partial y}=\frac{-xy}{(x+y)\sqrt{xy}}=-\frac{\sqrt{xy}}{(x+y)} \qquad ...(2)$$

Adding (1) and (2) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Example 10 : If $v=(x^2+y^2+z^2)^{-1/2}$ then prove that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$

Solution : Given $v=(x^2+y^2+z^2)^{-1/2}$

$$\therefore \frac{\partial v}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\times 2x$$

$$\frac{\partial^2 v}{\partial x^2}=-\frac{3}{2}(x^2+y^2+z^2)^{-5/2}\times 2x^2-(x^2+y^2+z^2)^{-3/2}\cdot 1$$

or $$\frac{\partial^2 v}{\partial x^2}=(x^2+y^2+z^2)^{-5/2}\left[3x^2-(x^2+y^2+z^2)\right]$$

$$\frac{\partial^2 v}{\partial x^2}=(x^2+y^2+z^2)^{-5/2}\left[2x^2-y^2-z^2\right] \qquad ...(1)$$

Similarly $$\frac{\partial^2 v}{\partial y^2}=(x^2+y^2+z^2)^{-5/2}\left[2y^2-x^2-z^2\right] \qquad ...(2)$$

and $$\frac{\partial^2 v}{\partial z^2} = \left(x^2 + y^2 + z^2\right)^{-5/2}\left[2z^2 - x^2 - y^2\right] \quad \ldots(3)$$

Adding (1), (2), and (3), we are left with

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \left(x^2 + y^2 + z^2\right)^{-5/2}\left[2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2\right]$$

or $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0\left(x^2 + y^2 + z^2\right)^{-5/2}(0)$$

or $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

Example 11 : If $u = e^{xyz}$ then show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right).e^{xyz}$

Solution : $u = e^{xyz}$

$$\therefore \frac{\partial u}{\partial z} = xy\, e^{xyz}$$

and $$\frac{\partial^2 u}{\partial y \partial z} = xy \times xz\, e^{xyz} + x\, e^{xyz}$$

or $$\frac{\partial^2 u}{\partial y \partial z} = \left(x^2yz + x\right)e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (2xyz + 1)\, e^{xyz} + \left(x^2yz + x\right) yz\, e^{xyz}$$

or $$\frac{\partial^3 u}{\partial x \partial y \partial z} = \left(2xyz + 1 + x^2y^2z^2 + xyz\right)e^{xyz}$$

or $$\frac{\partial^3 u}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}$$

Example 12 : If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$ then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x + y + z)^2}$$

Solution : Given $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$

$$\therefore \frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$

And $$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

or $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{\left(x^3+y^3+z^3-3xyz\right)}$$

$$=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

or $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\frac{3}{(x+y+z)} \quad ...(1)$$

Now $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad \text{From (1)}$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

or $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=-\frac{9}{(x+y+z)^2}$$

Example 13 : If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ Prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Where u is a function of x, y and z.

Solution : Given $$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \quad ...(1)$$

Differeniating partially w. r. t. x, we get

$$\frac{2x}{a^2+u}-\frac{x^2}{\left(a^2+u\right)^2}\cdot\frac{\partial u}{\partial x}-\frac{y^2}{\left(b^2+u\right)^2}\cdot\frac{\partial u}{\partial x}-\frac{z^2}{\left(c^2+u\right)^2}\cdot\frac{\partial u}{\partial x}=0$$

or $$\frac{2x}{a^2+u}-\left(\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}\right)\cdot\frac{\partial u}{\partial x}=0$$

Similarly, $$\frac{2y}{b^2+u}-\left(\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}\right)\cdot\frac{\partial u}{\partial y}=0$$

And $$\frac{2z}{c^2+u}-\left(\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}\right)\cdot\frac{\partial u}{\partial z}=0$$

Putting $$\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}=P \qquad ...(2)$$

in the above equation we get,

$$\frac{\partial u}{\partial x}=\frac{2x}{\left(a^2+u\right)}\times\frac{1}{P} \qquad ...(3)$$

$$\frac{\partial u}{\partial y}=\frac{2y}{\left(b^2+u\right)}\times\frac{1}{P} \qquad ...(4)$$

$$\frac{\partial u}{\partial z}=\frac{2x}{\left(c^2+u\right)}\times\frac{1}{P} \qquad ...(5)$$

Squaring and adding above we get,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}$$

or $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{P^2}\times P \qquad \text{from (1)}$$

or $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{P} \qquad ...(6)$$

Now multiplying equations (2), (3), and (4) by x, y and z respectively and adding we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2}{P}\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right)$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2}{P}\times 1=\frac{2}{p} \qquad \text{from (1)} \qquad \ldots(7)$$

$\therefore$ from (6) and (7) we have,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Example 14 : If $x^x y^y z^z = c$ show that for x = y = z $\frac{\partial^2 z}{\partial x \partial y}=-(x\log ex)^{-1}$

Where z is a function of x and y.

Solution : Given $x^x y^y z^z = c$

Taking log on both sides, we get

$$x\log x + y\log y + z\log z = c \qquad \ldots(1)$$

Differentiating partially w. r. t. x, we get

$$x\times\frac{1}{x}+\log x+\frac{\partial z}{\partial x}\log z+z\times\frac{1}{z}\cdot\frac{\partial z}{\partial x}=0$$

or $$(1+\log x)+\frac{\partial z}{\partial x}(1+\log z)=0$$

or $$\frac{\partial z}{\partial x}=-\frac{(1+\log x)}{(1+\log z)} \qquad \ldots(2)$$

Similarly, $$\frac{\partial z}{\partial y}=-\frac{(1+\log y)}{(1+\log z)} \qquad \ldots(3)$$

Differentiating (3) partially w. r. t. x we get,

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{-0\cdot(1+\log z)+\frac{1}{z}\cdot\frac{\partial z}{\partial x}(1+\log y)}{(1+\log z)^2}$$

or $$\frac{\partial^2 z}{\partial x \partial y}=\frac{1}{z}\times\frac{1+\log y}{(1+\log z)^2}\times\frac{\partial z}{\partial x}$$

$$=\frac{1}{z}\times\frac{(1+\log y)}{(1+\log z)^2}\left\{-\left(\frac{1+\log x}{1+\log z}\right)\right\}$$

$$=-\frac{(1+\log x)(1+\log x)}{x(1+\log x)(1+\log x)} \qquad \text{Putting } x=y=z$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e + \log x)} \qquad (\because \log e = 1)$$

$$= -\frac{1}{x\log(ex)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$

Example 15 : If $u = f(r)$, and $r^2 = x^2 + y^2$ show that $\frac{\partial^2 u}{\partial x^2} = f''(r) + \frac{1}{r} f'(r)$

Solution : Given $u = f(r)$,

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} \qquad \because r^2 = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r} \text{ (From(1)) or } \quad 2r\frac{\partial r}{\partial x} = 2x \frac{\partial r}{\partial x} = \frac{x}{r} \quad \ldots(1)$$

Now,
$$\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{\partial r}{\partial x} \cdot \frac{x}{r} + f'(r)\left(\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right)$$

$$= f''(r) \cdot \frac{x^2}{r^2} + f'(r)\left(\frac{r - \frac{x^2}{r}}{r^2}\right) \qquad \text{(From(1))}$$

$$= f''(r) \cdot \frac{x^2}{r^2} + f'(r)\left(\frac{r^2 - x^2}{r^3}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{x^2}{r^2} + f'(r)\, \frac{y^2}{r^3}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{y^2}{r^2} + f'(r)\, \frac{x^2}{r^3}$$

Adding above we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{x^2 + y^2}{r^2} + f'(r)\, \frac{(x^2 + y^2)}{r^3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r) \qquad (\because r^2 = x^2 + y^2)$$

EXERCISE 1.1

Q. 1. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for following functions -

(i) $u = \tan^{-1}\left(y/x\right)$ (ii) $u = \log \frac{x^2 + y^2}{xy}$

(iii) $u = x^y$ (iv) $u = x \sin y + y \sin x$

(v) $u = \frac{ay - bx}{by - ax}$ (vi) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Q. 2. If $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Q. 3. If $u = \sin^{-1}\frac{x^2 + y^2}{x + y}$ prove that $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Q. 4. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Q. 5. If $u = \sin^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{x}{y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Q. 6. If $u = \cos^{-1}\left(\frac{x - y}{x + y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Q. 7. If $u = f(x - by) + \phi(x + by)$ prove that $b^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$.

Q. 8. If $u = \sin(y + ax) - (y + ax)^2$ prove that $\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}$.

Q. 9. If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, find the value of $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$.

Q. 10. If $u = (x^2 + y^2 + z^2)^{1/2}$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

Q. 11. $u = \log r$, where $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}$.

Q. 12. If $u = e^x(x \cos y - y \sin y)$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Q. 13. If $u = a\log\left(x^2+y^2\right)+b\tan^{-1}\left(y/x\right)$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

Q. 14. If $u=\tan^{-1}\frac{xy}{\left(1+x^2+y^2\right)}$, show that $\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{\left(1+x^2+y^2\right)^{3/2}}$.

Q. 15. If $u=\log\sqrt{x^2+y^2+z^2}$, show that $\left(x^2+y^2+z^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$

Q. 16. If $u=x^y$, prove that $\frac{x}{y}\frac{\partial u}{\partial x}+\frac{1}{\log x}\cdot\frac{\partial u}{\partial x}=24$

Q. 17. If $u=\begin{vmatrix} x^2 & y^2 & z^2\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}$ then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Q. 18. If $u=\left(1-2xy+y^2\right)^{1/2}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u^2u^2$.

Q. 19. If $u=\tan^{-1}\left(y/x\right)$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

Q. 20. If $u=\log x^2+y^2+z^2$, prove that $x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial z\partial x}=z\frac{\partial^2 u}{\partial x\partial y}$.

Q. 21. If $u=e^{ax+by}f\left(ax-by\right)$, prove that $b\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial y}=2abu$

Q. 22. If $u=\frac{1}{x^2+y^2+z^2}$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{x}{\left(x^2+y^2+z^2\right)^2}$

Q. 23. If $u=\log\left(x^2+y^2\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.$

Q. 24. If $u=\phi\left(x,y,z\right)=0$, show that $\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_y=-1.$ Hence find,

$\left(\frac{\partial y}{\partial z}\right)$, x constant, $\left(\frac{\partial z}{\partial x}\right)$, y constant, $\left(\frac{\partial x}{\partial y}\right)$, z constant.

Q. 25. Find the partial differential coefficient of x^3y^2 w. r. t. x and y.

1.2 HOMOGENEOUS FUNCTION

A function $f(x,y)$ of two independent variables x and y is said to be homogeneous, if it can be expressed in the form,

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \ldots + a_n y^n \quad \ldots(1)$$

Here each term in the expression has degree n. Thus $f(x,y)$ is a homogeneous function of degree n in variables x and y.

We can write $f(x,y)$ as :

$$f(x,y) = x^n \left[a_0 + a_1 \left({y}/{x} \right) + a_2 \left({y}/{x} \right)^2 + \ldots + \left({y}/{x} \right)^n \right]$$

$$f(x,y) = x^n f\left({y}/{x} \right)$$

Thus every homogeneous function of independent variables x and y can be expressed as $x^n f\left({y}/{x} \right)$.

1.3 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS.

If $f(x,y)$ be a homogeneous function of x and y of degree n, then,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof : As $f(x,y)$ be a homogeneous functions of degree n, then it can be written as

$$f(x,y) = x^n f\left({y}/{x} \right) \qquad (1)$$

Differentiating (1) partially w. r. t. x.

$$\frac{f(x,y)}{\partial x} = nx^{n-1} f\left({y}/{x} \right) - x^n f'\left({y}/{x} \right) \cdot \frac{y}{x^2}$$

$$\frac{\partial f}{\partial x} = nx^{n-1} f\left({y}/{x} \right) - x^{n-2} f'\left({y}/{x} \right) \qquad \ldots(2)$$

Similarly,

$$\frac{\partial f}{\partial x} = x^n f\left({y}/{x} \right) \cdot \frac{1}{x}$$

$$\frac{\partial f}{\partial x} = x^{n-1} f'\left({y}/{x} \right) \qquad (3)$$

Now, multiplying (2) and (3) respectively by x and y we get,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n f\left({y}/{x} \right) - x^{n-1} y f'\left({y}/{x} \right) + x^{n-1} y f'\left({y}/{x} \right)$$

or $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n f\left({y}/{x} \right)$$

or $\qquad x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=n\cdot f$ $\qquad$ from (1)

Remark : If $f(x_1,x_2\ldots x_n)$ be a homogeneous function of degree n, of independent variables $x_1,x_2\ldots x_n$ then

$$x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+\ldots x_n\frac{\partial f}{\partial x_n}=nf$$

SOLVED EXAMPLES

Example 1. : Verify Euler's Theorem for the function

$$f(x,y)=ax^2+2hxy+by^2$$

Solution : Here, $f(x,y)=ax^2+2hxy+by^2$, is a homogeneous function of degree 2,

So, $\qquad x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f$

Now, $\qquad f(x,y)=ax^2+2hxy+by^2$ $\qquad$ ...(1)

Differentiating partially w. r. t. x,

$$\frac{\partial f}{\partial x}=2ax+2hy \qquad \text{...(2)}$$

Similarly, $\frac{\partial f}{\partial y}=2hx+2by$ $\qquad$ ...(3)

Multiplying (2) and (3) by x and y respectively and adding we get,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2\left[ax^2+hxy+hxy+by^2\right]$$

or $\qquad x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2\left[ax^2+2hxy+by^2\right]$

or $\qquad x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f$ $\qquad$ from (1)

Example 2. : If $u=\left(y/z\right)+\left(z/x\right)+\left(x/y\right)$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

Solution : Here, u is a homogeneous function in three variables of degree $1-1=0$. So, from Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0\cdot u=0$$

Example 3. : If $u = \sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show by Euler's Theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Solution : Here, $\sin u = \left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) = f\,(let)$

$\therefore\ f$ is a homogeneous function in two variables of degree $\dfrac{1}{2}-\dfrac{1}{2}=0.$

So, from Euler's Theorem, we have,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0\cdot f = 0$$

or $\quad x\dfrac{\partial}{\partial x}(\sin u) + y\dfrac{\partial}{\partial y}(\sin u) = 0 \qquad \because f = \sin u$

or $\quad x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = 0$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0 \qquad \because \cos u \neq 0$

Example 4. : If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

Solution : Here, $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right) = f(let)$ is a homogeneous function in x and y of degree $3-1=2$.

So, from Euler's Theorem, we have,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

Now $\quad x\dfrac{\partial}{\partial x}(\tan u) + y\dfrac{\partial}{\partial y}(\tan u) = 2\tan u$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u\cos^2 u$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u\cos^2 u$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

Example 5. : If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos u-1)\sin 2u$$

Solution : As in example (4)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \qquad ...(1)$$

Differentiating above partially w. r. t. x we get

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x} \qquad ...(2)$$

Again differentiating (1) partially w. r. t. y, we get

$$x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 y}{\partial y^2}=2\cos 2u\frac{\partial u}{\partial x} \qquad ...(3)$$

Multitplying (2) and (3) respectively by x and y, and adding we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\frac{\partial u}{\partial x}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\sin 2u \qquad \text{(from example 4)}$$

Example 6. : If u, be a homogeneous function of degree n, show that :

(i) $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=(n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}=(n-1)\dfrac{\partial u}{\partial y}$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}+x^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u$

Solution :

(i) Since u is a homogeneous function of degree n in variable x and y then by Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad ...(1)$$

Differentiating (1) partially w. r. t. x, we get-

$$\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\cdot 1\right)+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \qquad ...(2)$$

(ii) Similarly differentiating (1) partially w. r. t. y we get

$$x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot 1=n\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \qquad ...(3)$$

(iii) Multiplying (2) by x and (3) by y, and adding we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \qquad \text{From (1)}$$

Example 7. : If $u=f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Solution : Since u is a homogeneous function in x and y of degree 1 – 1 = 0. So, by Euler's theorem, we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

$$u=f\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial u}{\partial x}=f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad \text{or} \quad x\frac{\partial u}{\partial x}=\frac{-y}{x}f'\left(\frac{y}{x}\right) \qquad ...(1)$$

and $$\frac{\partial u}{\partial y}=f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{or} \quad y\frac{\partial u}{\partial x}=\frac{y}{x}f'\left(\frac{y}{x}\right) \qquad ...(2)$$

Adding (1) and (2), we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Example 8. : If $u=A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}\ldots n$, terms. Where $\alpha_1+\beta_1=\alpha_2+\beta_2=\ldots=n$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Solution : Given $u=A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}y^{\beta_2}\ldots\ldots$ ...(1)

Differentiating (1) partially with respect to x, we get,

$$\frac{\partial u}{\partial x} = A_1\alpha_1 x^{\alpha_1 - 1} y^{\beta_1} + A_2\alpha_2 x^{\alpha_2 - 1} y^{\beta_2} \ldots \qquad \ldots(2)$$

Multiplying above by x we get,

$$x\frac{\partial u}{\partial x} = A_1\alpha_1 x^{\alpha_1} y^{\beta_1} + A_2\alpha_2 x^{\alpha_2} y^{\beta_2} + \ldots \qquad \ldots(3)$$

Similarly, differentiating (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial x} = A_1\beta_1 x^{\alpha_1} y^{\beta_1 - 1} + A_2\beta_2 x^{\alpha_2} y^{\beta_2 - 1} + \ldots$$

Multiplying above by y we get,

$$y\frac{\partial u}{\partial y} = A_1\beta_1 x^{\alpha_1} y^{\beta_1} + A_2\beta_2 x^{\alpha_2} y^{\beta_2} + \ldots \qquad \ldots(4)$$

On adding (3) and (4) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = A_1(\alpha_1 + \beta_2) x^{\alpha_1} y^{\beta_1} + A_2(\alpha_2 + \beta_2) x^{\alpha_2} y^{\beta_2} + \ldots$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\left[A_1 x^{\alpha_1} y^{\beta_1} + A_2 x^{\alpha_2} y^{\beta_2} + \ldots\right]$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad (\text{from (1) } \alpha_1 + \beta_1 + \alpha_2 + \beta_2 = \ldots = \ldots = n)$$

Example 9. : Verify Euler's Theorem for the following functions :

(i) $f(x, y, z) = axy + byz + czx$

(ii) $f(x, y) = x^3 + 3x^2 y + 3xy^2 + y^3$

(iii) $f(x, y) = x^3 \log\left(\frac{y}{x}\right)$

(iv) $f(x, y, z) = 3x^2 yz + 4xy^2 z + 5y^4$

Solution :(i) Given $f(x, y, z) = axy + byz + czx$ ...(1)

Differentiating (1) partially with respect to x, we get,

$$\frac{\partial f}{\partial x} = ay + cz \qquad \ldots(2)$$

Similarly, $$\frac{\partial f}{\partial y} = ax + bz \qquad \ldots(3)$$

or $$\frac{\partial f}{\partial z} = by + cx \qquad \ldots(4)$$

Multiplying (2), (3) and (4) by x, y, z respectively and adding, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=axy+czx+axy+byz+byz+czx$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=2(axy+byz+czx) \qquad \text{from (1)}$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=2f \qquad \text{from (1)}$$

(ii) Given $f(x,y)=x^3+3x^2y+3xy^2+y^2$ ...(1)

Differentiating (1) partially with respect to x, we get,

$$\frac{\partial f}{\partial x}=3x^2+6xy+3y^2 \qquad ...(2)$$

Similarly, $$\frac{\partial f}{\partial y}=3x^2+6xy+3y^2 \qquad ...(3)$$

Now, multiplying (2) and (3) by x, y respectively and adding, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3x^2y+6xy+3y^2+3x^2y+6xy+3y^2$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3(x^3+3x^2y+3xy^2+y^3)$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f \qquad \{\text{from (1)}\}$$

(iii) Given $f(x,y)=x^3\log\left(\frac{y}{x}\right)$ ...(1)

Differentiating (1) partially with respect to x, we get,

$$\frac{\partial f}{\partial x}=3x^2\log\left(\frac{y}{x}\right)+x^3\left(\frac{x}{y}\right)\times\left(\frac{-y}{x^2}\right)=3x^2$$

$$\frac{\partial f}{\partial x}=3x^2\log\left(\frac{y}{x}\right)-x^2 \qquad ...(2)$$

Similarly, $$\frac{\partial f}{\partial y}=x^3\cdot\frac{x}{y}\times\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial y}=\frac{x^2}{y} \qquad ...(3)$$

Now, multiplying (2) and (3) respectively by x, and y and adding, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3x^3\log\left(\frac{y}{x}\right)-\cancel{x^3}+\cancel{x^3}$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3\left(x^3\log\left(\frac{y}{x}\right)\right)$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f \qquad \text{from (1)}$$

(iv) Given $f(x,y,z)=3x^2yz+4xy^2z+5y^4$ ...(1)

Differentiating (1) partially with respect to x, we get,

$$\frac{\partial f}{\partial x}=6xyz+4y^2z \qquad ...(2)$$

Similarly, $$\frac{\partial f}{\partial y}=3x^2z+8xyz+20y^3 \qquad ...(3)$$

and $$\frac{\partial f}{\partial z}=3x^2y+4xy^2 \qquad ...(4)$$

Now, multiplying (2), (3) and (4) by x, y and z respectively and adding, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=6x^2yz+4xy^2z+3x^2yz+8xy^2z+20y^4$$
$$+3x^2yz+4xy^2z$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=4\left(3x^2yz+4xy^2z+5y^4\right)$$
$$=4f \qquad \text{from (1)}$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=4f$$

Example 10.: If $u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

Solution : Given $u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\therefore\ \tan u=\frac{x^3+y^3}{x+y} \qquad ...(1)$$

Differentiating above partially w. r. t. x we get

$$\sec^2 u\frac{\partial u}{\partial x}=\frac{3x^2(x+y)-\left(x^3+y^3\right)\cdot 1}{(x+y)^2}$$

$$\sec^2 u\frac{\partial u}{\partial x}=\frac{3x^3+3x^2y-x^3-y^3}{(x+y)^2}$$

or $$\sec^2 u\frac{\partial u}{\partial x}=\frac{2x^3+3x^2y-y^3}{(x+y)^2} \qquad ...(2)$$

Similarly, $$\sec^2 u\frac{\partial u}{\partial y}=\frac{2xy^2+2y^3-x^3}{(x+y)^2} \qquad ...(3)$$

Multitplying (2) and (3) respectively by x and y, and adding we get,

$$\sec^2 u\left[x\frac{\partial u}{\partial x}\cdot y\frac{\partial u}{\partial y}\right]=x\frac{\left(2x^3+3x^2y-y^3\right)}{(x+y)^2}+y\frac{\left(2y^3+3xy^2-x^3\right)}{(x+y)^2}$$

or $$\sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{2x^4+3x^3y-y^3x+2y^4+3xy^3-x^3y}{(x+y)^2}$$

$$=\frac{2x^4+2y^4+2x^3y+2xy^3}{(x+y)^2}$$

$$=\frac{2\left[x^3(x+y)+y^3(x+y)\right]}{(x+y)^2}$$

$$=\frac{2(x+y)\left(x^3+y^3\right)}{(x+y)^2}=\frac{x^3+y^3}{(x+y)}$$

$$=2\tan u \qquad \text{(from (1))}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u\times\cos^2 u=2\sin u\cos u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u$$

Example 11.: If $u=\cos^{-1}\left(\frac{x}{y}\right)+\cot^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Solution : We have $$u=\cos^{-1}\left(\frac{x}{y}\right)+\cot^{-1}\left(\frac{y}{x}\right) \qquad ...(1)$$

Differentiating above partially w. r. t. x we get

$$\frac{\partial u}{\partial x}=\frac{-1}{\sqrt{1-x^2/y^2}}\times\frac{1}{y}+1-\frac{1}{1+y^2/x^2}\times\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial x}=\frac{-y}{\sqrt{y^2-x^2}}\times\frac{1}{y}+\frac{x^2}{x^2+y^2}\times\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{y^2 - x^2}} + \frac{y}{x^2 + y^2} \qquad ...(2)$$

Similarly, $\frac{\partial u}{\partial y} = -\frac{1}{\sqrt{1 - x^2/y^2}} \times \left(-\frac{x}{y^2}\right) - \frac{1}{1 + y^2/x^2} \times \left(\frac{1}{x}\right)$

$$= \frac{xy}{\sqrt{y^2 - x^2}} \times \frac{1}{y^2} - \frac{x^2}{x^2 + y^2} \times \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{x}{y\sqrt{y^2 - x^2}} - \frac{x}{x^2 + y^2} \qquad ...(3)$$

Multiplying equations (2) and (3) respectively by x and y and adding we get -

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} + \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Example 12.: If $u = \frac{xy}{(x+y)}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

Solution : Given $u = \frac{xy}{(x+y)}$ ...(1)

Differentiating above partially w. r. t. x we get

$$\frac{\partial u}{\partial x} = \frac{y(x+y) - 1 \times xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

or $x\frac{\partial u}{\partial x} = \frac{xy^2}{(x+y)^2}$ ...(2)

Similarly, $\frac{\partial u}{\partial y} = \frac{x(x+y) - xy}{(x+y)^2}$

or $\frac{\partial u}{\partial y} = \frac{x^2 + xy - xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$

$$y\frac{\partial u}{\partial y} = \frac{x^2 y}{(x+y)^2} \qquad ...(3)$$

Adding (2) and (3) we get -

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{xy^2}{(x+y)^2}+\frac{x^2y}{(x+y)^2}$$

$$=\frac{xy(x+y)}{(x+y)^2}$$

$$=\frac{xy}{(x+y)}=u \qquad \text{[from (1)]}$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$$

Remark : Since $u=\dfrac{xy}{(x+y)}$ is homogeneous of degree $2-1-1=0$, therefore,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1\times u=u$$

Example 13.: If $u=f\left(x/y\right)+y\phi\left(y/x\right)$, show that

(i) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial x}=y\phi\left(y/x\right)$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0$

Solution : Given $u=f\left(x/y\right)+y\phi\left(y/x\right)$ ...(1)

Differentiating (1) partially w. r. t. x, we get

$$\frac{\partial u}{\partial x}=f'\left(x/y\right)\times\frac{1}{y}+y\phi\left(x/y\right)\cdot\left(-y/x^2\right)$$

or $$x\frac{\partial u}{\partial x}=\frac{x}{y}f'\left(x/y\right)-\frac{y^2}{x}\phi'\left(x/y\right) \qquad ...(2)$$

Differentiating (1) partially w. r. t. y, we get

$$\frac{\partial u}{\partial y}=f'\left(x/y\right)\left(-\frac{x}{y^2}\right)+1\cdot\phi\left(y/x\right)+y\phi'\left(y/x\right)\cdot\left(\frac{1}{x}\right)$$

or $$y\frac{\partial u}{\partial y}=-\frac{x}{y}f'\left(x/y\right)+y\cdot\phi\left(y/x\right)+\frac{y^2}{x}\phi'\left(y/x\right) \qquad ...(3)$$

Adding (2) and (3), we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=y\phi\left(y/x\right) \qquad ...(4)$$

This proves result (i)

Now, differentiating (4) partially w. r. t. x we get

$$x\frac{\partial^2 u}{\partial x^2}+1\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=y\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad ...(5)$$

Differentiating (4) partially w. r. t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot 1=\phi'\left(\frac{y}{x}\right)+y\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

or $$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=y\phi\left(\frac{y}{x}\right)+\frac{y^2}{x}+\phi'\left(\frac{y}{x}\right) \quad ...(6)$$

Adding (5) and (6) we have,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=y\phi\left(\frac{y}{x}\right) \quad ...(7)$$

From equation (4) and (7) we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

This proves (ii)

Example 14.: If $u=x^m f\left(\frac{x}{y}\right)+x^n g\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+mnu=(m+n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

Solution : Let $v=x^m f\left(\frac{x}{y}\right)$ and $w=x^n g\left(\frac{y}{x}\right)$

Then $u=v+w$ ...(1)

As $v=x^m f\left(\frac{x}{y}\right)$ is a homogeneous function of x and y of degree m, therefore, by

Euler's theorem we have $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=m(m-1)v$ ...(2)

(see example 6)

Also $w=x^n g\left(\frac{y}{x}\right)$ is a homogeneous function of x and y of degree n, so, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)w \quad ...(3)$$

(see solved example 6)

Adding (2) and (3) we have,

$$x^2\frac{\partial^2}{\partial x^2}+(v+w)+2xy\frac{\partial^2}{\partial x\partial y}(v+w)+y^2\frac{\partial^2}{\partial y^2}(v+w)$$

$$=m(m-1)v+n(n-1)w \quad \ldots(4)$$

Since v and w are homogeneous function of x and y of degree m and n respectively, then by Euler's theorem :

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x\frac{\partial w}{\partial x}+yx\frac{\partial w}{\partial y}=mv+nw$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=mv+nw \qquad \text{from (1)} \qquad \ldots(5)$$

Again $$m(m-1)v+n(n-1)w$$

$$=m^2v-mv+n^2w-nw$$

$$=(m^2v+n^2w)-(mv+nw)$$

$$=m(m+n)v+n(m+n)w-mn(v+w)-(mv+nw)$$

$$=(m+n-1)\,(mv+nw)-mnu \qquad \text{from (1)}$$

Substituting these values in (4) we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(m+n-1)(mv+nv)-mnu$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+mnu=(m+n-1)(mv+nv)$$

$$=(m+n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \qquad \text{from (5)}$$

Example 15.: If $u=\log\left(\frac{x^2+y^2}{xy}\right)+\sin^{-1}\left(y/x\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Solution : Given $u=\log\left(\frac{x^2+y^2}{xy}\right)+\sin^{-1}\left(y/x\right)$ $\ldots(1)$

Differentiating (1) partially w. r. t. x

$$\frac{\partial u}{\partial x}=\frac{xy}{x^2+y^2}\times\frac{2x(xy)-y(x^2+y^2)}{x^2y^2}+\frac{1}{\sqrt{1-y^2/x^2}}\left(-y/x^2\right)$$

$$\frac{\partial u}{\partial x}=\frac{x^2y-y^3}{xy(x^2+y^2)}-\frac{xy}{\sqrt{x^2-y^2}}\times\frac{1}{x^2}$$

$$\frac{\partial u}{\partial x}=\frac{y\left(x^2-y^2\right)}{xy\left(x^2+y^2\right)}-\frac{y}{x\sqrt{x^2-y^2}}$$

or $$x\frac{\partial u}{\partial x}=\frac{\left(x^2-y^2\right)}{x^2+y^2}-\frac{y}{\sqrt{x^2-y^2}} \qquad \ldots(2)$$

Differentiating (1) partially w. r. t. y, we have,

$$\frac{\partial u}{\partial x}=\frac{xy}{x^2+y^2}\times\frac{\left(2y(xy)-x\left(x^2+y^2\right)\right)}{x^2y^2}+\frac{1}{\sqrt{1-y^2/x^2}}\cdot\left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial x}=\frac{xy\left(y^2x-x^3\right)}{\left(x^2+y^2\right)x^2y^2}+\frac{x}{\sqrt{x^2-y^2}}\times\frac{1}{x}$$

or $$\frac{\partial u}{\partial x}=\frac{x\left(y^2-x^2\right)}{xy\left(x^2+y^2\right)}+\frac{1}{\sqrt{x^2-y^2}}$$

or $$y\frac{\partial u}{\partial y}=\frac{y^2-x^2}{x^2+y^2}+\frac{y}{\sqrt{x^2-y^2}} \qquad \ldots(3)$$

Adding (2) and (3) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x^2-y^2}{x^2+y^2}-\frac{y}{\sqrt{x^2-y^2}}+\frac{y^2-x^2}{x^2+y^2}+\frac{y}{\sqrt{x^2-y^2}}$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x^2-y^2}{x^2+y^2}-\frac{y}{\sqrt{x^2-y^2}}+\frac{x^2-y^2}{x^2+y^2}+\frac{y}{\sqrt{x^2-y^2}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

EXERCISE 3.1

Q. 1. : Verify Euler's theorem for the following functions.

(i) $z=x^2\left(x^2-y^2\right)^3/\left(x^2+y^2\right)^3$

(ii) $z=\left(x^{1/4}-y^{1/4}\right)/\left(x^{1/5}+y^{1/5}\right)$

(iii) $z=x^n\sin(y/x)$ (iv) $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

Q. 2. : If $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Q. 3. : If z is a homogeneous function of x and y of degree n, then show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Q. 4. : If $u=x^3+y^3+z^3+3xyz$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u.$

Q. 5. : If $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$

Q. 6. : If $u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u.$

Q. 7. : If $u=x\,f\left(\frac{y}{x}\right)+\phi\left(\frac{y}{x}\right)$ prove that $x^2\frac{\partial u}{\partial x}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$

Q. 8. : If $u=f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

Q. 9. : If $u=\cos^{-1}\left({x+y}\big/{\sqrt{x}+\sqrt{y}}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$

Q. 10. : If $u=\sec^{-1}\left({x^3+y^3}\big/{x+y}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\cot u.$

1.4 TOTAL DIFFERENTIAL COEFFICIENT

Let $u=f(x.y)$ ...(1)

be a function of two variables such that $x=\phi(t)$ and $y=\psi(t)$, i. e. x and y are functions of t. If we substitute these values of x and y in (1) we get

$u=f(\phi(t),\psi(t))$ which may be treated as function of the single variable t,

then ordinary derivative $\frac{du}{dt}$, is called total coefficient of u with respect to t.

Sometimes, we find it very difficult to express u in term of t alone by eliminating x and y. So we are to find $\frac{du}{dx}$ without actually substituting the values of x and y in terms of t in $u=f(x,y)$.

Let δx, δy and δu be the small increments in x, y and u respectively corresponding to small increment in t.

i. e. $x+\delta x=\phi(t+\delta t)$ or $y+\delta y=\psi(t+\delta t)$ and $u+\delta u=f(x+\delta x,y+\delta y)$

Now, from definition

$$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{(u + \delta u) - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(x + \delta x,\ y + \delta y) - f(x, y)}{\delta t}$$

Adding and subtracting $f(x + \delta x, y)$ numerator,

$$= \lim_{\delta t \to 0} \frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta t} + \frac{f(x + \delta x - y) - f(x, y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta t}$$

$$+ \lim_{\delta t \to 0} \frac{f(x + \delta y - y) - f(x, y)}{\delta t} \qquad ...(1)$$

$$= \lim_{\delta t \to 0} \left(\frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta y} \cdot \frac{\delta y}{\delta t} \right.$$

$$\left. + \lim_{\delta t \to 0} \frac{f(x + \delta x - y) - f(x, y)}{\delta x} \cdot \frac{\delta x}{\delta t} \right)$$

$$\because \quad \lim_{\delta t \to 0} \frac{\delta y}{\partial t} = \frac{dy}{dt} \ \ and \ \ \lim_{\delta t \to 0} \frac{\delta x}{\partial t} = \frac{dx}{dt}$$

Also $\delta t \to 0,\ \delta x \to 0 \ \ and \ \ \delta y \to 0$

$$\therefore \ \lim_{\delta t \to 0} \left(\frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta y} \right.$$

$$\lim_{\delta y \to 0} \left(\frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta y} = \frac{\delta}{\partial y} f(x + \delta x, y) \right.$$

By definition (as $x + \delta x$ remains unchanged while y changes from y to $y + \delta y$, i. e. $x + \delta x$ is treated as constant for this limit)

$$\lim_{\delta y \to 0} \left(\frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta y} = \frac{\delta}{\partial y} f(x + \delta x, y) = \frac{\delta u}{\partial y} \right.$$

$$\text{as } \ \delta x \to 0, \quad \text{and} \quad u = f(x, y)$$

Similarly, $$\lim_{\delta x \to 0} \left(\frac{f(x + \delta x,\ y + \delta y) - f(x + \delta x, y)}{\delta x} \right.$$

$$= \lim_{\delta x \to 0} \left(\frac{f(x + \delta x,\ y) - f(x, y)}{\delta x} \right.$$

$$= \frac{\partial f(x, y)}{\partial x} = \frac{\partial u}{\partial x} \qquad \because \quad u = f(x, y)$$

$\therefore$ from (1) we have,

$$\frac{du}{dt}=\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial u}{\partial x}+\frac{dx}{dt}$$

or
$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}+\frac{dy}{dt} \qquad \ldots(2)$$

In general, if $u=f(x_1,x_2,\ldots,x_n)$ and $x_1,x_2,\ldots,x_n$ are function of *t*, then –

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt}+\ldots\frac{\partial u}{\partial x_n}\frac{dx_n}{dt}$$

Remark : If $u=f(x,y)$ is a function of *x* and *y*, and *t* is a function of *x*, then from the result (2) above, the total differential coefficient of *f* with respect to *x*, is given by

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \quad \text{or} \quad \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad \ldots(3)$$

In general if $u=f(x_1,x_2,\ldots,x_n)$ be function of $x_1,x_2,\ldots,x_n$ and $x_2,x_3,\ldots,x_n$ are function of x_1 alone, the the total differential coefficient of *u* with respect to x_1, is given by –

$$\frac{df}{dx}=\frac{\partial f}{\partial x_1}+\frac{\partial x}{\partial x_2}\cdot\frac{dx_2}{dx_1}+\frac{\partial f}{\partial x_3}\cdot\frac{dx_3}{dx_2}+\ldots+\frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dx_1}$$

or
$$\frac{du}{dx}=\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dx_1}+\frac{\partial u}{\partial x_3}\cdot\frac{dx_3}{dx_1}+\ldots+\frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dx_1}$$

1.4.1 First Differential Coefficient of an Implicit Function :

Let $f(x,y)=c$, where c as a constant, and y is a function of x, then from the result (3), we have,

$$\frac{d}{dx}(c)=\frac{d}{dx}\cdot(f(x,y))$$

$$0=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$0=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$$

1.4.2 Second Differential Coefficient of an Implicit Function :

From $$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} \qquad \ldots(1)$$

Let, $p = -\dfrac{\partial f}{\partial x}$, $q = \dfrac{\partial f}{\partial y}$ and $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$

Therefore, from (1) we have

$$\frac{dy}{dx} = -\frac{p}{q} \quad \ldots(2)$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{-p}{q}\right) = \frac{-\left(\frac{dp}{dx} \times q - p\frac{dq}{dx}\right)}{q^2} \quad \ldots(3)$$

Now,

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \cdot \frac{dy}{dx}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x} \cdot \left(\frac{dy}{dx}\right)$$

$$= r + s\left({}^{-p}\!/\!{}_{q}\right)$$

$$\therefore \frac{dp}{dx} = \frac{r^2 - sp}{q} \quad \ldots(4)$$

Similarly, $\dfrac{dq}{dx} = \dfrac{\partial q}{\partial x} + \dfrac{\partial q}{\partial y} \cdot \dfrac{dy}{dx}$

$$\frac{dq}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \cdot \frac{dy}{dx}$$

$$\frac{dq}{dx} = \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dx}$$

$$\frac{dq}{dx} = s + t\left({}^{-p}\!/\!{}_{q}\right)$$

$$= \frac{sq - pt}{q} \quad \ldots(5)$$

Substituting these values in (3) we get

$$\frac{d^2 y}{dx^2} = -\frac{\frac{q(qr - sp)}{q} - p \cdot \frac{sq - pt}{q}}{q^2}$$

or $$\frac{d^2y}{dx^2} = -\frac{q^2r - qsp - psq + p^2t}{q^3}$$

or $$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$

SOLVED EXAMPLES

Example 1. : If $u = x^2y^2$, where $x^2 + xy + y^2 = 1$, find $\frac{du}{dx}$.

Solution : Given $u = x^2y^2$ ...(1)

We know that $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$ ...(2)

Now, from (1), $\frac{\partial u}{\partial x} = 2xy^2$ ...(3)

And, $\frac{\partial u}{\partial y^2} = 2x^2y$ ...(4)

Also, $x^2 + xy + y^2 = 1$

Differentiating above w. r. t. x we get,

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$ ...(5)

Now from equations (3), (4) and (5), equation (2) becomes –

$$\frac{du}{dx} = 2xy^2 + 2x^2y\left(-\frac{2x + y}{x + 2y}\right)$$

or $$\frac{du}{dx} = \frac{2xy^2 \times (x + 2y) - 2x^2y(2x + y)}{(x + 2y)}$$

or $$\frac{du}{dx} = \frac{2x^2y^2 + 4xy^3 - 4x^3y - 2x^2y^2}{(x + 2y)}$$

or $$\frac{du}{dx} = \frac{4xy(y^2 - x^2)}{(x + 2y)}$$

Example 2. : If $u = \sqrt{x^2 + y^2}$, and $x^2 + y^2 + 3axy = 5a^2$, find $\frac{du}{dx}$ where $x = a, y = a$.

Solution : Given $u = \sqrt{x^2 + y^2}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{2y}{\sqrt{x^2 + y^2}}$$

$$\frac{du}{dx}=\frac{x}{\sqrt{x^2+y^2}} \quad \ldots(1) \text{ and } \frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2}} \quad \ldots(2)$$

Now given $x^2+y^2+3axy=5a^2$

Differentiating above w. r. t. x we get,

$$2x+2y\frac{dy}{dx}+3ay+3ax\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{2x+3ay}{2y+3ax} \quad \ldots(3)$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}$$

Putting the values of $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ and $\frac{\partial y}{\partial x}$ from equation (1), (2) and (3) respectively in (4),

we get –

$$\frac{du}{dx}=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\left(-\frac{2x+3ay}{2y+3ax}\right)$$

When $x=a$, and $y=a$

$$\frac{du}{dx}=\frac{a}{a\sqrt{2}}+\frac{a}{a\sqrt{2}}\left(-\frac{2a+3a^2}{2a+3a^2}\right)$$

$$=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

$$\therefore \frac{du}{dx}=0$$

Example 3. :Find $\frac{du}{dx}$ if $u=\sin\left(x^3+y^3\right)$, where $a^3x^3+b^3y^3=c^3$.

Solution : Given $u=\sin\left(x^3+y^3\right)$

$$\therefore \frac{\partial u}{\partial x}=\cos\left(x^3+y^3\right)\times 3x^2 \quad \text{and} \quad \frac{\partial u}{\partial y}=\cos\left(x^3+y^3\right)\times 3y^2$$

We also have, $a^3x^3+b^3y^3=c^3$

Differentiating above w. r. t. x we get

$$3a^2x^2+3b^3y^2\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{a^3x^2}{b^3y^2}$$

$$\because \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \ldots(1)$$

Putting the values of $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ and $\frac{\partial y}{\partial x}$ in (1), we get –

$$\frac{du}{dx} = 3x^2\cos\left(x^3+y^3\right) + 3y^2\cos\left(x^3+y^3\right)\left(-\frac{a^3x^2}{b^3y^2}\right)$$

$$\frac{du}{dx} = 3\cos\left(x^3+y^3\right)\left[x^2 - \frac{a^3x^2y^2}{b^3y^2}\right]$$

$$\frac{du}{dx} = 3\cos\left(x^3+y^3\right)\left[\frac{b^3x^2 - a^3x^2}{b^3}\right]$$

$$\frac{du}{dx} = 3\frac{x^2}{b^3}\cos\left(x^3+y^3\right)\cdot\left(b^3-a^3\right)$$

$$\frac{du}{dx} = 3\left(b^3-a^3\right)\frac{x^2}{b^3}\cos\left(x^3+y^3\right)$$

Example 4. : Find $\frac{dy}{dx}$ if $x^y + y^x$.

Solution : Let $f(x,y) = x^y + y^x$ ...(1)

Then $f(x,y) = c$ which is an implicit function in x and y,

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \qquad ...(2)$$

Differentiating (1) partially w. r. t. x. we get,

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating (1) partially w. r. t. y. we get,

$$\frac{\partial f}{\partial y} = x^y \log y + xy^{x-1}$$

Putting these values in (2), we get –

$$\frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log y + xy^{x-1}}\right]$$

Example 5. : Find $\frac{dy}{dx}$ If $f(x,y) = 0$; and $\phi(y,z) = 0$ then show that

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$$

Solution : $\because\ f(x,y) = 0 \quad \therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$ ...(1)

And $\phi(y,z) = 0 \therefore \frac{dz}{dy} = -\frac{\partial \phi/\partial y}{\partial \phi/\partial z}$ ...(2)

Multiplying (1) and (2) we get,

$$\frac{dy}{dx}\cdot\frac{dz}{dy}=\frac{\partial f/\partial x}{\partial f/\partial y}\times\frac{\partial\phi/\partial y}{\partial\phi/\partial z}$$

or
$$\frac{\partial f}{\partial y}\cdot\frac{\partial\phi}{\partial z}\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}=\frac{\partial f}{\partial x}\times\frac{\partial\phi}{\partial y}$$

Example 6. : Find $\frac{du}{dx}$ If $u=x^2-y^2+\sin yz$, $y=e^z$ *and* $z=\log n$.

Solution : As u is a function of x, y and z and y and z are functions of x, we have ,

$$\frac{du}{dx}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dx} \quad ...(1)$$

Now, $u=x^2-y^2+\sin yz$ ...(2)

Differentiating (2) partially w. r. t. x

$$\frac{\partial u}{\partial x}=2x$$

Similarly, $\frac{\partial u}{\partial y}=-2y+2\cos yz$ and $\frac{\partial u}{\partial z}=y\cos yz$

From $y=e^x$ and $z=\log x$

$$\frac{dy}{dx}=e^x \qquad \frac{dz}{dx}=\frac{1}{x}$$

Substituting these values in equation (1)we get,

$$\frac{du}{dx}=2x\left(-2y+z\cos yz\right)e^x+y\cos yz\frac{1}{x}$$

or
$$\frac{du}{dx}=\frac{1}{x}\left[2x^2+xe^x\left(z\cos yz-2y\right)+y\cos yz\right]$$

Example 7. : If $u=f\{(x-z),(z-x),(x-y)\}$, then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$

Solution : Given $u=f\{(x-z),(z-x),(x-y)\}$

Let $X=y-z,\ Y=z-x,$ *and* $Z=x-y$ ...(1)

Then $u=f(x,y,z)$

$$\therefore\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} \quad ...(2)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\cdot\frac{dZ}{dy} \quad ...(3)$$

And
$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\cdot\frac{dZ}{dz} \quad ...(4)$$

From (1), $\frac{\partial X}{\partial x}=0,\quad \frac{\partial X}{\partial y}=1,\quad \frac{\partial X}{\partial z}=-1$

$$\frac{\partial Y}{\partial x}=-1, \qquad \frac{\partial Y}{\partial y}=0, \qquad \frac{\partial Y}{\partial z}=1$$

And $$\frac{\partial Z}{\partial x}=1, \qquad \frac{\partial Z}{\partial y}=-1, \qquad \frac{\partial Z}{\partial z}=0$$

Putting these values in equation (2), (3), and (4) we get,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}(0)+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1)=\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Y}(0)+\frac{\partial u}{\partial Z}(0)=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}$$

And $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}(-1)+\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Z}(0)=-\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$$

On adding above we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}+\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Example 8. : If $u=x^2-y^2+\sin yz, \quad y=e^x, \quad z=\log x$, find $\frac{du}{dx}$.

Solution : Given $u=x^2-y^2+\sin yz, \quad y=e^x, \quad z=\log x$

$$\therefore \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} \qquad \ldots(1)$$

Now, $$\frac{\partial u}{\partial x}=2x$$

$$\frac{\partial u}{\partial y}=-2y+2\cos yz, \qquad \frac{\partial y}{\partial x}=e^x$$

$$\frac{\partial u}{\partial z}=y\cos yz, \text{ and} \qquad \frac{\partial z}{\partial x}=\frac{1}{x}$$

Putting these values in equation (1), we get,

$$\frac{du}{dx}=2x+(-2y+z\cos yz)e^x+y\cos yz\times\frac{1}{x}$$

or $$\frac{du}{dx}=\frac{2x^2+(2\cos yz-2y)xe^x+y\cos yz}{x}$$

Example 9. : If $u=f\left(x^2+2yz,\ y^2+2zx,\ z^2+xy\right)$, prove that

$$\left(y^2-zx\right)\frac{\partial u}{\partial x}+\left(x^2-yz\right)\frac{\partial u}{\partial y}+\left(z^2-xy\right)\frac{\partial u}{\partial z}=0.$$

Solution : Given $u=f\left(x^2+2yz,\ y^2+2zx,\ z^2+xy\right)$

Let $X = x^2 + 2yz, \quad Y = y^2 + 2zx, \qquad Z = z^2 + 2xy, \qquad \ldots(1)$

Now,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} \qquad \ldots(2)$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} \qquad \ldots(3)$$
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} \qquad \ldots(4)$$

From equation (1), we have,
$$\left.\begin{array}{lll} \dfrac{\partial X}{\partial x} = 2x, & \dfrac{\partial X}{\partial y} = 2z, & \dfrac{\partial X}{\partial z} = 2y \\ \dfrac{\partial Y}{\partial x} = 2z, & \dfrac{\partial Y}{\partial y} = 2y, & \dfrac{\partial Y}{\partial z} = 2x \\ \dfrac{\partial Z}{\partial x} = 0, & \dfrac{\partial Z}{\partial y} = 0, & \dfrac{\partial Z}{\partial z} = 0 \end{array}\right\} \qquad \ldots(5)$$

Multiplying equation (2), (3) and (4) by $x^2 + 2yz, y^2 + 2zx,$ and $z^2 + 2xy$ respectively and putting the values from equation (5) to the equation (2), (3) and (4) and adding, we get,

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = \left\{\frac{\partial u}{\partial X}(2x) + \frac{\partial u}{\partial Y}(2z) + \frac{\partial u}{\partial Z}(0)\right\}\left(y^2 - zx\right)$$
$$+\left\{\frac{\partial u}{\partial X}(2z) + \frac{\partial u}{\partial Y}(2y) + \frac{\partial u}{\partial Z}(0)\right\}\left(x^2 - yz\right)$$
$$+\left\{\frac{\partial u}{\partial X}(2y) + \frac{\partial u}{\partial Y}(2x) + \frac{\partial u}{\partial Z}(0)\right\}\left(z^2 - xy\right)$$
$$= \frac{\partial u}{\partial X}\left\{2x\left(y^2 - zx\right) + (2z)\left(y^2 - zx\right) + 2y\left(z^2 - xy\right)\right\}$$
$$+ \frac{\partial u}{\partial Y}\left\{2z\left(y^2 - zx\right) + 2y\left(x^2 - yz\right) + 2x\left(z^2 - xy\right)\right\}$$
$$+ \frac{\partial u}{\partial Z}$$
$$= \frac{\partial u}{\partial X}\left\{2xy^2 - 2zx^2 + 2zx^2 - 2yz^2 + -2yz^2 - 2xy^2\right\} + \frac{\partial u}{\partial Y}\left\{2zy^2 - 2z^2x\right.$$
$$\left. + 2x^2y - 2y^2z + 2xz^2 - 2x^2y\right.$$
$$= \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(0)$$
$$= 0$$
$$\therefore \left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0$$

Example 10. : If $z = f(x, y), x = e^{u} + e^{-v}, \quad y = e^{-u} - e^{v}$, then show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Solution : Given $z = f(x, y), x = e^{u} + e^{-v}, \quad y = e^{-u} - e^{v}$ ...(1)

We have,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \qquad \text{...(2)} \qquad \text{from (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \qquad \text{...(3)}$$

and $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad \text{...(4)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \qquad \text{...(5)}$$

We also have

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= e^{u}, \quad \frac{\partial y}{\partial u} = -e^{-u} \\ \frac{\partial x}{\partial v} &= -e^{-v}, \quad \frac{\partial y}{\partial v} = -e^{v} \end{aligned}\right\} \qquad \text{...(6)}$$

Subtracting eqation (3) from eqation (2) we have,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\left(\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}\right) + \frac{\partial z}{\partial y}\cdot\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \qquad \text{...(7)}$$

From equation (4), (5), (6) and (7), we have

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\left(e^{u} + e^{v}\right) + \frac{\partial z}{\partial y}\cdot\left(-e^{-u} + e^{v}\right)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot(x) - \frac{\partial z}{\partial y}\cdot(y) \qquad \text{From equation (1)}$$

or $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Example 11. : If $z = f(x, y)$, *when* $x = uv$, *and* $y = \frac{u}{v}$, then show that

$$\frac{\partial z}{\partial y} = v\cdot\frac{\partial z}{\partial u} - \frac{v^2}{2u}\cdot\frac{\partial z}{\partial v} \quad \text{and} \quad \frac{\partial z}{\partial x} = \frac{1}{uv}\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

Solution : Given $z = f(x, y)$, *when* $x = uv$, *and* $y = \frac{u}{v}$

We have $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$ ...(1) and $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$ ...(2)

Now,

$$x = uv \quad \text{and} \quad y=\frac{u}{v}$$

$$\therefore \frac{\partial x}{\partial u}=v \qquad \frac{\partial y}{\partial u}=\frac{1}{v}$$

and $\frac{\partial x}{\partial v}=u$ or $\frac{\partial u}{\partial y}=v$

or $\frac{\partial u}{\partial x}=\frac{1}{v}$ $\quad \frac{\partial y}{\partial v}=\frac{-u}{v^2}$ or $\frac{\partial v}{\partial y}=-\frac{v^2}{u}$

or $\frac{\partial v}{\partial x}=\frac{1}{u}$

Putting these values in equation (1) and (2), we get,

$$\frac{\partial z}{\partial x}=\frac{1}{v}\frac{\partial z}{\partial u}+\frac{1}{u}\frac{\partial z}{\partial v} \quad \text{and} \quad \frac{\partial z}{\partial y}=v\frac{\partial z}{\partial u}-\frac{v^2}{u}\cdot\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial x}=\frac{1}{uv}\left(u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right) \quad \text{and} \quad \frac{\partial z}{\partial y}=v\frac{\partial z}{\partial u}-\frac{v^2}{u}\cdot\frac{\partial z}{\partial v}$$

EXERCISE 4.1

Q. 1. : Find $\frac{dy}{dx}$ if $ax^2+2hxy+by^2=1$ (Use partial derivatives.)

Q. 2. : If $u=x\log xy$ where $x^3+y^3+3xy=1$ find $\frac{du}{dx}\cdot 0$

Q. 3. : If $u=x^2y$ where $x^2+xy+y^2=1$ find $\frac{du}{dx}\cdot$

Q. 4. : Find $\frac{du}{dx}$ If $u=\sin\left(x^2+y^2\right)$ where $a^2x^2+b^2y^2=c^2$

Q. 5. : If $u=2(ax+by)^2-\left(x^2+y^2\right)$ and $a^2+b^2=1$, find the value of $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}$.

1.5 CHANGE OF VARIABLES

If $u=f(x,y)$ be a function of x and y and x and y are functions of two variables $t_1, \; t_2,$ i. e.

$$x=\phi(t_1,t_2), \quad \text{and} \quad y=\psi(t_1,t_2)$$

Then we have,

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1} \qquad (t_2 \text{ constant}) \qquad \ldots(1)$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2} \qquad (t_1 \text{ constant}) \qquad \ldots(2)$$

In case if $x = \phi(t_1, t_2)$, and $y = \psi(t_1, t_2)$ can be easily expressed as $t_1 = f_1(x, y)$ and $t_2 = f_2(x, y)$ then we have ,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial x}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial y}$$

SOLVED EXAMPLES

Example 1.: If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

Solution : If $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore\ x^2 + y^2 = r^2 \qquad \ldots(1)$$

On differentiating (1) partially w. r. t. x, we have,

$$2x = 2r\frac{\partial r}{\partial x}$$

or $$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \ldots(2)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \ldots(3)$$

Squaring and adding (2) and (3) we have,

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2}$$

$$= \frac{x^2 + y^2}{r^2} \qquad \text{from (1)}$$

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$$

Example 2.: If $r = r\cos\theta,\ y = r\sin\theta$ *and* $r^2 = x^2 + y^2$ prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$$

Solution : Given $r = r\cos\theta,\ y = r\sin\theta$ *and* $r^2 = x^2 + y^2$ ...(1)

On differentiating (1) partially w. r. t. x, we have,

$$2r\frac{\partial r}{\partial x} = 2x \qquad \text{Similarly,} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

or $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore\ \frac{\partial^2 r}{\partial x^2} = \frac{1 \times r - \frac{\partial r}{\partial x}x}{r^2} \qquad \text{and} \qquad \frac{\partial^2 r}{\partial y^2} = \frac{1 \times r - \frac{\partial r}{\partial y}y}{r^2}$$

$$= \frac{r - \frac{x}{r} \times x}{r^2} \qquad \text{or} \qquad \frac{\partial^2 r}{\partial y^2} = \frac{r - \frac{y}{r} \times y}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2 - x^2}{r^3} \quad ...(2) \qquad \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3} \quad ...(3)$$

Adding (2) and (3) we get,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3}$$

or $$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2 + r^2 - y^2}{r^3} = \frac{2r^2 - r^2}{r^3} \qquad \text{from (1)}$$

or $$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2}{r^3} = \frac{1}{r} \qquad ...(4)$$

We also have,

$$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left\{\left(\frac{x}{r}\right)^2 + \left(\frac{x}{r}\right)^2\right\}$$

or $$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left\{\frac{x^2 + y^2}{r^2}\right\}$$

or $$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}$$

or $$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r} \qquad \ldots(5)$$

From (4) and (5) we have the proof.

Example 3.: If $x = r\cos\theta$, $y = r\sin\theta$ show that $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$; $\frac{\partial x}{r\partial\theta}=r\frac{\partial\theta}{\partial x}$ and find the value of $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}$

Solution : Given $x = r\cos\theta$, $y = r\sin\theta$ then we have

$$r^2 = x^2 + y^2 \qquad \ldots(1) \qquad \theta = \tan^{-1}\frac{y}{x} \qquad \ldots(2)$$

From equation (1) , we have,

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\frac{r\cos\theta}{r} \qquad (\because x = r\cos\theta)$$

or $$\frac{\partial r}{\partial x}=\cos\theta \qquad \ldots(3)$$

Further $x = r\cos\theta$

Differentiating above partially w. r. t. r, we have

$$\frac{\partial x}{\partial r}=\cos\theta \qquad \ldots(4)$$

From equation (3) and (4) we have,

$$\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$$

Again,

$$x = r\cos\theta \qquad \text{and} \qquad \theta=\tan^{-1}\frac{y}{x}$$

$$\therefore \frac{\partial x}{\partial\theta}=-r\sin\theta \qquad \therefore \frac{\partial\theta}{\partial x}=\frac{1}{1+\left(y/x\right)^2}\left(-y/x^2\right)$$

$$\frac{\partial\theta}{\partial x}=\frac{x^2}{x^2+y^2}\left(-y/x^2\right)$$

$$\frac{\partial\theta}{\partial x}=-y/r^2 \qquad (\because r^2=x^2+y^2)$$

or $$r\frac{\partial\theta}{\partial x}=-\sin\theta$$

$\therefore$ from the above equation we have,

$$\frac{1}{r}\cdot\frac{\partial x}{\partial \theta}=r\frac{\partial \theta}{\partial x}$$

Further $\frac{\partial \theta}{\partial x}=-\frac{y}{x^2+y^2}$

$$\therefore \quad \frac{\partial^2 \theta}{\partial x^2}=-\frac{0\left(x^2+y^2\right)-2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 \theta}{\partial x^2}=-\frac{2xy}{\left(x^2+y^2\right)^2} \quad \ldots(1)$$

$$\frac{\partial \theta}{\partial y}=\frac{1}{1+y^2/x^2}\times\frac{1}{x} \quad \ldots(2)$$

$$\therefore \quad \frac{\partial^2 \theta}{\partial y^2}=-\frac{0\left(x^2+y^2\right)-2xy}{\left(x^2+y^2\right)^2}$$

or $$\frac{\partial^2 \theta}{\partial y^2}=-\frac{2xy}{\left(x^2+y^2\right)^2}$$

Adding (1) and (2), we have,

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=\frac{2xy}{\left(x^2+y^2\right)^2}-\frac{2xy}{\left(x^2+y^2\right)^2}$$

$$\therefore \quad \frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0$$

Example 4.: If $x=r\cos\theta,\ y=r\sin\theta$ *and* $z=f(x,y)$ prove that

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cos\theta-\frac{1}{r}\ \frac{\partial z}{\partial \theta}\sin\theta$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\sin\theta-\frac{1}{r}\ \frac{\partial z}{\partial \theta}\cos\theta$$

and $$\frac{r^2\left(r^n\cos n\theta\right)}{\partial x\partial y}=-n\,(n-1)\,r^{n-1}\sin(n-2)\theta$$

Solution : Here, z is a function of x and y where as x and y are functions of r and θ.

$$\therefore \quad \frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \quad \ldots(1)$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \quad \ldots(2)$$

We also have

$$x = r\cos\theta,\quad y = r\sin\theta\quad r^2 = x^2 + y^2\quad and\quad \theta = \tan^{-1} y/x$$

$$\therefore\ \frac{\partial r}{\partial x} = \cos\theta,\ \frac{\partial r}{\partial y} = \sin\theta,\ \frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}\quad and\quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial z}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial\theta}\quad \ldots(3)\ \text{and}\ \frac{\partial z}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial z}{\partial\theta}\quad \ldots(4)$$

We have,

$$\frac{\partial z}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial z}{\partial\theta}$$

Putting $z = r^n\cos n\theta$ in above, we get,

$$\frac{\partial}{\partial y}\left(r^n\cos n\theta\right) = \sin\theta\frac{\partial}{\partial r}\left(r^n\cos n\theta\right) + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\left(r^n\cos n\theta\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(nr^{n-1}\cos n\theta\right) + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\left(r^n\cos n\theta\right)$$

$$= nr^{n-1}\left(\cos n\theta\cdot\sin\theta - \sin n\theta\cdot\cos n\theta\right)$$

$$\therefore\ \frac{\partial}{\partial y}\left(r^n\cos n\theta\right) = nr^{n-1}\left[\sin(\theta - n\theta)\right]\quad \ldots(5)$$

Now, $\dfrac{\partial^2}{\partial x\partial y}\cdot\left(r^n\cos n\theta\right) = n\dfrac{\partial}{\partial x}\left[r^{n-1}\sin(\theta - n\theta)\right] = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial y}\left(r^n\cos n\theta\right)\right]$

$$= n\left[\cos\theta\frac{\partial}{\partial r}\left(r^{n-1}\sin(1-n)\theta\right) - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(r^{n-1}\sin(1-n)\theta\right)\right]\ \text{from (5)}$$

Substituting $r^{n-1}\sin(1-n)\theta$ for z in (3)

$$= n\left[\cos\theta(n-1)\,r^{n-2}\sin(1-n)\theta - \frac{\sin\theta}{r}\cdot r^{n-1}(1-n)\cos(1-n)\theta\right]$$

$$\frac{\partial^2}{\partial x\partial y}\cdot\left(r^n\cos n\theta\right) = n\,(n-1)\,r^{n-2}\left[-\cos\theta\sin(n-1)\theta + \sin\theta\cos(n-1)\theta\right]$$

$$\frac{\partial^2}{\partial x\partial y}\cdot\left(r^n\cos n\theta\right) = -n\,(n-1)\,r^{n-2}\left[\sin(n-1)\theta\cos\theta - \cos(n-1)\sin\theta\right]$$

$$= -n\,(n-1)\,r^{n-2}\left[\sin(n-1)\theta - \theta\right]$$

$$\frac{\partial^2}{\partial x\partial y}\cdot\left(r^n\cos n\theta\right) = n\,(n-1)\,r^{n-2}\left[\sin(n-1)\theta\right]\quad \text{Proved !}$$

Example 5.: If $u = f(x, y)$, *and* $x = r\cos\theta$, $y = r\sin\theta$ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial z}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Solution : We have, $x = r\cos\theta,\ y = r\sin\theta$ ...(1)

$$r^2 = x^2 + y^2 \quad ...(2), \qquad \theta = \tan^{-1} y/x \quad ...(3)$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \quad ...(4), \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta \quad ...(5)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-y/x^2\right) \text{ and } \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(1/x\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} \qquad \frac{\partial \theta}{\partial y} = \frac{r\cos\theta}{r^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{r\sin\theta}{r^2} \qquad \frac{\partial \theta}{\partial y} = \frac{r\cos\theta}{r^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad ...(6), \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \quad ...(7)$$

We know that,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

or
$$\frac{\partial(u)}{\partial x} = \cos\theta \times \frac{\partial(u)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial(u)}{\partial \theta} \quad ...(8)$$

Replacing u by $\frac{\partial u}{\partial x}$ in the above,

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

Using polar equivalent of $\frac{\partial u}{\partial x}$ in (8)

$$= \cos\theta\left(\cos\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\right) - \frac{\sin\theta}{r}\left\{-\sin\theta\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta \partial r}\cos\theta\right.$$

$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)\Bigg\}$$

$$\frac{\partial^2 u}{\partial x^2}=\cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2}-\sin\theta\left\{\frac{1}{r}\cdot\frac{\partial^2 u}{\partial\theta\partial r}-\frac{1}{r^2}\cdot\frac{\partial u}{\partial\theta}\right\}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial^2 u}{\partial\theta\partial r}-\frac{1}{r}\left(\sin\theta\frac{\partial^2 u}{\partial\theta}+\frac{\partial u}{\partial\theta}\cos\theta\right)\right]$$

$$\frac{\partial^2 u}{\partial x^2}=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}+\frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial r}$$

$$+\frac{2\cos\theta\sin\theta}{r^2}\cdot\frac{\partial u}{\partial\theta}\qquad\ldots(9)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2}=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{\cos^2\theta}{r}\cdot\frac{\partial u}{\partial r}$$

$$-\frac{2\cos\theta\sin\theta}{r^2}\cdot\frac{\partial u}{\partial\theta}\qquad\ldots(10)$$

Adding equation (9) and (10) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(\cos^2+\sin^2\theta\right)\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\left(\sin^2\theta+\cos^2\right)\cdot\frac{\partial^2 u}{\partial\theta^2}$$

$$+\frac{1}{r}\left(\sin^2\theta+\cos^2\right)\cdot\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}++\frac{1}{r}\cdot\frac{\partial u}{\partial r}$$

Which is required transformed equation.

Example 6. : If in the above example (5) if $u=\left(Ar^2+Br^{-n}\right)\sin n\theta$ then prove that

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}++\frac{1}{r}\cdot\frac{\partial u}{\partial r}=0$$

Solution : $u=\left(Ar^n+Br^{-n}\right)\sin n\theta$

$$\therefore\ \frac{\partial u}{\partial r}=n\left(Ar^{n-1}+Br^{-n-1}\right)\sin n\theta\quad\text{and}\quad\frac{\partial u}{\partial\theta}=\left(Ar^n+Br^{-n}\right)n\cos n\theta$$

and $$\frac{\partial^2 u}{\partial r^2}=n\left[(n-1)\,Ar^n+Br^{-n-2}(n+1)\right]\sin n\theta\quad\text{and}$$

$$\frac{\partial^2 u}{\partial\theta^2}=\left(Ar^n+Br^{-n}\right)n^2(-\sin n\theta)$$

$$\therefore\ \frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}++\frac{1}{r}\cdot\frac{\partial u}{\partial r}=n\left[(n-1)\ Ar^{n-2}+Br^{-n-2}(n+1)\right]\sin n\theta$$

$$-\frac{1\times n^2}{r^2}\left[\left(Ar^n+Br^{-n}\right)\sin n\theta\right]+\frac{1}{r}\times n\left(Ar^{n-1}+Br^{-n-1}\right)\times\sin n\theta$$

$$=\sin n\theta\left[n(n-1)\ Ar^{n-2}+n(n+1)Br^{-n-2}-n^2Ar^{n-2}-n^2Br^{-n-2}\right.$$

$$\left.+nAr^{n-2}-nBr^{-n-2}\right]$$

$$=\sin n\theta\times\left[A\left(n^2-n-n^2+n\right)r^{n-2}+B\left(n^2+n-n^2-n\right)r^{-n-2}\right]$$

$$=0$$ **Proved !**

MISCELLANEOUS EXAMPLES

Example 1. : If $u=\log(\tan x+\tan y+\tan z)$ prove that

$$(\sin 2x)\frac{\partial u}{\partial x}+(\sin 2y)\frac{\partial u}{\partial y}+(\sin 2z)\frac{\partial u}{\partial z}=2$$

Solution : Given $u=\log(\tan x+\tan y+\tan z)$

$$\therefore\quad \frac{\partial u}{\partial x}=\frac{\sec^2 x}{\tan x+\tan y+\tan z}$$

$$\therefore\quad \sin 2x\frac{\partial u}{\partial x}=\frac{\sec^2 x\sin 2x}{\tan x+\tan y+\tan z}$$

or $$\sin 2x\frac{\partial u}{\partial x}=\frac{2\tan x}{\tan x+\tan y+\tan z}\qquad\ldots(1)$$

Similarly,

$$\sin 2y\frac{\partial u}{\partial y}=\frac{2\tan y}{\tan x+\tan y+\tan z}\qquad\ldots(2)$$

And $$\sin 2z\frac{\partial u}{\partial z}=\frac{2\tan z}{\tan x+\tan y+\tan z}\qquad\ldots(3)$$

Adding (1), (2) and (3)

$$(\sin 2x)\frac{\partial u}{\partial x}+(\sin 2y)\frac{\partial u}{\partial y}+(\sin 2z)\frac{\partial u}{\partial z}=2\left[\frac{\tan x+\tan y+\tan z}{\tan x+\tan y+\tan z}\right]$$

$$=2$$

Example 2. : If $z=xyf\left(x/y\right)$ show that

(i) $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

(ii) If z is constant, then $\dfrac{f'(y/x)}{f(y/x)} = \dfrac{x\left(y + x\dfrac{dy}{dx}\right)}{y\left(y - x\dfrac{dy}{dx}\right)} + y\dfrac{\partial z}{\partial y} = 2z$

Solution : Given $z = xyf\left(\frac{x}{y}\right)$ ...(1)

(i) As z is homogeneous function of x and y in degree 2, so from Euler's theorem we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad \text{This proves (i)}$$

(ii) As z is constant, then from (1) $xy\, f\left(\frac{y}{x}\right)$ is also constant, which may be written as $\phi(x, y) = c(\text{constant})$.

then,

$$\frac{\partial y}{\partial x} = -\frac{\partial \phi/\partial x}{\partial \phi/\partial y}$$

$\because$ Now $\phi(x, y) = x\,y\,f\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial \phi}{\partial x} = y\,f\left(\frac{y}{x}\right) + x\,y\,f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$$

or $$\frac{\partial \phi}{\partial x} = y\,f\left(\frac{y}{x}\right) - \frac{y^2}{x}\,f'\left(\frac{y}{x}\right)$$

and $$\frac{\partial \phi}{\partial y} = x\,f\left(\frac{y}{x}\right) - xy\,f'\left(\frac{y}{x}\right) \times \frac{1}{x}$$

or $$\frac{\partial \phi}{\partial y} = x\,f\left(\frac{y}{x}\right) + y\,f'\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y\,f(y/x) - (y^2/x)\,f'(y/x)}{x\,f(y/x) - y\,f'(y/x)}$$

$$x\,f(y/x)\frac{dy}{dx} + y\,f'(y/x)\frac{dy}{dx} + y\,f(y/x) - (y^2/x)\,f'(y/x) = 0$$

$$\left(x\frac{dy}{dx} + y\right)f(y/x) + f'(y/x)\left(y\frac{dy}{dx} - \frac{y^2}{x}\right) = 0$$

or $$f'(y/x)\left(\frac{y^2}{x} - y\frac{dy}{dx}\right) = \left(x \times \frac{dy}{dx} + y\right) f(y/x)$$

or $$\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\dfrac{dy}{dx}\right)}{y\left(y - x\dfrac{dy}{dx}\right)}$$

Example 3. : Verify Euler's theorem for $y^n \sin\left(y/x\right)$.

Solution : Given $u = y^n \sin\left(y/x\right)$, then u is a homogeneous function of x and y of degree n.

Then,
$$\frac{\partial u}{\partial x} = y^n \cos\left(y/x\right)\left(-y/x^2\right).$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -\frac{y^{n+1}}{x}\cos\left(y/x\right). \qquad \ldots(1)$$

And
$$\frac{\partial u}{\partial y} = xy^{n-1}\sin\left(y/x\right) + y^n \cos\left(y/x\right)\times\frac{1}{x}$$

$$\therefore \quad y\frac{\partial u}{\partial y} = ny^n \sin\left(y/x\right) - \frac{y^{n-1}}{x}\cos\left(y/x\right) \qquad \ldots(2)$$

Adding (1) and (2) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y^{n+1}}{x}\cancel{\cos}\left(\cancel{y/x}\right) + ny^n \sin\left(y/x\right) + \frac{y^{n-1}}{x}\cancel{\cos}\left(\cancel{y/x}\right)$$

or
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ny^n \sin\left(y/x\right)$$

or
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$ Thus Euler's theorem is verified !

Example 4. : If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, find $\frac{d^2y}{dx^2}$.

Solution : Let $\phi(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad \ldots(1)$

$$\because \; p = \frac{\partial\phi}{\partial x} = 2ax + 2hy + 2g, \quad q = \frac{\partial\phi}{\partial y} = 2hx + 2by + 2f$$

$$r = \frac{\partial^2\phi}{\partial x^2} = 2a \quad s = \frac{\partial^2\phi}{\partial x\partial y} = 2h, \quad t = \frac{\partial^2\phi}{\partial y^2} = 2b$$

We know that,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\left[q^2r - 2pqs + p^2t\right]}{q^3} \qquad \ldots(2)$$

Putting these values of p, q, r, s and t in equation (2), we get

$$\frac{\partial^2 y}{\partial x^2} = -\frac{(2hx+2by+2f)^2 2a - 2(2ax+2hy+2g)(2hx+2by+2f)\,2h + (2ax+2hy+2g)^2\,2b}{(2hx+2by+2f)^3}$$

$$= -\frac{8(hx+by+f)^2 a - 2(ax+hy+g)(hx+by+f)h + (ax+hy+g)^2 h}{8(hx+by+f)^3}$$

$$= -\frac{\begin{matrix} a(h^2x^2+b^2y^2+f^2+2bhxy+2bfy+2fhx)-2ah^2x^2-2abhxy-2afhx \\ -2h^3xy-2bh^2y^2-2h^2fy-2gh^2x-2bghy-2fhg+a^2x^2h+h^3y^2 \\ +g^2h+2h^2xy+2h^2gy+2aghx \end{matrix}}{(hx+by+f)^3}$$

$$= -\frac{(h^2-ab)\left[ax^2+2hxy+by^2+2gx+2fy\right]-af^2-bg^2+2fgh}{(hx+by+f)^3}$$

$$= \frac{-(c)(h^2-ab)-af^2-bg^2+2fgh}{(hx+by+f)^3} \qquad \text{from (1)}$$

$$= \frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}$$

Example 5. : If $u = 3(lx+my+nz)^2 - (x^2+y^2+n^2)$ *and* $l^2+m^2+n^2=1$

Show that $\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}+\frac{\partial^2 y}{\partial z^2}=0$

Solution : Given $u = 3(lx+my+nz)^2 - (x^2+y^2+z^2)$

$\because \frac{\partial u}{\partial x} = 6l(lx+my+xz)-2x$ and $\frac{\partial^2 u}{\partial x^2} = 6l^2-2$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 6m^2-2 \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = 6n^2-2$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 6l^2-2+6m^2-2+6x^2-2$$

$$= 6(l^2+m^2+x^2-1)$$

$$= 6(1-1) \qquad (\because l^2+m^2+n^2=1)$$

$$= 0$$

Example 6. : If $\tan u = \frac{\cos x}{\sin hy}$ *and* $\tan hv = \frac{\sin x}{\cos hy}$

Show that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

Solution : Given $\tan u = \frac{\cos x}{\sin hy}$

$$\because \sec^2 u \frac{\partial u}{\partial x} = -\frac{\sin x}{\sin hy}$$

$$\because \frac{\partial u}{\partial x} = -\frac{\sin x}{\sin hy \sec^2 u} = -\frac{\sin x}{\sin hy\left(1+\tan^2 u\right)}$$

or $$\frac{\partial u}{\partial x} = \frac{-\sin x}{\sin hy\left(1+\dfrac{\cos^2 x}{\sin h^2 y}\right)} = -\frac{\sin x \sin hy}{\sin^2 h^2 y + \cos^2 x} \qquad \cdots(1)$$

Now, $$\tan hv = \frac{\sin x}{\cos hy} = \sin x \sec hy$$

$$\sec h^2 v \frac{\partial v}{\partial y} = -\sin x \sec hy \tan hy$$

or $$\frac{\partial v}{\partial y} = \frac{-\sin x \sec hy \tan hy}{\sec h^2 v} = \frac{-\sin x \sec hy \tan hy}{1-\tan h^2 v}$$

$$= \frac{-\sin x \sec hy \tan hy}{1-\sin h^2 x/\cos h^2 y} = \frac{-\sin x \sin hy}{\cos h^2 y - \sin^2 x}$$

$$= \frac{-\sin x \sin hy}{1+\left(\sin^2 hy\right)-\left(1-\cos^2 x\right)} = \frac{-\sin x \sin hy}{\cos h^2 y + \cos^2 x} = \frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}$$

Similarly we can prove $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

EXERCISE

Q. 1. Verify $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ when $f = \log\left(\dfrac{x^2+y^2}{xy}\right), x \neq 0, \ y \neq 0.$

Q. 2. If $u = \left(1-2xy+y^2\right)^{-1/2}$ Show that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = y^2u^3$

Q. 3. If $\dfrac{1}{u} = \dfrac{1}{x} + f\left(\dfrac{x-y}{xy}\right)$ Show that $x^2\dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} = u^2$

Q. 4. If $u = \left(1-2xy+y^2\right)^{-1/2}$ Show that $\dfrac{\partial}{\partial x}\left\{\left(1-x^2\right)\dfrac{\partial u}{\partial x}\right\} + \dfrac{\partial}{\partial y}\left\{y^2\dfrac{\partial u}{\partial y}\right\} = 0$

Q. 5. If $u = ze^{ax+by}$ above z is a homogeneous function of x and y of degree n. Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (ax + by + n)u$$

Q. 6. If $u = A^{-1/2} \cdot e^{-x^2/4a^2t}$ Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.

Q. 7. If $x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = f(x,y)$, Prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Q. 8. If $x = \xi\cos\alpha - \eta\sin\theta,\ \ y = \xi\sin\alpha + \eta\cos\alpha$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}$

Q. 9. If $u = f(y - x, x - y)$ Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Q. 10. If $\theta = t^n e^{-r^2/4t}$ find what value of x will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial v}\right) = \frac{\partial \theta}{\partial t}$? (Ans. : –3/2)

Q. 11. If $u = \log\left(x^2 + y^2 + z^2\right)$ show that $x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$

Q. 12. If $z = e^{ax+by} f(ax - by)$ show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab^2$

Q. 13. Show that the function $u = \log\frac{1}{r}$, where $r = \sqrt{(x-a)^2 + (y-b)^2}$ satisfies the equation $x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$.

Q. 14. If $r^2 = x^2 + y^2 + z^2$ and $v = r^m$ show that $v_{xx} + v_{yy} + v_{zz} = m(m-1)r^{m-2}$.

Q. 15. If $u = \cos^{-1}\frac{x - y}{\sqrt{x} + \sqrt{y}}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Q. 16. If $f(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x+\log y}{x^2+y^2}$ show that $x\frac{\partial t}{\partial x}+y\frac{\partial t}{\partial y}+2f(x,y)=0$

Q. 17. If $\left(x^{1/3}+y^{1/3}\right)\text{cosec}^2 u = x^{1/2}+y^{1/2}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u\left(12+\sec^2 u\right)}{144}$$

Q. 18. If $f(x,y)=x^2y^4\sin^{-1}\left(y/x\right)$ is a homogeneous function of degree 6. Hence or otherwise find the value of $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$.

Q. 19. If v be a function of r alone, where $r^2=x_1^1+x_2^2+\ldots+x_n^2$. Show that

$$\frac{\partial^2 v}{\partial x_2^1}+\frac{\partial^2 v}{\partial x_2^2}+\ldots+\frac{\partial^2 v}{\partial x_n^2}=\frac{\partial^2 v}{\partial r^2}+\frac{n-1}{r}\cdot\frac{\partial v}{\partial r}$$

Q. 20. If $u=f(r)$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$ where $r^2=x^2+y^2$.

2 Maxima and Minima

Chapter Includes:

2.1 INTRODUCTION

2.1.1 Increasing and Decreasing Functions

A function $y = f(x)$ is said to be an increasing function of x in an interval, say $a \leq x \leq b$, if y increases as x increases. i.e. if $a \leq x_1 < x_2 \leq b$, then $f(x_1) \leq f(x_2)$.

A function $y = f(x)$ is said to be a decreasing function of x in an interval, say $a \le x \le b$, if y decreases as x increases. i.e. if $a \le x_1 < x_2 \le b$, then $f(x_1) \ge f(x_2)$.

2.1.2. Sign of the derivative

Let f be an increasing function defined in a closed interval [a,b]. Then for any two values x_1 and x_2 in [a, b] with $x_1 < x_2$, we have $f(x_1) \le f(x_2)$.

$\therefore f(x_1) \le f(x_2)$ and $x_2 - x_1 > 0$

$$\Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} \ge 0$$

$$\Rightarrow \quad \underset{x_2 \to x_1}{\text{Lt}} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \ge 0, \text{ if this limit exists.}$$

$\Rightarrow \quad f'(x) \ge 0$ for all $x \in [a,b]$.

Similarly, if f is decreasing on [a,b] then $f'(x) \le 0$, if the derivative exists.

The converse holds with the additional condition, that f is continuous on $[a, b]$.

Note

Let f be continuous on $[a,b]$ and has derivative at each point of the open interval (a,b), then

(i) If $f'(x) > 0$ for every $x \in (a,b)$, then f is strictly increasing on $[a,b]$

(ii) If $f'(x) < 0$ for every $x \in (a,b)$, then f is strictly decreasing on $[a,b]$

(iii) If $f'(x) = 0$ for every $x \in (a,b)$, then f is a constant function on $[a,b]$

(iv) If $f'(x) \ge 0$ for every $x \in (a,b)$, then f is increasing on $[a,b]$

(v) If $f'(x) \le 0$ for every $x \in (a,b)$, then f is decreasing on $[a,b]$

The above results are used to test whether a given function is increasing or decreasing.

2.1.3 Stationary Value of a Function

A function $y = f(x)$ may neither be an increasing function nor be a decreasing function of x at some point of the interval $[a,b]$. In such a case, $y = f(x)$ is called stationary at that point. At a stationary point $f'(x) = 0$ and the tangent is parallel to the x - axis.

Example 1

If $y = x - \frac{1}{x}$, prove that y is a strictly increasing function for all real values of x. ($x \neq 0$)

Solution :

We have $y = x - \frac{1}{x}$

Differentiating with respect to x, we get

$\frac{dy}{dx} = 1 + \frac{1}{x^2} > 0$ for all values of x, except $x = 0$

$\therefore$ y is a strictly increasing function for all real values of x. ($x \neq 0$)

Example 2

If $y = 1 + \frac{1}{x}$, show that y is a strictly decreasing function for all real values of x. ($x \neq 0$)

Solution :

We have $y = 1 + \frac{1}{x}$

$\frac{dy}{dx} = 0 - \frac{1}{x^2} < 0$ for all values of x. ($x \neq 0$)

$\therefore$ y is a strictly decreasing function for all real values of x. ($x \neq 0$)

Example 3

Find the ranges of values of x in which $2x^3 - 9x^2 + 12x + 4$ is strictly increasing and strictly decreasing.

Solution :

Let $y = 2x^3 - 9x^2 + 12x + 4$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$
$$= 6(x^2 - 3x + 2)$$
$$= 6(x - 2)(x - 1)$$

$\frac{dy}{dx} > 0$ when $x < 1$ or $x > 2$

x lies outside the interval (1, 2).

$\frac{dy}{dx} < 0$ when $1 < x < 2$

$\therefore$ The function is strictly increasing outside the interval [1, 2] and strictly decreasing in the interval (1, 2)

Example 4

Find the stationary points and the stationary values of the function $f(x) = x^3 - 3x^2 - 9x + 5$.

Solution :

Let $y = x^3 - 3x^2 - 9x + 5$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

At stationary points, $\frac{dy}{dx} = 0$

$$\therefore 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x + 1)(x - 3) = 0$$

The stationary points are obtained when $x = -1$ and $x = 3$

when $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$

when $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$

$\therefore$ The stationary values are 10 and -22

The stationary points are $(-1, 10)$ and $(3, -22)$

Example 5

For the cost function $C = 2000 + 1800x - 75x^2 + x^3$ find when the total cost (C) is increasing and when it is decreasing. Also discuss the behaviour of the marginal cost (MC)

Solution :

Cost function $C = 2000 + 1800x - 75x^2 + x^3$

$$\frac{dC}{dx} = 1800 - 150x + 3x^2$$

$$\frac{dC}{dx} = 0 \Rightarrow 1800 - 150x + 3x^2 = 0$$

$$\Rightarrow 3x^2 - 150x + 1800 = 0$$

$$\Rightarrow x^2 - 50x + 600 = 0$$

$$\Rightarrow (x - 20)(x - 30) = 0$$

$$\Rightarrow x = 20 \text{ or } x = 30$$

For ,

(i) $0 < x < 20$, $\frac{dC}{dx} > 0$ | (i) $x = 10$ then $\frac{dC}{dx}$ $s=600 > 0$

(ii) $20 < x < 30$, $\frac{dC}{dx} < 0$ | (ii) $x = 25$ then $\frac{dC}{dx} = -75 < 0$

(iii) $x > 30$; $\frac{dC}{dx} > 0$ | (iii) $x = 40$ then $\frac{dC}{dx} = 600 > 0$

$\therefore$ C is increasing for $0 < x < 20$ and for $x > 30$.

C is decreasing for $20 < x < 30$

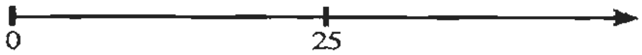
$$MC = \frac{d}{dx}(C)$$

$$\therefore \quad MC = 1800 - 150x + 3x^2$$

$$\frac{d}{dx}(MC) = -150 + 6x$$

$\frac{d}{dx}(\text{MC}) = 0 \quad \Rightarrow 6x = 150$

$\Rightarrow x = 25.$



For,

(i) $0 < x < 25$, $\frac{d}{dx}(\text{MC}) < 0$ | (i) $x = 10$ then $\frac{d}{dx}(\text{MC}) = -90 < 0$

(ii) $x > 25$, $\frac{d}{dx}(\text{MC}) > 0$ | (ii) $x = 30$ then $\frac{d}{dx}(\text{MC}) = 30 > 0$

$\therefore$ MC is decreasing for $x < 25$ and increasing for $x > 25$.

2.1.4 Maximum and Minimum Values

Let f be a function defined on $[a,b]$ and c an interior point of $[a,b]$ (i.e.) c is in the open interval (a,b). Then

(i) $f(c)$ is said to be a maximum or relative maximum of the function f at $x = c$ if there is a neighbourhood $(c-\delta, c+\delta)$ of c such that for all $x \in (c-\delta, c+\delta)$ other than c, $f(c) > f(x)$

(ii) $f(c)$ is said to be a minimum or relative minimum of the function f at $x = c$ if there is a neighbourhood $(c-\delta, c+\delta)$ of c such that for all $x \in (c-\delta, c+\delta)$ other than c, $f(c) < f(x)$.

(iii) $f(c)$ is said to be an extreme value of f or extremum at c if it is either a maximum or minimum.

2.1.5 Local and Global Maxima and Minima

Consider the graph (Fig. 2.1) of the function $y = f(x)$.

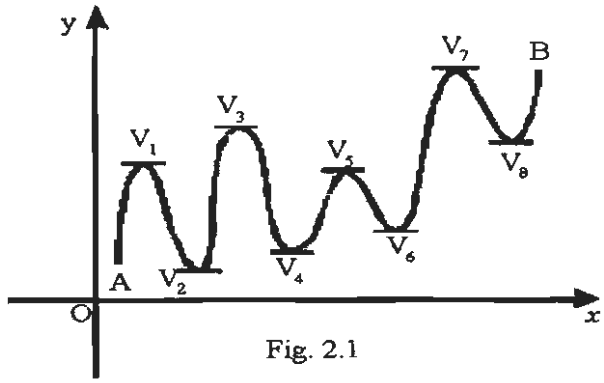


Fig. 2.1

The function $y = f(x)$ has several maximum and minimum points. At the points $V_1, V_2, \ldots V_8$, $\frac{dy}{dx} = 0$. In fact the function has maxima at V_1, V_3, V_5, V_7 and minima at V_2, V_4, V_6, V_8. Note that maximum value at V_5 is less than the minimum value at V_8. These maxima and minima are called local or relative maxima and minima. If we consider the part of the curve between A and B then the function has absolute maximum or global maximum at V_7 and absolute minimum or global minimum at V_2.

Note

By the terminology maximum or minimum we mean local maximum or local minimum respectively.

2.1.6. Criteria for Maxima and Minima.

	Maximum	Minimum
Necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient condition	$\frac{dy}{dx} = 0$; $\frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0$; $\frac{d^2y}{dx^2} > 0$

2.1.7 Concavity and Convexity

Consider the graph (Fig. 2.2) of the function $y = f(x)$.

Let PT be the tangent to the curve $y = f(x)$ at the point P. The curve (or an arc of the curve) which lies above the tangent line PT is said to be concave upward or convex downward.

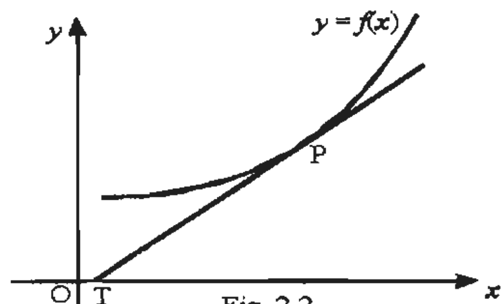


Fig. 2.2

The curve (or an arc of the curve) which lies below the tangent line PT (Fig. 2.3) is said to be convex upward or concave downward.

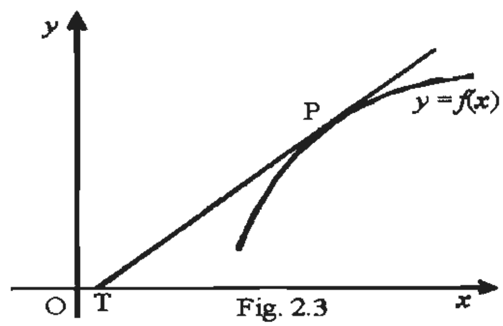


Fig. 2.3

2.1.8 Conditions for Concavity and Convexity.

Let $f(x)$ be twice differentiable. Then the curve $y = f(x)$ is

(i) concave upward on any interval if $f''(x) > 0$

(ii) convex upward on any interval if $f''(x) < 0$

2.1.9 Point of Inflection

A point on a curve $y = f(x)$, where the concavity changes from upto down or vice versa is called a Point of Inflection.

For example, in $y = x^{\frac{1}{3}}$ (Fig. 2.4) has a point of inflection at $x = 0$

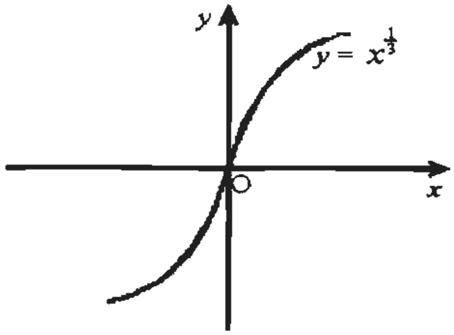


Fig. 2.4

2.1.10 Conditions for point of inflection

A point $(c, f(c))$ on a curve $y = f(x)$ is a point of inflection (i) if $f''(c) = 0$ or $f''(c)$ is not defined and (ii) if $f''(x)$ changes sign as x increases through c i.e. $f'''(c) \neq 0$ when $f'''(x)$ exists

Example 6

Investigate the maxima and minima of the function $2x^3 + 3x^2 - 36x + 10$.

Solution :

Let $y = 2x^3 + 3x^2 - 36x + 10$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 6x^2 + 6x - 36 \qquad \text{--------(1)}$$

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 + 6x - 36 = 0$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x + 3)\ (x - 2) = 0$$

$$\Rightarrow x = -3, 2$$

Again differentiating (1) with respet to x, we get

$$\frac{d^2y}{dx^2} = 12x + 6$$

when $x = -3$, $\frac{d^2y}{dx^2} = 12(-3) + 6 = -30 < 0$

$\therefore$ It attains maximum at $x = -3$

$\therefore$ Maximum value is $y = 2(-3)^3 + 3(-3)^2 - 36(-3) + 10 = 91$

when $x = 2$, $\frac{d^2y}{dx^2} = 12(2) + 6 = 30 > 0$

$\therefore$ It attains minimum at $x = 2$

$\therefore$ Minimum value is $y = 2(2)^3 + 3(2)^2 - 36(2) + 10 = -34$

Example 7

Find the absolute (global) maximum and minimum values of the function $f(x) = 3x^5 - 25x^3 + 60x + 1$ in the interval $[-2, 1]$

Solution :

Given $f(x) = 3x^5 - 25x^3 + 60x + 1$

$f^1(x) = 15x^4 - 75x^2 + 60$

The necessary condition for maximum and minimum is

$f'(x) = 0$

$\Rightarrow 15x^4 - 75x^2 + 60 = 0$

$\Rightarrow x^4 - 5x^2 + 4 = 0$

$\Rightarrow x^4 - 4x^2 - x^2 + 4 = 0$

$\Rightarrow (x^2 - 1)(x^2 - 4) = 0$

$\therefore \quad x = \pm 1, -2, \qquad (2 \notin [-2, 1])$

$f''(x) = 60x^3 - 150x$

$f''(-2) = 60(-2)^3 - 150(-2) \quad = -180 < 0$

$\therefore f(x)$ is maximum.

$f''(-1) = 60(-1)^3 - 150(-1) \quad = 90 > 0$

$\therefore f(x)$ is minimum.

$f''(1) = 60(1)^3 - 150(1) \quad = -90 < 0$

$\therefore f(x)$ is maximum.

The maximum value when $x = -2$ is

$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1 = -15$

The minimum value when $x = -1$ is

$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1 = -37$

The maximum value when $x = 1$ is

$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1 = 39$

$\therefore$ Absolute maximum value = 39.

and Absolute minimum value = −37

Example 8

What is the maximum slope of the tangent to the curve $y = -x^3 + 3x^2 + 9x - 27$ and at what point is it?

Solution :

We have $y = -x^3 + 3x^2 + 9x - 27$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

$\therefore$ Slope of the tangent is $-3x^2 + 6x + 9$

Let $M = -3x^2 + 6x + 9$

Differenating with respect to x, we get

$$\frac{dM}{dx} = -6x + 6 \text{ ------------(1)}$$

Slope is maximum when $\frac{dM}{dx} = 0$ and $\frac{d^2M}{dx^2} < 0$

$$\frac{dM}{dx} = 0 \Rightarrow -6x + 6 = 0$$

$$\Rightarrow x = 1$$

Again differentiating (1) with respect to x, we get

$$\frac{d^2M}{dx^2} = -6 < 0, \quad \therefore \text{ M is maximum at } x = 1$$

$\therefore$ Maximum value of M when $x = 1$ is

$$M = -3(1)^2 + 6(1) + 9 = 12$$

When $x = 1$; $y = -(1)^3 + 3(1)^2 + 9(1) - 27 = -16$

$\therefore$ Maximum slope = 12

The required point is (1, -16)

Example 9

Find the points of inflection of the curve $y = 2x^4 - 4x^3 + 3$.

Solution :

We have $y = 2x^4 - 4x^3 + 3$

Differentiate with respect to x , we get

$$\frac{dy}{dx} = 8x^3 - 12x^2$$

$$\frac{d^2y}{dx^2} = 24x^2 - 24x$$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad 24x(x-1) = 0$$

$$\Rightarrow \quad x = 0, 1$$

$$\frac{d^3y}{dx^3} = 48x - 24$$

when $x = 0, 1 \quad \frac{d^3y}{dx^3} \neq 0.$

$\therefore$ points of inflection exist.

when $x = 0$, $\quad y = 2(0)^4 - 4(0)^3 + 3 = 3$

when $x = 1$, $\quad y = 2(1)^4 - 4(1)^3 + 3 = 1$

$\therefore$ The points of inflection are (0, 3) and (1, 1)

Example 10

Find the intervals on which the curve $f(x) = x^3 - 6x^2 + 9x - 8$ is convex upward and convex downward.

Solution :

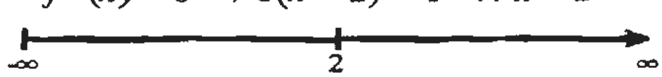
We have $f(x) = x^3 - 6x^2 + 9x - 8$

Differentiating with respect o x,

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12$$

$$f''(x) = 0 \Rightarrow 6(x-2) = 0 \quad \therefore x = 2$$



	For
(i) $-\infty < x < 2, f''(x) < 0$	(i) $x = 0$ then $f''(x) = -12 < 0$
(ii) $2 < x < \infty, f''(x) > 0$	(ii) $x = 3$ then $f''(x) = 6 > 0$

$\therefore$ The curve is convex upward in the interval $(-\infty, 2)$

The curve is convex downward in the interval $(2, \infty)$

Exercise 2.1

1) Show that the function $x^3 + 3x^2 + 3x + 7$ is an increasing function for all real values of x.

2) Prove that $75 - 12x + 6x^2 - x^3$ always decreases as x increases.

3) Separate the intervals in which the function $x^3 + 8x^2 + 5x - 2$ is increasing or decreasing.

4) Find the stationary points and the stationary values of the function $f(x) = 2x^3 + 3x^2 - 12x + 7$.

5) For the following total revenue functions, find when the total revenue (R) is increasing and when it is decreasing. Also discuss the behaviour of marginal revenue (MR).
(i) $R = -90 + 6x^2 - x^3$ (ii) $R = -105x + 60x^2 - 5x^3$

6) For the following cost functions, find when the total cost (C) is increasing and when it is decreasing. Also discuss the behaviour of marginal cost (MC).
(i) $C = 2000 + 600x - 45x^2 + x^3$ (ii) $C = 200 + 40x - \frac{1}{2}x^2$.

7) Find the maximum and minimum values of the function
(i) $x^3 - 6x^2 + 7$ (ii) $2x^3 - 15x^2 + 24x - 15$
(iii) $x^2 + \frac{16}{x}$ (iv) $x^3 - 6x^2 + 9x + 15$

8) Find the absolute (global) maximum and minimum values of the function $f(x) = 3x^5 - 25x^3 + 60x + 15$ in the interval $[-\frac{3}{2}, 3]$.

9) Find the points of inflection of the curve $y = x^4 - 4x^3 + 2x + 3$.

10) Show that the maximum value of the function $f(x) = x^3 - 27x + 108$ is 108 more than the minimum value.

11) Find the intervals in which the curve $y = x^4 - 3x^3 + 3x^2 + 5x + 1$ is convex upward and convex downward.

12) Determine the value of output q at which the cost function $C = q^2 - 6q + 120$ is minimum.

13) Find the maximum and minimum values of the function $x^5 - 5x^4 + 5x^3 - 1$. Discuss its nature at $x = 0$.

14) Show that the function $f(x) = x^2 + \frac{250}{x}$ has a minimum value at $x = 5$.

15) The total revenue (TR) for commodity x is $TR = 12x + \frac{x^2}{2} - \frac{x^3}{3}$.
Show that at the highest point of average revenue (AR), AR = MR (where MR = Marginal Revenue).

2.2 APPLICATION OF MAXIMA AND MINIMA

The concept of zero slope helps us to determine the maximum value of profit functions and the minimum value of cost functions. In this section we will analyse the practical application of Maxima and Minima in commerce.

Example 11

A firm produces x tonnes of output at a total cost $C = (\frac{1}{10}x^3 - 5x^2 + 10x + 5)$. At what level of output will the marginal cost and the average variable cost attain their respective minimum?

Solution :

$$\text{Cost } C(x) = \text{Rs.}(\frac{1}{10}x^3 - 5x^2 + 10x + 5)$$

$$\text{Marginal Cost} = \frac{d}{dx}(C)$$

$$MC = \frac{3}{10}x^2 - 10x + 10$$

$$\text{Average variable cost} = \frac{\text{Variable cost}}{x}$$

$$AVC = (\frac{1}{10}x^2 - 5x + 10)$$

(i) Let $y = MC = \frac{3}{10}x^2 - 10x + 10$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{3}{5}x - 10$$

Marginal cost is minimum when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{3}{5}x - 10 = 0 \text{ or } x = \frac{50}{3}$$

when $x = \frac{50}{3}$, $\frac{d^2y}{dx^2} = \frac{3}{5} > 0$ $\therefore$ MC is minimum.

$\therefore$ Marginal cost attains its minimum at $x = \frac{50}{3}$ units.

(ii) Let $z = AVC = \frac{1}{10}x^2 - 5x + 10$

Differentiating with respect to x, we get

$$\frac{dz}{dx} = \frac{1}{5}x - 5$$

AVC is minimum when $\frac{dz}{dx} = 0$, and $\frac{d^2z}{dx^2} > 0$

$$\frac{dz}{dx} = 0 \Rightarrow \frac{1}{5}x - 5 = 0 \Rightarrow x = 25.$$

when $x = 25$, $\frac{d^2z}{dx^2} = \frac{1}{5} > 0$ $\therefore$ AVC is minimum at $x = 25$ units.

$\therefore$ Average variable cost attains minimum at $x = 25$ units.

Example 12

A certain manufacturing concern has total cost function $C = 15 + 9x - 6x^2 + x^3$. Find x, when the total cost is minimum.

Solution :

Cost $C = 15 + 9x - 6x^2 + x^3$

Differentiating with respect to x, we get

$$\frac{dC}{dx} = 9 - 12x + 3x^2 \qquad \text{----------(1)}$$

Cost is minimum when $\frac{dC}{dx} = 0$ and $\frac{d^2C}{dx^2} > 0$

$$\frac{dC}{dx} = 0 \Rightarrow 3x^2 - 12x + 9 = 0$$

$$x^2 - 4x + 3 = 0$$

$$\Rightarrow x = 3, \ x = 1$$

Differentiating (1) with respect to x we get

$$\frac{d^2C}{dx^2} = -12 + 6x$$

when $x = 1$; $\frac{d^2C}{dx^2} = -12 + 6 = -6 < 0 \therefore$ C is maximum

when $x = 3$, $\frac{d^2C}{dx^2} = -12 + 18 = 6 > 0 \quad \therefore$ C is minimum

$\therefore$ when $x = 3$, the total cost is minimum

Example 13

The relationship between profit P and advertising cost x is given by $P = \frac{4000x}{500+x} - x$. Find x which maximises P.

Solution :

Profit $P = \frac{4000\,x}{500 + x} - x$

Differentiating with respect to x we get

$$\frac{dP}{dx} = \frac{(500+x)4000 - (4000x)(1)}{(500+x)^2} - 1$$

$$= \frac{2000000}{(500+x)^2} - 1 \qquad \text{-----------(1)}$$

Profit is maximum when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$$\frac{dP}{dx} = 0 \Rightarrow \frac{2000000}{(500+x)^2} - 1 = 0$$

$$\Rightarrow 2000000 = (500 + x)^2$$

$$\Rightarrow 1000 \times \sqrt{2} = 500 + x$$

$$500 + x$$

$$x = 914.$$

Differentiating (1) with respect to x we get

$$\frac{d^2P}{dx^2} = -\frac{4000000}{(500+x)^3}$$

$\therefore$ when $x = 914$; $\frac{d^2P}{dx^2} < 0$ $\therefore$ Profit is maximum.

Example 14

The total cost and total revenue of a firm are given by $C = x^3 - 12x^2 + 48x + 11$ and $R = 83x - 4x^2 - 21$. Find the output (i) when the revenue is maximum (ii) when profit is maximum.

Solution :

(i) Revenue $R = 83x - 4x^2 - 21$

Differentiating with respect to x,

$$\frac{dR}{dx} = 83 - 8x$$

$$\frac{d^2R}{dx^2} = -8$$

Revenue is maximum when $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$

$$\frac{dR}{dx} = 0 \Rightarrow 83 - 8x = 0 \quad \therefore x = \frac{83}{8}$$

Also $\frac{d^2R}{dx^2} = -8 < 0$. $\therefore$ R is maximum

$\therefore$ When the output $x = \frac{83}{8}$ units, revenue is maximum

(ii) Profit $P = R - C$

$$= (83x - 4x^2 - 21) - (x^3 - 12x^2 + 48x + 11)$$

$$= -x^3 + 8x^2 + 35x - 32$$

Differentiating with respect to x,

$$\frac{dP}{dx} = -3x^2 + 16x + 35$$

$$\frac{d^2P}{dx^2} = -6x + 16$$

Profit is maximum when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$$\therefore \frac{dP}{dx} = 0 \Rightarrow -3x^2 + 16x + 35 = 0$$

$$\Rightarrow 3x^2 - 16x - 35 = 0$$

$$\Rightarrow (3x + 5)\ (x - 7) = 0$$

$$\Rightarrow x = \frac{-5}{3} \text{ or } x = 7$$

when $x = \frac{-5}{3}$, $\frac{d^2P}{dx^2} = -6(\frac{-5}{3}) + 16 = 26 > 0$ $\therefore$ P is minimum

when $x = 7$, $\frac{d^2P}{dx^2} = -6(7) + 16 = -26 < 0$ $\therefore$ P is maximum

$\therefore$ when $x = 7$ units, profit is maximum.

Example 15

A telephone company has a profit of Rs. 2 per telephone when the number of telephones in the exchange is not over 10,000. The profit per telephone decreases by 0.01 paisa for each telephone over 10,000. What is the maximum profit?

Solution :

Let x be the number of telephones.

The decrease in the profit per telephone

$= (x - 10{,}000)(0.01)$, $x > 10{,}000$.

$= (0.01x - 100)$

The profit per telephone

$= 200 - (0.01x - 100)$

$= (300 - 0.01x)$

The total profit for x telephones

$= x(300 - 0.01x)$

$= 300x - 0.01x^2$

Let the total profit $P = 300x - 0.01x^2$

Differentiating with respect to x, we get

$$\frac{dP}{dx} = 300 - 0.02x \qquad \text{---------(1)}$$

Conditions for the maximum profit are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2P}{dx^2} < 0$$

$$\frac{dP}{dx} = 0 \Rightarrow 300 - 0.02x = 0$$

$$\Rightarrow x = \frac{300}{0.02} = 15{,}000.$$

Differentiating (1) with respect to x we get

$$\frac{d^2P}{dx^2} = -0.02 < 0 \quad \therefore \quad \text{P is maximum}$$

$\therefore$ when $x = 15{,}000$, the maximum profit

$P = (300 \times 15{,}000) - (0.01) \times (15{,}000)^2$ paise

$= \text{Rs. } (45{,}000 - 22{,}500) = \text{Rs. } 22{,}500$

$\therefore$ Maximum profit is Rs. 22,500.

Example 16

The total cost function of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ where x is the output. A tax at Rs. 2 per unit of output is imposed and the producer adds it to his cost. If the market

demand function is given by $p = 2530 - 5x$, where Rs. p is the price per unit of output, find the profit maximising output and price.

Solution :

Total Revenue (R) $= px$

$$= (2530 - 5x)x = 2530x - 5x^2$$

Total cost after the imposition of tax is

$$C + 2x = \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x$$

$$= \frac{1}{3}x^3 - 5x^2 + 30x + 10$$

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$= (2530x - 5x^2) - (\frac{1}{3}x^3 - 5x^2 + 30x + 10)$$

$$P = -\frac{1}{3}x^3 + 2500x - 10$$

Differentiating P with respect to x,

$$\frac{dP}{dx} = -x^2 + 2500 \qquad \text{----------(1)}$$

Conditions for maximum profit are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2P}{dx^2} < 0$$

$$\frac{dP}{dx} = 0 \quad \Rightarrow 2500 - x^2 = 0$$

$$\Rightarrow x^2 = 2500 \quad \text{or } x = 500$$

Differentiating (1) with respect to x

$$\frac{d^2P}{dx^2} = -2x$$

When $x = 50$, $\frac{d^2P}{dx^2} = -50 < 0$ $\therefore$ P is maximum

$\therefore$ Profit maximising output is 50 units

When $x = 50$, price $p = 2530 - (5 \times 50)$

$$= 2530 - 250 = \text{Rs. } 2280$$

2.2.1 Inventory Control

Inventory is defined as the stock of goods. In practice raw materials are stored upto a capacity for smooth and efficient running of business.

2.2.2 Costs Involved in Inventory Problems

(i) **Holding cost or storage cost or inventory carrying cost. (C_1)**

The cost associated with carrying or holding the goods in stock is known as holding cost per unit per unit time.

(ii) **Shortage cost (C_2)**

The penalty costs that are incurred as a result of running out of stock are known as shortage cost.

(iii) **Set up cost or ordering cost or procurement cost : (C_3)**
This is the cost incurred with the placement of order or with the initial preparation of production facility such as resetting the equipment for production.

2.2.3 Economic Order Quantity (EOQ)

Economic order quantity is that size of order which minimises total annual cost of carrying inventory and the cost of ordering under the assumed conditions of certainty with the annual demands known. Economic order quantity is also called Economic lot size formula.

2.2.4 Wilson's Economic Order Quantity Formula

The formula is to determine the optimum quantity ordered (or produced) and the optimum interval between successive orders, if the demand is known and uniform with no shortages.

Let us have the following assumptions.

(i) Let R be the uniform demand per unit time.

(ii) Supply or production of items to the inventory is instantaneous.

(iii) Holding cost is Rs. C_1 per unit per unit time.

(iv) Let there be n orders (cycles) per year, each time q units are ordered (produced).

(v) Let Rs C_3 be the ordering (set up) cost per order (cycle). Let t be the time taken between each order.

Diagramatic representation of this model is given below :

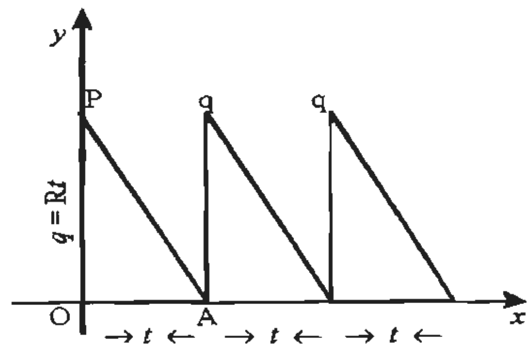


Fig. 2.5

If a production run is made at intervals t, a quantity q = Rt must be produced in each run. Since the stock in small time dt is Rt dt, the stock in period t is

$$\int_0^t Rt\,dt = \frac{1}{2}Rt^2$$

$$= \frac{1}{2}qt \qquad (\because Rt = q)$$

= Area of the inventory triangle OAP (Fig. 4.5).

Cost of holding inventory per production run $= \frac{1}{2}C_1\,Rt^2$.

Set up cost per production run $= C_3$.

$\therefore$ Total cost per production run $= \frac{1}{2}C_1\,Rt^2 + C_3$

Average total cost per unit time

$$C(t) = \frac{1}{2} C_1 Rt + \frac{C_3}{t} \text{ ---------(1)}$$

$C(t)$ is minimum if $\frac{d}{dt}C(t) = 0$ and $\frac{d^2}{dt^2}C(t) > 0$

Differentiating (1) with respect to t we get

$$\frac{d}{dt}C(t) = \frac{1}{2}C_1 R - \frac{C_3}{t^2} \text{ ---------(2)}$$

$$\frac{d}{dt}C(t) = 0 \quad \Rightarrow \frac{1}{2} C_1 R - \frac{C_3}{t^2} = 0$$

$$\Rightarrow t = \sqrt{\frac{2C_3}{C_1R}}$$

Differentiating (2) with respect to t, we get

$$\frac{d^2}{dt^2}C(t) = \frac{2C_3}{t^3} > 0, \text{ when } t = \sqrt{\frac{2C_3}{C_1R}}$$

Thus $C(t)$ is minimum for optimum time interval

$$t_0 = \sqrt{\frac{2C_3}{C_1R}}$$

Optimum quantity q_0 to be produced during each production run,

$$EOQ = q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}}$$

This is known as the Optimal Lot - size formula due to Wilson.

Note : (i) Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{EOQ} = R\sqrt{\frac{C_1}{2C_3R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

(ii) Minimum average cost per unit time, $C_0 = \sqrt{2C_1C_3R}$

(iii) Carrying cost $= \frac{q_0}{2} \times C_1$, Ordering cost $= \frac{R}{q_0} \times C_3$

(iv) At EOQ, Ordering cost = Carrying cost.

Example 17

A manufacturer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and no shortages are allowed. The inventory holding cost is 20 paise per unit per month and the set up cost per run is Rs.350. Determine (i) the optimum run size q_0 (ii) optimum scheduling period t_0 (iii) minimum total variable yearly cost.

Solution :

$$\text{Supply rate } R = \frac{12{,}000}{12} = 1{,}000 \text{ units / month.}$$

$$C_1 = 20 \text{ paise per unit per month}$$

C_3 = Rs. 350 per run.

(i) $q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times350\times1000}{0.20}}$

= 1,870 units / run.

$t_0 = \sqrt{\frac{2C_3}{C_1R}} = \sqrt{\frac{2\times350}{0.20\times1000}}$ = 56 days

(iii) $C_0 = \sqrt{2C_1C_3R} = \sqrt{2\times0.20\times12\times350\times(1000\times12)}$

= Rs.4,490 per year.

Example 18

A company uses annually 24,000 units of raw materials which costs Rs. 1.25 per unit, placing each order costs Rs. 22.50 and the holding cost is 5.4% per year of the average inventory. Find the EOQ, time between each order, total number of orders per year. Also verify that at EOQ carrying cost is equal to ordering cost

Solution :

Requirement = 24,000 units / year

Ordering Cost (C_3) = Rs.22.50

Holding cost (C_1) = 5.4% of the value of each unit.

$= \frac{5.4}{100} \times 1,25$

= Re.0.0675 per unit per year.

$EOQ = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2\times2400\times22.5}{0.0675}}$ = 4000 units.

Time between each order $= t_0 = \frac{q_0}{R} = \frac{4000}{24000} = \frac{1}{6}$ year

Number of order per year $= \frac{R}{q_0} = \frac{24000}{4000} = 6$

At EOQ carrying cost $= \frac{q_0}{2} \times C_1 = \frac{4000}{2} \times 0.0675$ = Rs.135

Ordering cost $= \frac{R}{q_0} \times C_3 = \frac{24000}{4000} \times 22.50$ = Rs.135

Example 19

A manufacturing company purchases 9000 parts of a machine for its annual requirements. Each part costs Rs.20. The ordering cost per order is Rs.15 and carrying charges are 15% of the average inventory per year.

Find (i) economic order quantity

(ii) time between each order

(iii) minimum average cost

Solution :

Requirement $R = 9000$ parts per year

$C_1 = 15\%$ unit cost

$= \frac{15}{100} \times 20 =$ Rs.3 each part per year.

$C_3 =$ Rs.15 per order

$$EOQ = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times15\times9000}{3}}$$

$= 300$ units.

$$t_0 = \frac{q_0}{R} = \frac{300}{9000} = \frac{1}{30} \text{ year}$$

$$= \frac{365}{30} = 12 \text{ days (approximately).}$$

Minimum Average cost $= \sqrt{2C_1C_3R}$

$$= \sqrt{2\times3\times15\times9000} = \text{Rs.}900$$

EXERCISE 2.2

1) A certain manufacturing concern has the total cost function $C = \frac{1}{5}x^2 - 6x + 100$. Find when the total cost is minimum.

2) A firm produces an output of x tons of a certain product at a total cost given by $C = 300x - 10x^2 + \frac{1}{3}x^3$. Find the output at which the average cost is least and the corresponding value of the aveage cost.

3) The cost function, when the output is x, is given by $C = x(2e^x + e^{-x})$. Show that the minimum average cost is $2\sqrt{2}$.

4) A firm produces x tons of a valuable metal per month at a total cost C given by $C =$ Rs.$(\frac{1}{3}x^3 - 5x^2 + 75x + 10)$. Find at what level of output, the marginal cost attains its minimum.

5) A firm produces x units of output per week at a total cost of Rs. $(\frac{1}{3}x^3 - x^2 + 5x + 3)$. Find the level at which the marginal cost and the average variable cost attain their respective minimum.

6) It is known that in a mill the number of labourers x and the total cost C are related by $C = \frac{3}{2(x-4)} + \frac{3}{32}x$. What value of x will minimise the cost?

7) $R = 21x - x^2$ and $C = \frac{x^3}{3} - 3x^2 + 9x + 16$ are respectively the sales revenue and cost function of x units sold.

Find (i) At what output the revenue is maximum? What is the total revenue at this point?

(ii) What is the marginal cost at a minimum?

(iii) What output will maximise the profit?

8) A firm has revenue function $R = 8x$ and a production cost function $C = 150000 + 60\left(\frac{x^2}{900}\right)$. Find the total profit function and the number of units to be sold to get the maximum profit.

9) A radio manufacturer finds that he can sell x radios per week at Rs p each, where $p = 2(100 - \frac{x}{4})$. His cost of production of x radios per week is Rs. $(120x + \frac{x^2}{2})$. Show that his profit is maximum when the production is 40 radios per week. Find also his maximum profit per week.

10) A manufacturer can sell x items per week at a price of $p = 600 - 4x$ rupees. Production cost of x items works out to Rs. C where $C = 40x + 2000$. How much production will yield maximum profit?

11) Find the optimum output of a firm whose total revenue and total cost functions are given by $R = 30x - x^2$ and $C = 20 + 4x$, x being the output of the firm.

12) Find EOQ for the data given below. Also verify that carrying costs is equal to ordering costs at EOQ.

Item	Monthly Requirements	Ordering cost per order	Carrying cost Per unit.
A	9000	Rs. 200	Rs. 3.60
B	25000	Rs. 648	Rs. 10.00
C	8000	Rs. 100	Rs. 0.60

13) Calculate the EOQ in units and total variable cost for the following items, assuming an ordering cost of Rs.5 and a holding cost of 10%

Item	Annual demand	Unit price (Rs.)
A	460 Units	1.00
B	392 Units	8.60
C	800 Units	0.02
D	1500 Units	0.52

14) A manufacturer has to supply his customer with 600 units of his products per year. Shortages are not allowed and storage cost amounts to 60 paise per unit per year. When the set up cost is Rs. 80 find,

(i) the economic order quantity.

(ii) the minimum average yearly cost

(iii) the optimum number of orders per year

(iv) the optimum period of supply per optimum order.

15) The annual demand for an item is 3200 units. The unit cost is Rs.6 and inventory carrying charges 25% per annum. If the cost of one procurement is Rs.150, determine (i) Economic order quantity. (ii) Time between two consecutive orders (iii) Number of orders per year (iv) minimum average yearly cost.

2.3 PARTIAL DERIVATIVES

In differential calculus, so far we have discussed functions of one variable of the form $y = f(x)$. Further one variable may be expressed as a function of several variables. For example, production may be treated as a function of labour and capital and price may be a function of supply and demand. In general, the cost or profit depends upon a number of independent variables, for example, prices of raw materials, wages on labour, market conditions and so on. Thus a dependent variable y depends on a number of independent variables $x_1, x_2, x_3..x_n$. It is denoted by $y = f(x_1, x_2, x_3..x_n)$ and is called a function of n variables. In this section, we will restrict the study to functions of two or three variables and their derivatives only.

2.3.1. Definition

Let $u = f(x, y)$ be a function of two independent variables x and y. The derivative of $f(x, y)$ with respect to x, keeping y constant, is called partial derivative of u with respect to x and is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or u_x. Similarly we can define partial derivative of f with respect to y.

Thus we have

$$\frac{\partial f}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} = \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

provided the limit exists.

(Here y is fixed and Δx is the increment of x)

Also $$\frac{\partial f}{\partial y} = \underset{\Delta y \to 0}{\text{Lt}} = \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

provided the limit exists.

(Here x is fixed and Δy is the increment of y).

2.3.2 Successive Partial Derivatives.

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are in general functions of x and y. So we can differentiate functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ partially with respect to x and y. These derivatives are called second order partial derivatives of $f(x, y)$. Second order partial derivatives are denoted by $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

Note

If f, f_x, f_y are continuous then $f_{xy} = f_{yx}$

2.3.3 Homogeneous Function

A function $f(x, y)$ of two independent variable x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for $t > 0$.

2.3.4 Euler's Theorem on Homogeneous Function

Theorem : Let f be a homogeneous function in x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\ f.$$

Corrollary : In general if $f(x_1, x_2, x_3 \ldots x_m)$ is a homogeneous function of degree n in variables $x_1, x_2, x_3 \ldots x_m$, then,

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} + \ldots + x_m\frac{\partial f}{\partial x_m} = n\ f.$$

Example 20

It $u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$, find each of the following.

(i) $\frac{\partial u}{\partial x}$ **(ii)** $\frac{\partial u}{\partial y}$ **(iii)** $\frac{\partial^2 u}{\partial x^2}$ **(iv)** $\frac{\partial^2 u}{\partial y^2}$ **(v)** $\frac{\partial^2 u}{\partial x \partial y}$ **(vi)** $\frac{\partial^2 u}{\partial y \partial x}$

Solution :

$$u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$$

(i) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(1000 - x^3 - y^2 + 4x^3y^6 + 8y)$

$$= 0 - 3x^2 - 0 + 4(3x^2)y^6 + 0$$

$$= -3x^2 + 12x^2y^6.$$

(ii) $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(1000 - x^3 - y^2 + 4x^3y^6 + 8y)$

$$= 0 - 0 - 2y + 4x^3(6y^5) + 8$$

$$= -2y + 24x^3y^5 + 8$$

(iii) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$$= \frac{\partial}{\partial x}(-3x^2 + 12x^2y^6)$$
$$= -6x + 12(2x)y^6$$
$$= -6x + 24xy^6.$$

(iv) $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}(-2y + 24x^3y^5 + 8)$$
$$= -2 + 24x^3(5y^4) + 0$$
$$= -2 + 120x^3y^4$$

(v) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}(-2y + 24x^3y^5 + 8)$$
$$= 0 + 24(3x^2)y^5 + 0$$
$$= 72x^2y^5.$$

(vi) $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$
$$= \frac{\partial}{\partial y}(-3x^2 + 12x^2y^6)$$
$$= 0 + 12x^2(6y^5) = 72x^2y^5$$

Example 21

If $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 5$ find (i) $f_x(1, -1)$ (ii) $f_{yy}(1, 1)$ (iii) $f_{xy}(2, 1)$

Solution :

(i) $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 5$

$$f_x = \frac{\partial}{\partial x}(f) = \frac{\partial}{\partial x}(3x^2 + 4y^3 + 6xy - x^2y^3 + 5)$$
$$= 6x + 0 + 6(1)y - (2x)y^3 + 0$$
$$= 6x + 6y - 2xy^3.$$
$$f_x(1, -1) = 6(1) + 6(-1) - 2(1)(-1)^3 = 2$$

(ii) $$f_y = \frac{\partial}{\partial y}(f) = \frac{\partial}{\partial y}(3x^2 + 4y^3 + 6xy - x^2y^3 + 5)$$
$$= 12y^2 + 6x - 3x^2y^2$$
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}(12y^2 + 6x - 3x^2y^2)$$
$$= 24y - 6x^2y$$
$$\therefore f_{yy}(1, 1) = 18$$

(iii) $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\,(12y^2 + 6x - 3x^2y^2)$

$= 6 - 6xy^2$

$\therefore\ f_{xy}(2, 1) = -6$

Example 22

If $u = \log\sqrt{x^2 + y^2 + z^2}$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{x^2 + y^2 + z^2}$$

Solution :

We have $u = \frac{1}{2}\log\,(x^2 + y^2 + z^2)$ -----------(1)

Differentiating (1) partially with respect to x,

$$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2 + y^2 + z^2} = \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2 + z^2}\right) = \frac{(x^2 + y^2 + z^2)(1) - x(2x)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{x^2 + y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2}$$

Differentiating (1) partially with respect to y we get,

$$\frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2 + z^2)(1) - y(2y)}{(x^2 + y^2 + z^2)^2} = \frac{-y^2 + z^2 + x^2}{(x^2 + y^2 + z^2)^2}$$

Differentiating (1) partially with respect to z we get,

$$\frac{\partial u}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{(x^2 + y^2 + z^2)(1) - z(2z)}{(x^2 + y^2 + z^2)^2} = \frac{-z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-x^2 + y^2 + z^2 - y^2 + z^2 + x^2 - z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}$$

Example 23

Verify Euler's theorem for the function
$u(x, y) = x^3 + y^3 + x^2y$.

Solution :

We have $u(x, y) = x^3 + y^3 + x^2y$ ---------(1)

$u(tx, ty) = t^3x^3 + t^3y^3 + t^2x^2\,(ty)$

$$= t^3 (x^3 + y^3 + x^2y) = t^3\, u(x, y)$$

$\therefore$ u is a homogeneous function of degree 3 in x and y.

We have to verify that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy$$

$$\therefore \quad x\frac{\partial u}{\partial x} = 3x^3 + 2x^2y$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = 3y^2 + x^2$$

$$\therefore \quad y\frac{\partial u}{\partial y} = 3y^3 + x^2y$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3 + 2x^2y + 3y^3 + x^2y$$

$$= 3(x^3 + x^2y + y^3) = 3u$$

Thus Euler's Theorem is verified, for the given function.

Example 24

Using Euler's theorem if $u = \log\frac{x^4 + y^4}{x - y}$

show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Solution :

$$u = \log\frac{x^4 + y^4}{x - y}$$

$$\Rightarrow \quad e^u = \frac{x^4 + y^4}{x - y}$$

This is a homogeneous function of degree 3 in x and y

$\therefore$ By Euler's theorem,

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3e^u$$

$$x\, e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 3e^u$$

dividing by e^u we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Example 25

Without using Euler's theorem prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4u$, if $u = 3x^2yz + 4xy^2z + 5y^4$

Solution :

We have $u = 3x^2yz + 4xy^2z + 5y^4$ --------(1)

Differentiating partially with respect to x, we get

$$\frac{\partial u}{\partial x} = 3(2x)yz + 4(1)y^2z + 0$$

$$= 6xyz + 4y^2z$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = 3x^2(1)z + 4x(2y)z + 20y^3$$
$$= 3x^2z + 8xyz + 20y^3$$

Differentiating (1) partially with respect to z, we get

$$\frac{\partial u}{\partial z} = 3x^2y(1) + 4xy^2(1) + 0$$
$$= 3x^2y + 4xy^2$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$
$$= 6x^2yz + 4xy^2z + 3x^2yz + 8xy^2z + 20y^4 + 3x^2yz + 4xy^2z$$
$$= 12x^2yz + 16xy^2z + 20y^4$$
$$= 4(3x^2yz + 4xy^2z + 5y^4) = 4u.$$

Example 26

The revenue derived from selling x calculators and y adding machines is given by $R(x, y) = -x^2+8x-2y^2+6y+2xy+50$. If 4 calculators and 3 adding machines are sold, find the marginal revenue of selling (i) one more calculator (ii) one more adding machine.

Solution :

(i) The marginal revenue of selling one more calculator is R_x.

$$R_x = \frac{\partial}{\partial x}(R) = \frac{\partial}{\partial x}(-x^2 + 8x - 2y^2 + 6y + 2xy + 50)$$
$$= -2x + 8 - 0 + 0 + 2(1)(y)$$
$$R_x(4, 3) = -2(4) + 8 + 2(3) = 6$$

$\therefore$ At (4, 3), revenue is increasing at the rate of Rs.6 per caculator sold.

$\therefore$ Marginal revenue is Rs. 6.

(ii) Marginal Revenue of selling one more adding machine is R_y

$$R_y = \frac{\partial}{\partial y}(R) = \frac{\partial}{\partial y}(-x^2 + 8x - 2y^2 + 6y + 2xy + 50)$$
$$= 0 + 0 - 4y + 6 + 2x(1)$$
$$= -4y + 6 + 2x$$
$$R_y(4, 3) = -4(3) + 6 + 2(4) = 2$$

Thus at (4, 3) revenue is increasing at the rate of approximately Rs.2 per adding machine.

Hence Marginal revenue is Rs.2.

EXERCISE 2.3

1) If $u = 4x^2 - 3y^2 + 6xy$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

2) If $u = x^3 + y^3 + z^3 - 3xyz$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$

3) If $z = 4x^6 - 8x^3 - 7x + 6xy + 8y + x^3y^5$, find each of the follwoing

(i) $\frac{\partial u}{\partial x}$ (ii) $\frac{\partial u}{\partial y}$ (iii) $\frac{\partial^2 z}{\partial x^2}$ (iv) $\frac{\partial^2 z}{\partial y^2}$ (v) $\frac{\partial^2 z}{\partial x \partial y}$ (vi) $\frac{\partial^2 z}{\partial y \partial x}$

4) If $f(x, y) = 4x^2 - 8y^3 + 6x^5y^2 + 4x + 6y + 9$, evaluate the following.

(i) f_x (ii) $f_x(2, 1)$ (iii) f_y (iv) $f_y(0, 2)$

(v) f_{xx} (vi) $f_{xx}(2, 1)$ (vii) f_{yy} (viii) $f_{yy}(1, 0)$

(ix) f_{xy} (x) $f_{xy}(2, 3)$ (xi) $f_{yx}(2, 3)$

5) If $u = x^2y + y^2z + z^2x$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$.

6) If $u = \log\sqrt{x^2 + y^2}$, show that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{x^2 + y^2}$

7) If $u = x^3 + 3xy^2 + y^3$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

8) If $e^{\frac{-z}{x^2 - y^2}} = x - y$, prove that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2 - y^2$.

9) Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the function $u = xy + \sin xy$.

10) If $u = \log(x^2 + y^2 + z^2)$ prove that $x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$

11) Verify Euler's theorem for each of the following functions.

(i) $u = \frac{y}{x}$ (ii) $f = x^{\frac{2}{3}} + y^{\frac{2}{3}} - 3x^{\frac{1}{3}}y^{\frac{1}{3}}$

(iii) $z = \frac{x - y}{x + y}$ (iv) $u = \frac{1}{\sqrt{x^2 + y^2}}$

(v) $u = \frac{x^3 + y^3}{x^2 + y^2}$ (vi) $u = x \log\left(\frac{y}{x}\right)$

12) Use Euler's theorem to prove the following

(i) If $u = \frac{x^2 + y^2}{\sqrt{x + y}}$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

(ii) If $z = e^{x^3 + y^3}$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \log z$

(iii) If $f = \log\left(\frac{x^2 + y^2}{x + y}\right)$ then show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1$

(iv) If $u = \tan^{-1}\left(\frac{x^2 + y^2}{x - y}\right)$ then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$.

13) Without using Euler's theorem prove the following

(i) If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(ii) If $u = \log\frac{x^2 + y^2}{x + y}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

14) The cost of producing x washers and y dryers is given by $C(x, y) = 40x + 200y + 10xy + 500$. Presently, 50 washers and 90 dryers are being produced. Find the marginal cost of producing (i) one more washer (ii) one more dryer.

15) The revenue derived from selling x pens and y note books is given by $R(x, y) = 2x^2 + y^2 + 4x + 5y + 800$

At present, the retailer is selling 30 pens and 50 notebooks. Which of these two product lines should be expanded in order to yield the greater increase in revenue?

16) The annual profit of a certain hotel is given by $P(x, y) = 100x^2 + 4y^2 + 2x + 5y + 10000$. Where x is the number of rooms available for rent and y is the monthly advertising expenditures. Presently, the hotel has 90 rooms available and is spending Rs.1000 per month on advertising.

(i) If an additional room is constructed, how will this affect annual profit?

(ii) If an additional rupee is spent on monthly advertising expenditures, how will this affect annual profit?

2.4 APPLICATIONS OF PARTIAL DERIVATIVES

In this section we learn how the concept of partial derivatives are used in the field of Commerce and Economics.

2.4.1 Production Function

Production P of a firm depends upon several economic factors like investment or capital (K), labour (L), raw material (R), etc. Thus $P = f(K, L, R, ...)$. If P depends only on labour (L) and capital (K), then we write $P = f(L, K)$.

2.4.2 Marginal Productivities

Let $P = f(L, K)$ represent a production function of two variables L and K.

$\frac{\partial P}{\partial L}$ is called the 'Marginal Productivity of Labour' and $\frac{\partial P}{\partial K}$ is the 'Marginal Productivity of Capital'.

2.4.3 Partial Elasticities of Demand

Let $q_1 = f(p_1, p_2)$ be the demand for commodity A which depends upon the prices p_1 and p_2 of commodities A and B respectively.

The partial elasticity of demand q_1 with respect to p_1 is defined as

$$-\frac{p_1}{q_1}\frac{\partial q_1}{\partial p_1} = \frac{Eq_1}{Ep_1}$$

Similarly the partial elasticity of demand of q_1 with respect to price p_2 is $-\frac{p_2}{q_1} \cdot \frac{\partial q_1}{\partial p_2} = \frac{Eq_1}{Ep_2}$

Example 27

Find the marginal productivities of capital (K) and labour (L), if $P = 10K - K^2 + KL$, when $K = 2$ and $L = 6$

Solution :

We have $P = 10K - K^2 + KL$ -----------(1)

The marginal productivity of capital is $\frac{\partial P}{\partial K}$

$\therefore$ Differentiating (1) partially with respect to K we get

$$\frac{\partial P}{\partial K} = 10 - 2K + (1)\,L$$
$$= 10 - 2K + L$$

when K =2, and L = 6, $\frac{\partial P}{\partial K} = 10 - 2(2) + 6 = 12$

The marginal productitivy of labour is $\frac{\partial P}{\partial L}$

$\therefore$ Differentiating (1) partially with respect to L we get

$$\frac{\partial P}{\partial L} = K$$

when K = 2, and L = 6 $\frac{\partial P}{\partial L} = 2.$

$\therefore$ Marginal productivity of capital = 12 units

$\therefore$ Marginal productivity of labour = 2 units

Example 28

For some firm, the number of units produced when using x units of labour and y units of capital is given by the production function $f(x, y) = 80\, x^{\frac{1}{4}}\, y^{\frac{3}{4}}$. Find (i) the equations for both marginal productivities. (ii) Evaluate and interpret the results when 625 units of labour and 81 units of capital are used.

Solution :

Given $f(x, y) = 80\, x^{\frac{1}{4}}\, y^{\frac{3}{4}}$ ----------(1)

Marginal producitivity of labour is $f_x(x, y)$.

$\therefore$ Differentiating (1) partially with respect to x, we get

$$f_x = 80\ \frac{1}{4} x^{-\frac{3}{4}} y^{\frac{3}{4}} = 20 x^{-\frac{3}{4}} y^{\frac{3}{4}}$$

Marginal productivity of capital is $f_y(x, y)$

$\therefore$ Differentiating (1) partially with resepect to y we get

$$f_y = 80\ x^{\frac{1}{4}}\left(\frac{3}{4}\right) y^{-\frac{1}{4}} = 60 x^{\frac{1}{4}} y^{-\frac{1}{4}}$$

(ii) $f_x(625, 81) = 20(625)^{-\frac{3}{4}}(81)^{\frac{3}{4}}$

$$= 20\left(\frac{1}{125}\right)(27) = 4.32$$

i.e. when 625 units of labour and 81 units of capital are used, one more unit of labour results in 4.32 more units of production.

$f_y(625, 81) = 60\ (625)^{\frac{1}{4}}(81)^{-\frac{1}{4}}$

$$= 60(5)\left(\frac{1}{3}\right) = 100$$

(i.e.) when 625 units of labour and 81 units of capital are used, one more unit of capital results in 100 more units of production.

Example 29

The demand for a commodity A is $q_1 = 240 - p^2_1 + 6p_2 - p_1 p_2$. Find the partial Elasticities $\frac{Eq_1}{Ep_1}$ and $\frac{Eq_1}{Ep_2}$ when $p_1 = 5$ and $p_2 = 4$.

Solution :

Given $q_1 = 240 - p^2_1 + 6p_2 - p_1 p_2$

$$\frac{\partial q_1}{\partial p_1} = -2p_1 - p_2$$

$$\frac{\partial q_1}{\partial p_2} = 6 - p_1$$

(i) $$\frac{Eq_1}{Ep_1} = -\frac{p_1}{q_1}\frac{\partial q_1}{\partial p_1}$$

$$= \frac{-p_1}{240 - p_1^2 + 6p_2 - p_1 p_2}(-2p_1 - p_2)$$

when $p_1 = 5$ and $p_2 = 4$

$$\left(\frac{Eq_1}{Ep_1}\right) = \frac{-(5)(-10-4)}{240 - 25 + 24 - 20} = \frac{70}{219}$$

(ii) $$\frac{Eq_1}{Ep_2} = -\frac{p_2}{q_1}\frac{\partial q_1}{\partial p_2}$$

$$= \frac{-p_2(6 - p_1)}{240 - p_1^2 + 6p_2 - p_1 p_2}$$

when $p_1 = 5$ and $p_2 = 4$

$$\left(\frac{Eq_1}{Ep_2}\right) = \frac{-4(6-5)}{240 - 25 + 24 - 20} = \frac{-4}{219}$$

EXERCISE 2.4

1) The production function of a commodity is $P = 10L + 5K - L^2 - 2K^2 + 3KL$.

Find (i) the marginal productivity of labour

(ii) the marginal productivity of capital

(iii) the two marginal productivities when L = 1 and K = 2.

2) If the production of a firm is given by $P = 3K^2L^2 - 2L^4 - K^4$, prove that $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = 4P$.

3) If the production function is $Z = y^2 - xy + x^2$ where x is the labour and y is the capital find the marginal productivities of x and y when $x = 2$ and $y = 3$.

4) For some firm, the number of units produced when using x units of labour and y units of capital is given by the production function $f(x,y) = 100x^{\frac{1}{5}}\ y^{\frac{4}{5}}$. Find
(i) both marginal productivities.
(ii) interpret the results when 243 units of labour and 32 units of capital are used.

5) For the production function $p = 5(L)^{0.7}\ (K)^{0.3}$ find the marginal productivities of labour (L) and capital (K) when L = 10 and K = 3.

6) For the production function $P = C(L)^{\alpha}\ (K)^{\beta}$ where C is a positive constant and if $\alpha + \beta = 1$ show that

$$K\frac{\partial P}{\partial K} + L\frac{\partial P}{\partial L} = P.$$

7) The demand for a quantity A is $q_1 = 16 - 3p_1 - 2p_2^2$. Find (i) the partial elasticities $\frac{Eq_1}{Ep_1}, \frac{Eq_1}{Ep_2}$ (ii) the partial elasticities for $p_1 = 2$ and $p_2 = 1$.

8) The demand for a commodity A is $q_1 = 10 - 3p_1 - 2p_2$. Find the partial elasticities when $p_1 = p_2 = 1$.

9) The demand for a commodity X is $q_1 = 15 - p_1^2 - 3p_2$. Find the partial elasticities when $p_1 = 3$ and $p_2 = 1$.

10) The demand function for a commodity Y is $q_1 = 12 - p_1^2 + p_1p_2$. Find the partial elasticities when $p_1 = 10$ and $p_2 = 4$.

EXERCISE 2.5

Choose the correct answer

1) The stationary value of x for $f(x) = 3(x-1)(x-2)$ is
(a) 3 (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{-3}{2}$

2) The maximum value of $f(x) = \cos x$ is
(a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) 1

3) $y = x^3$ is always
(a) an increasing function of x (b) decreasing function of x
(c) a constant function (d) none of these.

4) The curve $y = 4 - 2x - x^2$ is
(a) concave upward (b) concave downward
(c) straight line (d) none of these.

5) If $u = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to
(a) y^2u (b) x^2u (c) $2xu$ (d) $2yu$

6) If $u = \log(e^x + e^y)$ then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ is equal to

(a) $\frac{1}{e^x + e^y}$ (b) $\frac{e^x}{e^x + e^y}$ (c) 1 (d) $e^x + e^y$

7) If $u = x^y$ $(x > 0)$ then $\frac{\partial u}{\partial y}$ is equal to

(a) $x^y \log x$ (b) $\log x$ (c) $y^x \log x$ (d) $\log y^x$

8) $f(x, y) = \frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}$ is a homogeneous function of degree

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$

9) If $f(x, y) = 2x + ye^{-x}$, then f_y (1, 0) is equal to

(a) e (b) $\frac{1}{e}$ (c) e^2 (d) $\frac{1}{e^2}$

10) If $f(x, y) = x^3 + y^3 + 3xy$ then f_{xy} is

(a) $6x$ (b) $6y$ (c) 2 (d) 3

11) If marginal revenue is Rs.25 and the elasticitiy of demand with respect to price is 2, then average revenue is

(a) Rs.50 (b) Rs.25 (c) Rs.27 (d) Rs.12.50

12) The elasticity of demand when marginal revenue is zero, is

(a) 1 (b) 2 (c) –5 (d) 0

13) The marginal revenue is Rs.40 and the average revenue is Rs.60. The elasticity of demand with respect to price is

(a) 1 (b) 0 (c) 2 (d) 3

14) If $u = x^2 - 4xy + y^2$ then $\frac{\partial^2 u}{\partial y^2}$ is

(a) 2 (b) $2xy$ (c) $2x^2y$ (d) $2xy^2$

15) If $z = x^3 + 3xy^2 + y^3$ then the marginal productivity of x is

(a) $x^2 + y^2$ (b) $6xy + 3y^2$ (c) $3(x^2 + y^2)$ (d) $(x^2 + y^2)^2$

16) If $q_1 = 2000 + 8p_1 - p_2$ then $\frac{\partial q_1}{\partial p_1}$ is

(a) 8 (b) –1 (c) 2000 (d) 0

17) The marginal productivity of labour (L) for the production function $P = 15K - L^2 + 2KL$ when L = 3 and K = 4 is

(a) 21 (b) 12 (c) 2 (d) 3

18) The production function for a firm is $P = 3L^2 - 5KL + 2k^2$. The marginal productivity of capital (K) when L = 2 and K = 3 is

(a) 5 (b) 3 (c) 6 (d) 2

19) The cost function $y = 40 - 4x + x^2$ is minimum when x

(a) $x = 2$ (b) $x = -2$ (c) $x = 4$ (d) $x = -4$

20) If R = 5000 units / year, C_1 = 20 paise, C_3 = Rs.20 then EOQ is

(a) 1000 (b) 5000 (c) 200 (d) 100

3 Integration

Chapter Includes:

1. Indefinite integrals as Antiderivatives
2. Methods of Integration
3. Basic Theorem on Integration
4. Some Special Integrals
5. Integration by Parts
6. Integration by Partial Fraction
7. Integration of Rational and Irrational Algebraic Functions
8. Integration of Transcendental Function
9. Definite Integral
10. Evaluation of Definite Integral by Substitution
11. General Properties of Definite Integral
12. Definite Integral as the Limit of a Sum
13. Application of Definite Integral to Find the Sum of Infinite Series

INTRODUCTION :

In our earlier classes we have read four fundamental operations, namely addition, subtraction, multiplication and division. It is admitted fact that subtraction is inverse process of addition where as division is reverse process of multiplication.

Similarly, integration is reverse process of differentiation. We have already discussed in earliar section that for a function f on an interval I, we can find its derivative f' at every point of the interval. Now, here question arises, if derivative of a function is known in an interval, can we find the function? The answer is 'yes' we can find the function, if we know that the derivative of a function is also a function, which may be a costant or the function of independent variable. The set of functions that give as a derivative are called antiderivatives of fax function. The formula that gives all the antiderivatives, is called the indefinite integral of the function and the processinvolved,is called integration. Integral calculus was developed for solving the problems of finding areas, enclosed by the curves volumes of solids of revolution. Integral calculus involves two types of integral namely indefinite and definite interals, first we will discuss the indefinite integral.

3.1 INDEFINITE INTEGRALS AS ANTIDERIVATIVES :

We know that

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = x \qquad \text{(i)}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \text{(ii)}$$

$$\frac{d}{dx}(e^x) = e^x \qquad \text{(iii)}$$

and $$\frac{d}{dx}(\log_e x) = \frac{1}{x} \qquad \text{(iv)}$$

We observe that in (i) the polynomial function x is the derived function of $\frac{x^2}{2}$, we say that $\frac{x^2}{2}$ is the anti-derivative (or integral) of x. Similarly from (ii), (iii) and (iv) $\sin x$, e^x and $\log_e x$ are the antiderivatives of (or integrals) of. $\cos x, e^x$ and $\frac{1}{x}$ respectively.

Since the derivative of a real number say C, treated as constant function, is zero and hence the equations (i), (ii), (iii) and (iv) can be rewritten as :

$$\frac{d}{dx}\left(\frac{x^2}{2} + C\right) = x$$

$$\frac{d}{dx}(\sin x + C) = \cos x$$

$$\frac{d}{dx}\left(e^{x}+C\right)=e^{x}$$

and $\frac{d}{dx}\left(\log_e x+C\right)=\frac{1}{x}$ Now choosing the the value of C arbirtary from the set of real numbers there exists infinitely many antiderivative of each of the functions mentioned above and as such the antiderivatives (or integrals) if these functions are not unique. The real number C is referred as arbitrary constant or constants of interation or parameter of integration.

More generally, if there is a function F of x such that

$$\frac{d}{dx}\left[F(x)\right]=f(x) \qquad \text{as } x\in I \text{ (an internal)}$$

Then for any real number C,

$$\frac{d}{dx}\left[F(x+C)\right]=f(x) \quad x\in I$$

Thus, $\left[F(x)+C;C\in R\right]$ denotes the family of antiderivative of $f(x)$, where C, denotes the the arbitrary constant or parameter integration.

Now, we introduce a new sumbol, namely $\int f(x)dx$, for integration (or antiderivative) of

$f(x)$, we write $\int f(x)dx=F(x)+C$ ...(1)

When C is any real number, referred as constant of integration. Due to uncertainity of values of C, the integration mentioned in equation (1), is called indifinite integral.

Notation : Given that. $\frac{dy}{dx}=f(x)$, we write

$$y=\int f(x)dx+C$$

Here,

x is called variable of integration.

$f(x)dx$ is called integration of $f(x)$, with respect to x.

$f(x)$ is called 'integrand' .

C is called constant of integration.

" The process of finding integral of a function, is know as interation."

3.2 METHODS OF INTEGRATION :

We already know the formulae for the deriavatives of so many important functions from which we can write directly the corresponding standard formulae for the integral of the functions to be integrated. Besides these,there are four methods to find the integration of functions, which are mentioned below :

(i) Integration by Substitution.

(ii) Integration by parts.

(iii) Integration by partial fraction.

(iv) Integration by Successive Reduction.

3.2.1 INTEGRATION OF STANDARD FUNCTIONS :

We know that,

$$\frac{d}{dx}\left(x^n + C\right) = nx^{n-1}$$

$$\therefore \int nx^{n-1}dx = x^n + C \quad \text{(from the definition of integration)}$$

And $$\frac{d}{dx}\int\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x}dx0 = \log_e x + C \quad (n \neq -1)$$

In this way we can find the integration of standard functions from differentiation. Below we give a table for derivatives and antiderivatives (or integrals) of some standard functions.

INTEGRAL (ANTI-DERIVATIVE)	DIFFERENTIATION
(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$	$\frac{d}{dx}\frac{x^{n+1}}{n+1} + C \quad x^n$
(ii) $\int \frac{1}{x} dx = \log_e^x + C$	$\frac{d}{dx}\log_e x = \frac{1}{x}$
(iii) $\int e^x dx = e^x + C$	$\frac{d}{dx}\left(e^x\right) = e^x$
(iv) $\int a^x dx = \frac{a^x}{\log_e a} + C$	$\frac{d}{dx}\left(a^x\right) = a^x \log_e a$
(v) $\int \sin x\, dx = -\cos x + C$	$\frac{d}{dx}(\cos x) = -\sin x$
(vi) $\int \cos x\, dx = \sin x + C$	$\frac{d}{dx}(\sin x) = \cos x$
(vii) $\int \sec^2 x\, dx = \tan x + C$	$\frac{d}{dx}(\tan x) = \sec^2 x$

(viii) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$	$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
(ix) $\int \sec x \tan x\,dx = \sec x + C$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
(x) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$	$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
(xi) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$	$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
(xii) $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$	$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
(xiii) $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$	$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
(xiv) $-\int -\frac{1}{1+x^2}\,dx = \cot^{-1} x + C$	$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$
(xv) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$	$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
(xvi) $\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1} x + C$	$\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$
(xvii) $\int \sin hx\,dx = \cos hx + C$	$\frac{d}{dx}(\cos hx) = \sin hx$
(xviii) $\int \cos hx\,dx = \sin hx + C$	$\frac{d}{dx}(\sin hx) = \cos hx$
(xix) $\int \sec h^2 x\,dx = \tan hx + C$	$\frac{d}{dx}(\tan hx) = \sec h^2 x$
(xx) $\int \sec hx \tan hx\,dx = -\sec hx + C$	$\frac{d}{dx}(\sec hx) = -\sec hx \tan hx$
(xxi) $\int \operatorname{cosec} h^2 x\,dx = -\cot hx + C$	$\frac{d}{dx}(\cot hx) = -\operatorname{cosec} h^2 x$
(xxii) $\int \operatorname{cosec} hx \cot hx\,dx = -\operatorname{cosec} hx + C$	$\frac{d}{dx}(\operatorname{cosec} hx) = -\operatorname{cosec} hx \cot hx$
(xxiii) $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$	$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$
(xxiv) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$	$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
(xxv) $\int \frac{dx}{x^2-1} = \frac{1}{2}\log\frac{x-1}{x+1},\ x > 1$	$\frac{d}{dx}\frac{1}{2}\log\left(\frac{x-1}{x+1}\right) = \frac{1}{x^2-1}$

3.3 BASIC THEOREM ON INTEGRATION

(i) The integral of product of constant and a function, is equal to the product of the constant and integral of the given function. i. e.

$$\int kf(x)\,dx = k\int f(x)\,dx, \text{ Where k is constant}$$

(ii) The integral of sum or difference of functions is equal to sum or difference of the integrals of the functions.

$$\int\{f_1(x) \pm f_2(x) + \ldots\}dx = \int f_1(x)dx \pm \int f_2(x)dx + \ldots$$

$$\int\{k_2 f_1(x) \pm k_2 f_2(x) + \ldots\}dx = k_1\int f_1(x)dx \pm k_2\int f_2(x)dx + \ldots$$

SOLVED EXAMPLES

Example 1. : Evaluate $\int\left\{\frac{2}{\sqrt{1-x^2}} + 4\sec^2 x\right\}dx$

Solution : We have $\int\left\{\frac{2}{\sqrt{1-x^2}} + 4\sec^2 x\right\}dx$

$$= 2\int\frac{1}{\sqrt{1-x^2}}dx + 4\int\sec^2 x dx$$

$$= 2\sin^{-1}x + 4\tan x + C$$

Example 2. : Evaluate $\int\left(x^2 + \frac{4}{x^3}\right)dx$

Solution : We have $\int\left(x^2 + \frac{4}{x^3}\right)dx = \int x^2 dx + 4\int\frac{1}{x^3}dx$

$$= \int x^2 dx + 4\int x^3 dx$$

$$= \frac{x^{2+1}}{2+1}dx + 4\frac{x^{-3+1}}{-3+1} + C$$

$$= \frac{x^3}{3} + \frac{4}{-2}x^{-2} + C$$

$$= \frac{x^3}{3} - \frac{2}{x^2} + C$$

Example 3. : Evaluate $\int\left\{(x+1)^2\sqrt{x}\right\}dx$

Solution : We have $\int\left\{(x+1)^2\sqrt{x}\right\}dx = \int\left(x^2+2x+1\right)\sqrt{x}\,dx$

$$= \int x^{5/2}dx + 2\int x^{3/2}dx + \int x^{1/2}dx$$

$$= \frac{x^{5/2+1}}{5/2+1} + \frac{x^{3/2+1}}{3/2+1} + \frac{x^{1/2+1}}{1/2+1} + C$$

$$= \frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2} + C$$

$$= \frac{2}{7}x^{7/2} + \frac{4}{5}\cdot x^{5/2} + \frac{2}{3}\cdot x^{3/2} + C$$

Example 4. : Evaluate $\int \frac{1+\sin^2 x}{\cos^2 x}dx$

Solution : We have

$$\int \frac{1+\sin^2 x}{\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\right)dx$$

$$= \int\left(\sec^2 x + \tan^2 x\right)dx$$

$$= \int\left(\sec^2 x + \sec^2 x - 1\right)dx$$

$$= \int\left(2\sec^2 x - 1\right)dx$$

$$= 2\int\left(2\sec^2 x\,dx - 2\right)\int dx$$

$$= 2\tan x - 2x + C$$

Example 5. : Evaluate $\int \frac{x+1}{x-2}dx$

Solution : We have

$$\int \frac{x+1}{x-2}dx = \int \frac{x-2+3}{x-2}dx$$

$$= \int\left(1 + \frac{3}{x-2}\right)dx$$

$$= \int dx + 3\int \frac{1}{x-2}dx$$

$$= x + 3\log(x-2) + C$$

Example 6. : Evaluate $\int \sqrt{1+\cos 2x}\, dx$

Solution : We have

$$\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x}\, dx$$

$$= \sqrt{2}\int \cos x\, dx$$

$$= \sqrt{2}\sin x + C$$

Example 7. : Evaluate $\int (\cos ec x + \cot x)^2\, dx$

Solution : We have

$$\int (\cos ec x + \cot x)^2\, dx = \int \left(\cos ec^2 x + 2\cos ec x \cot x + \cot^2 x\right) dx$$

$$= \int \left(\cos ec^2 x + \cos ec^2 x - 1 + 2\cos ec x \cot x\right) dx$$

$$= \int 2\cos ec^2 x\, dx + \int 2\cos ec x \cot x\, dx \int dx$$

$$= 2\int \cos ec^2 x\, dx + 2\int \cos ec x \cot x\, dx - \int dx$$

$$= -2\cot x - 2\cos ec x - x + C$$

Example 8. : Evaluate $\int \left(e^x + 2\sin x - 3\cos x\right) dx$

Solution : We have

$$\int \left(e^x + \sin x - \cos x\right) dx = \int e^x dx + \int \sin x\, dx - 2\int \cos x\, dx$$

$$= e^x - \cos x - \sin x + C$$

Example 9. : Evaluate $\int \frac{1}{1-\sin x}\, dx$

Solution : We have

$$\int \frac{1}{1-\sin x}\, dx = \int \frac{1+\sin x}{(1+\sin x)(1-\sin x)dx}\, dx \int \frac{1+\sin x}{\left(1-\sin^2 x\right)}\, dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$$

$$= \tan x + \sec x + C$$

Example 10.: Evaluate $\int\left(5^x+e^x+x^3\right)dx$

Solution : We have

$$\int\left(5^x+e^x+x^3\right)dx=\int 5^x dx+\int e^x dx-\int x^3\,dx$$

$$=\frac{5^x}{\log_e 5}+e^x+\frac{x^4}{4}+C$$

EXERCISE 3.1

Evaluate the following integrals.

Q. 1.: (i) $\int\left(x^2-\frac{1}{x^2}\right)dx$ (ii) $\int\sqrt{1-\sin 2x}\,dx$

Q. 2.: (i) $\int(\tan x-\cot x)^2 dx$ (ii) $\int\frac{dx}{\sqrt{x+a}-\sqrt{x}}$

Q. 3.: (i) $\int\left(\sin x/2-\cos x/2\right)^2 dx$ (ii) $\int(\operatorname{cosec} x-\cot x)^2\,dx$

Q. 4.: (i) $\int(\sin x+\cos x)^2\,dx$ (ii) $\int\frac{(x+a)^2}{\sqrt{x}}dx$

Q. 5.: (i) $\int\sec x(\sec x+\tan x)dx$ (ii) $\int\left(2x^2+3\sin x+5\sqrt{x}\right)dx$

Q. 6.: (i) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$ (ii) $\int\left(2x^{2/3}+2e^x-\frac{1}{x}\right)dx$

Q. 7.: (i) $\int\left(2x-3\cos x+e^x\right)dx$ (ii) $\int\left(2x^2+e^x\right)dx$

Q. 8.: (i) $\int\frac{4-3\sin x}{\cos^2 x}dx$ (ii) $\int\frac{1}{1+\cos x}$

Q. 9.: (i) $\int\frac{1-\cos 2x}{1+\cos 2x}dx$ (ii) $\int\sqrt{1-\cos 2x}\,dx$

Q. 10.: (i) $\int\frac{1}{1-\cos x}dx$ (ii) $\int\frac{1}{1+\sin x}dx$

ANSWERS

(1) (i) $\frac{x^3}{3}+\frac{1}{x}+C$

(ii) $(\sin x-\cos x)+C$

(2) (i) $\tan x-\cos x+C$

(ii) $\frac{2}{3a}(x+a)^{3/2}+\frac{2}{3a}x^{3/2}+C$

(3) (i) $x-\cos x+C$

(4) (i) $x-\frac{\cos 2x}{2}+C$

	(ii)	$-2\cot x - x + 2\operatorname{cosec} + C$		(ii)	$\frac{2}{7}x^{7/2} + \frac{6a}{5}x^{5/2} + 2a^2x^{3/2} + 2a^3x^{1/2} + C$
(5)	(i)	$\tan x + \sec x + C$	(6)	(i)	$\frac{x^2}{2} - 4\log x - 2x + C$
	(ii)	$\frac{2}{3}x^3 + 3\cos x + \frac{20}{3}x^{3/2} + C$		(ii)	$\frac{3}{7}x^{7/3} + 2e^x - \log x - 2x + C$
(7)	(i)	$x^2 - 3\sin x + e^x + C$	(8)	(i)	$4\tan x - 3\sec x + C$
	(ii)	$\frac{2}{3}x^3 + e^x + C$		(ii)	$-\cot x + \operatorname{cosec} x + C$
(9)	(i)	$\tan x - x + C$	(10)	(i)	$-\operatorname{cosec} x - \cot x + C$
	(ii)	$-\sqrt{2}\cos x - \log + C$		(ii)	$\tan x - \sec x + C$

3.3.1 INTEGRATION BY SUBSTITUTION (CHANGE OF VARIABLES)

When a function is such that it is difficult to find its integral directy from standard results, then we transform the given integration to the standard form by changing the independent variable of the integrand in to new variable and then we find the integral of the given function.

The method of finding integral by changing the variables of integrand into a new variable its called integration by substitution. This method is most powerful tool for finding the integrals.

Let the variable x of the integrand $\int f(x)\,dx$ is changed to a new variable t.

$$x = \phi(t), \quad then \quad \frac{dx}{dt} = \phi(t)$$

We write $\quad dx = \phi'(t)\cdot dt, \quad then$

$$\int f(x)\,dx = \int (\phi(t))\,\phi'(t)\cdot dt$$

The method of integration by substitution is similar to the differentiation of a function of a function.

Usually, we make a substitution for a function whose derivative also occurs in the integrand. The following solved example illustrates this method.

Example 1 : Find the following integrals.

(i) $\int \frac{3x^2}{1+x^2}$ (ii) $\int \frac{\left(\sin^{-1} x\right)^2}{\sqrt{1-x^{-1}}}$ (iii) $\int \frac{(1+\log x)^3}{x}\,dx$

Solution : (i)

We know that derivative of $1+x^3 = 3x^2$ $\left[i.e\frac{d}{dx}\left(1+x^3\right)=3x^2\right]$

The we use the substitution $1+x^3 = t$, so that $3x^2 dx = dt$.

$$\therefore \int \frac{3x^2}{1+x^3}dx = \int \frac{dt}{t} = \log t + C$$

$$= \log\left(1+x^3\right)+C$$

(ii) We know that derivative of $\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ then we use the substitution,

$$\sin^{-1} x = t \quad \text{so that} \quad \frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\therefore \int \frac{\left(\sin^{-1} x\right)^2}{\sqrt{1-x^2}}dx = \int t^2 dt = \frac{t^3}{3}+C$$

$$= \frac{\left(\sin^{-1} x\right)^3}{3}+C$$

(iii) We know that derivative of $1+\log x = \frac{1}{x}$ then we use the substitution,

$$1+\log x = t, \quad \text{so that} \quad \frac{1}{x}dx = dt$$

$$\therefore \int \frac{(1+\log x)^3}{x}dx = \int t^3 dt = \frac{t^4}{4}+C$$

$$= \frac{(1+\log x)^4}{4}+C$$

3.3.2 SOME FUNCTIONS WHICH ARE INTEGRATED BY SUBSTITUTION

(i) **Integration of a function of the form $f(ax \pm b)$**

Let $f(t)$ be a standard function which can be integrated by standard formulae and $\int f(t)dt = F(t)+C$, then to find the value of $\int f = (ax \pm b)dx$

We put, $ax \pm b = t$ so that $axdx = t$ or $dx = \frac{dt}{a}$, then

$$\int f(ax \pm b)dx = \int \frac{f(t)}{a}dt = \frac{1}{a}\frac{F(t)}{a}+C = \frac{1}{a}F(ax \pm b)+C$$

(ii) Integration of a function of the form $\int f(\phi(x))\phi'(x)dx$.

In such type of functions we put

$$\phi(x) = t \text{ so that } \int f\phi(x)\phi'(x)dx = dt$$

Now, if $\int f(t)dt$ can be evaluated from standard formulated then s is changed to x and the required integral is obtained.

(iii) Integration of a function of the form $\int \frac{\phi'(x)dx}{f(\phi(x))}dx$.

For the above functions we put

$$\phi(x) = t \text{ so that } \phi'(x)dx = dt$$

and $\int \frac{\phi'(x)dx}{f(\phi(x))}dx = \int \frac{dt}{f(t)}$ we evaluate this integral and then changing the variable t into the variable x, we get the required integration.

Remark : The integration by substitution is a descriptive method of integration. This method is also used in the integration by partial fraction, integration by parts. So this method can not be limited for certain functions.

3.3.3 SUBSTITUTION FOR SOME IMPORTANT FUNTIONS

(i) $x = a\tan\theta$ is used for the funtions $a^2 + x^2, \sqrt{a^2 + x^2}, \frac{1}{a^2 + x^2}, \frac{1}{\sqrt{a^2 + x^2}}$ and the functions of these types given in the other forms.

(ii) $x = a\sin\theta$ or $x = a\cos\theta$ is used for the funtions $a^2 - x^2, \sqrt{a^2 - x^2}, \frac{1}{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}$ and the functions of these types given in the other forms.

(iii) $x = a\sec\theta$ is used for the funtions $x^2 - a^2, \sqrt{x^2 - a^2}, \frac{1}{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}}$ and the functions of these types given in the other forms.

(iv) $x = a - a\sin\theta$ or $x = 2a\sin^2\theta$ is used for the funtions $\sqrt{2ax - x^2}$, and the functions of these types given in the other forms.

The fundamental concept of this substitution is to change the function in such form which can be integrated easily.

3.3.4 INTEGRATION OF $\tan x$, $\cot x$, $\sec x$ *and* $\operatorname{cosec} x$.

(i) $\tan x = \dfrac{\sin x}{\cos x}$

$$\therefore \int \tan x dx = \int \frac{\sin x}{\cos x}$$ Putting $\cos x = t$, $-\sin x dx = dt$

$$\therefore \int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-dt}{t} = dt$$

$$= -\log t = \log \cos x$$

$$= \log \frac{1}{\cos x}$$

$$= \log \sec x + C$$

$$\therefore \int \tan x dx = \log \sec x + C$$

(ii) $\int \cot x dx = \int \dfrac{\cos x}{\sin x} dx,$ Putting $\sin x = t$, $\cos x dx = dt$

$$\therefore \int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{dt}{t}$$

$$= \log \sin x + C$$

$$\therefore \int \cot x dx = \log \sin x + C$$

(iii) $\int \sec x dx = \int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx,$

$$= \int \frac{\sec^2 x + \sec x + \tan x}{\sec x + \tan x} dx$$ Putting $\sec x + \tan x = t$

$$\left(\sec x \tan x + \sec^2\right) dx = dt$$

$$\therefore \int \sec x dx = \int \frac{dt}{t} = \log_e t$$

$$= \log \sin x + C$$

$$\therefore \int \cot x dx = \log \sin x + C$$

$$= \log(\sec x + \tan x) + C$$

or, $$= \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + C$$ (on simplification)

$$\therefore \int \sec x dx = \log(\sec x + \tan x) + C$$

$$= \log \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)+C$$

(iii) $\int \cos ecx dx = \int \frac{\cos ecx(\cos ecx + \cot x)}{\cos ecx + \cot x} dx,$

$$= \int \frac{\cos ec^2 x + \cos ecx \times \cot x}{\cos ecx + \cot x} dx \quad \text{Putting } \cos ecx + \cot x = t,$$

$$\left(-\cos ecx \cot x - \cos ec^2 x\right) dx = dt$$

$$-(\cos ec^2 x + \cos x \cot x)dx = dt$$

$$= -\int \frac{dt}{t}$$

$$= -\log(\cos ecx + \cot x) = \log(\cos ecx + \cot x)^{-1}$$

$$\therefore \int \cos ecx dx = \log(\cos ecx - \cot x) + C$$

$$= \log \tan x/2 + C \qquad \text{(on simplification)}$$

$$\therefore \int \cos ecx dx = \log(\cos ecx - \cot x) + C$$

or, $= \log \tan \frac{x}{2} + C$

Alternatively $= \int \frac{\cos ecx(\cos ecx - \cot x)}{\cos ecx - \cot x} = dx$

$$= \int \frac{\cos ec^2 x - \cos ecx \cot x}{\cos ecx - \cot x} = dx \qquad \text{Putting } \cos ecx - \cot x = t$$

$$= \int \frac{dt}{t} \qquad \left(\therefore \cos ecx \cot x - \cos ec^2 x\right)dx = dt$$

$$= \log t + c$$

$$= \log t(\cos ec - \cot) + c$$

$$= \log t(\cos ec - \cot) + c$$

$$= \log \tan \frac{x}{2} + c$$

3.3.5 INTEGRATION OF $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ *and* $\cot^2 x$.

(i) $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x,$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos 2x$$

Now for $\frac{1}{2}\int \cos 2x$ putting $2x = t$, so that $2dx = dt$, $dx = \frac{dt}{2}$

$$\therefore \int \cos 2x = \log_e(\operatorname{cosec} x + \cot x) + C \quad \operatorname{cosec}^2 x - \cos x \cot x = dt$$

$$= -\int \frac{dt}{t} = \frac{1}{2}\int \cos t \cdot dt = \frac{\sin t}{2} = \frac{\sin 2x}{2}$$

$$\therefore \int \sin^2 x = \frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + C$$

(ii) $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx,$

$$\therefore \int \cos^2 x dx = \frac{1}{2}\cdot x + \frac{1}{2}\frac{\sin 2x}{2} + C$$

(iii) $\int \tan^2 x dx = \int \left(\sec^2 x - 1\right) dx,$

$$= \int \sec^2 x \, dx - \int dx$$

$$= \tan x - x + C$$

$$\therefore \int \tan^2 x dx = \tan x - x + C$$

(iv) $\int \cot^2 x dx = \int \left(\operatorname{cosec}^2 x - 1\right) dx,$

$$= \int \left(\operatorname{cosec}^2 x - 1\right) dx$$

$$= \int \operatorname{cosec}^2 x \, dx - \int dx$$

$$= -\cot x - x + C$$

$$\therefore \int \cot^2 x dx = -\cot x - x + C$$

3.3.6 INTEGRATION OF $\sin^3 x$, $\cos^3 x$, $\tan^3 x$ *and* $\cot^3 x$.

(i) $\int \sin^3 x dx = \int \frac{3\sin x - \sin 3x}{4} dx,$ $\qquad \left(\because \sin 3x = 3\sin x - 4\sin^3 x\right)$

$$= \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3x dx$$

$$= \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

$$\therefore \int \sin^3 x dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C$$

Second Method :

$$\int \sin^3 x dx = \int \sin^2 \sin x dx = \int \left(1-\cos^2 x\right) \sin x dx$$

$$= \int \left(1-t^2\right) dt \qquad \text{Putting } \cos x = t, \quad -\sin dx = dt$$

$$= \int dt - \int t^2 \, dt$$

$$= -t + \frac{t^3}{3} + C$$

$$\therefore \int \sin^3 x dx = -\cos x + \frac{(\cos x)^3}{3} + C$$

We see that the integration of $\sin^3 x$ from the different method are $\frac{\cos 3x}{12} - \frac{3\cos x}{4}$ and $-\cos x - \frac{\cos^3 x}{3}$. These two integrals are same but written in different forms.

(ii) $\int \cos^3 x dx,$ $\qquad \because \cos 3x = 4\cos^3 x - 3\cos x$

$$\because \cos^3 x = \frac{3\cos x + \cos 3x}{4}$$

$$\int \cos^3 x dx = \int \frac{3\cos x + \cos 3x}{4} dx$$

$$= \frac{3}{4}\int \cos x dx + \frac{1}{4}\int \cos 3x dx$$

$$= \frac{3}{4}\sin x + \frac{1}{4}\frac{\sin 3x}{3} + C$$

$$\therefore \int \cos^3 x dx = \frac{\sin 3x}{12} + \frac{3}{4}\sin x + C$$

Alternately,

$$\int \cos^3 x dx = \cos^2 x \cos x \, dx$$

$$= \int \left(1-\sin^2 x\right) \cos x \, dx \qquad \text{Putting } \sin x = t, \quad \cos x dx = dt,$$

$$= \int \left(1-t^2\right) dt$$

$$= \int dt - \int t^2 dt$$

$$= t - \frac{t^3}{3} + C$$

$$= \sin x - \frac{(\sin x)^3}{3} + C$$

$$\therefore \int \cos^3 x dx = \sin x - \frac{1}{3}\sin^3 x + C$$

(iii) $\int \tan^3 x dx = \int \tan^2 x \cdot \tan x \, dx$

$$= \int \left(\sec^2 x - 1\right) \cdot \tan x \, dx$$

$$= \int \left(\sec^2 x \tan x dx\right) dx - \int \tan x \, dx$$

$$= \int \sec^2 x \tan x dx - \log \sec x$$

Now, for $\int \sec^2 x \tan x dx$ Putting $\tan x = t$

$$\therefore \sec^2 x dx = dt$$

$$\therefore \int \sec^2 x \tan dx = \int t dt$$

$$= \frac{t^2}{2} = \frac{(\tan x)^2}{2}$$

$$\int \tan^3 x dx = \frac{t^2}{2} = \frac{\tan^2 x}{2} - \log \sec x + C$$

(iv) $\int \cot^3 x dx = \int \cot x \cdot \cot^2 x \, dx \quad = \int \cot x \left(\operatorname{cosec}^2 - 1\right) dx$

$$= \int \cot x \operatorname{cosec}^2 x dx - \int \cot x \cdot dx$$

$$= \int \cot x \operatorname{cosec}^2 x dx - \log \sin dx$$

Now, for $\int \cot x \operatorname{cosec}^2 x \, dx$ Putting $\cot x = t$

$$\operatorname{cosec}^2 x dx = dt$$

$$\therefore \int \cot x \operatorname{cosec}^2 x dx = -\int t dt$$

$$= -\frac{t^2}{2} = \frac{-(\cot x)^2}{2}$$

$$\therefore \int \cot^3 x dx = -\frac{\cot^2 x}{2} - \log \sin x + C$$

SOLVED EXAMPLES

1. Evaluate the following integrals.

(i) $\int \frac{x\tan^{-1}x^2}{1+x^4}dx$ (ii) $\int \cos^3 x\sin^2 x\,dx$ (iii) $\int \frac{\sin x}{\sin(x+a)}dx$

(iv) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ (v) $\int \sin x \sin(\cos x)\,dx$ (vi) $\int e^{\tan x}\sec^2 x\,dx$

(vii) $\int \frac{\cos x - \sin x}{\cos x + \sin x}dx$ (viii) $\int \frac{1}{1+\tan x}$ (ix) $\int \cot x \log \sin x$ (x) $\int \frac{\sin\left(\tan^{-1}x\right)}{1+x^2}dx$

Solutions :

(i) $\int \frac{x\tan^{-1}x^2}{1+x^4}dx$ Putting $x^2 = t,\ 2xdx = dt$

$$xdx = \frac{1}{2}dt$$

$$\int \frac{x\tan^{-1}x^2}{1+x^4}dx = \frac{1}{2}\int \frac{\tan^{-1}t}{1+t^2}dt$$

Now, for

$\int \frac{\tan^{-1}t}{1+t^2}dt$ Putting $\tan^{-1}t = u,$

$$= \frac{1}{1+t^2}dt = du$$

$$\int \frac{\tan^{-1}t}{1+t^2}dt = \int u\,du$$

$$= \frac{u^2}{2} = \frac{\left(\tan^{-1}t\right)^2}{2}$$

$$= \frac{\left(\tan^{-1}x^2\right)^2}{2}$$

$$\int \frac{x\tan^{-1}x^2}{1+x^4}dx = \frac{\left(\tan^{-1}x^2\right)^2}{2} + C$$

(ii) $\int \cos^3 x\sin^2 x\,dx = \int \cos x\cos^2 x\sin^2 x\,dx$

$$= \int \cos x\left(1-\sin^2 x\right)\sin^2 x dx$$ Putting $\sin x = t,$

$\therefore\ \cos x dx = dt$

$$\therefore\ \int \cos^3 x \sin^2 x\, dx = \int \left(1-t^2\right)t^2 dt$$

$$= \int t^2 dt - \int t^4 dt$$

$$= \frac{t^3}{3} - \frac{t^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$$\therefore\ \int \cos^3 x \sin^2 x\, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(iii) $\displaystyle\int \frac{\sin x}{\sin(x+a)} dx$

Putting $x + a = t$

$\therefore\ x = t - a$

$dx = dt$

$$\therefore\ \int \frac{\sin x}{\sin(x+a)} dx = \int \frac{\sin(t-a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} dt$$

$$= \int \frac{\sin t \cos a}{\sin t} dt - \int \frac{\cos t}{\sin t} \sin a\, dt$$

$$= \cos a \int dt - \sin a \int \cot t\, dt$$

$$= \cos a(t) - \sin a \log \sin t + C_1$$

$$= (x+a)\cos a - \sin a \cdot \log \sin(x+a) + C_1$$

$$= x\cos x + a\cos a - \sin a \log \sin(x+a) + C_1$$

$$\therefore\ \int \frac{\sin x}{\sin(x+a)} = x\cos x - \sin a \cdot \log \sin(x+a) + C$$

Where $C = a\cos a + C_1$

$$\therefore\ \int \frac{\sin x}{\sin(x+a)} dx = x\cos x - \sin a \cdot \log \sin(x+a) + C$$

(iv) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ Putting $e^x + e^{-x} = t$

$$\left(e^x + e^{-x}\right)dx = dt$$

$$\therefore \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(e^x + e^{-x}\right) + C$$

$$\therefore \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \log\left(e^x + e^{-x}\right) + C$$

(v) $\int \sin x \sin(\cos x)\,dx$ Putting $\cos x = t$

$$-\sin x\,dx = dt$$

$$\sin x\,dx = -dt$$

$$\int \sin x \sin(\cos x)\,dx = -\int \sin t\,dt$$

$$= \cos t + C$$

$$= \cos(\cos x) + C$$

$$\therefore \int \sin x \sin(\cos x)\,dx = \cos(\cos x) + C$$

(vi) $\int e^{\tan x}\sec^2 x\,dx$ Putting $\tan x = t$

$$\therefore \sec^2 x dx = dt$$

$$\int e^{\tan x}\sec^2 x\,dx = \int e^t dt$$

$$= e^t + C$$

$$= e^{\tan x} + C$$

$$\therefore \int e^{\tan x}\sec^2 x\,dx = e^{\tan x} + C$$

(vii) $\int \frac{\cos x - \sin x}{\cos x + \sin x}dx$ Putting $\cos x + \sin x = t$

$$(-\sin x + \cos x)dx = dt$$

or $$(\cos x - \sin x)dx = dt$$

$$\therefore \int \frac{\cos x - \sin x}{\cos x + \sin x}dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log(\cos x + \sin x) + C$$

$$\therefore \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx = \log(\cos x + \sin x) + C$$

(viii) $\int \frac{1}{1+\tan x}\,dx$

$$\int \frac{1}{1+\tan x}\,dx = \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x}{\cos x + \sin x}$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x + \sin x}$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x + \cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

For $\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Putting $\cos x + \sin x = t$

$$(-\sin x + \cos x)\,dx = u$$

or $(\cos x - \sin x)\,dx = dt$

$$\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx = \int \frac{dt}{t}$$

$$= \log t$$

$$= \log(\cos x + \sin x)$$

$$\therefore \int \frac{1}{1+\tan x}\,dx = \frac{1}{2}x + \log(\cos x + \sin x) + C$$

(ix) $\int \cot x \log \sin x\,dx$

Putting $\log \sin x = t$

$$\frac{1}{\sin x}\cos x\,dx = dt$$

or $\cot x\,dx = dt$

$$\therefore \int \cot x \log \sin x\,dx = \int t\,dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\log \sin x)^2}{2} + C$$

$$\therefore \int \cot x \log \sin x \, dx = \frac{(\log \sin x)^2}{2} + C$$

(x) $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$ Putting $\tan^{-1} x = t$

$$\frac{1}{1+x^2} dx = dt$$

$$\therefore \int \frac{\sin(\tan^{-1} x)}{1+x^2} dx = \int \sin t \, dt$$

$$= -\cos t + C$$

$$= -\cos(\tan^{-1} x) + C$$

$$\therefore \int \frac{\sin(\tan^{-1} x)}{1+x^2} dx = -\cos(\tan^{-1} x) + C$$

2. Evaluate the following integrals.

(i) $\int \sin 4x \cos 3x \, dx$ (ii) $\int \tan^3 x \sec^5 x \, dx$ (iii) $2x \sin(x^2+1)$

(iv) $\sin^2(2x+5)$ (v) $\sin^3(2x+1)$ (vi) $\cos^4 2x$

(vii) $\sin^4 x$ (viii) $\tan^4 x$ (ix) $\cos 2x \cos 4x \cos 6x$ (x) $\frac{1-\cos x}{1+\cos x}$

Solutions :

(i) $\int \sin 4x \cos 3x \, dx = \frac{1}{2} \int \{\sin(4x+3x) + \sin(4x-3x)\} dx$

$$= \frac{1}{2} \int \{\sin 7x + \sin(-x)\} dx$$

$$= \frac{1}{2} \int \{\sin 7x - \sin x\} dx$$

$$\therefore \int \sin 4x \cos 3x \, dx = \frac{1}{2} \int \sin 7x dx - \frac{1}{2} \int \sin x \, dx$$

$$= -\frac{1}{14} \cos 7x dx + \frac{1}{2} \cos x + C$$

$$\therefore \int \sin 4x \cos 3x = -\frac{1}{14} \cos 7x + \frac{1}{2} \cos x + C$$

(ii) $\int \tan^3 x \sec^5 x \, dx = \int \tan x \sec x (\tan^2 x \sec^4 x) dx$

$= \int \left(\sec^2 x\right)\left(\sec^4 x\right) \sec x \tan x \, dx$ Putting $\sec x = t$

$\therefore \sec x \tan x \; dx = dt$

$= \int \left(t^2 - 1\right) t^4 dt$

$= \int \left(t^6 - t^4\right) dt$

$= \int t^6 dt - \int t^4 dt$

$= \dfrac{t^7}{7} - \dfrac{t^5}{5} + C$

$= \dfrac{\sec^5 x}{7} - \dfrac{\sec^5 x}{5} + C$

$\therefore \int \tan^3 x \sec^5 x \, dx = \dfrac{\sec^5 x}{7} - \dfrac{\sec^5 x}{5} + C$

(iii) $\int 2x \sin\left(x^2 + 1\right) dx$ Putting $x^2 + 1 = t$

$\therefore 2x \, dx = dt$

$\therefore \int 2x \sin\left(x^2 + 1\right) dx = \int \sin t \, dt$

$= -\cos t + C$

$= -\cos\left(x^2 + 1\right) + C$

$\therefore \int 2x \sin\left(x^2 + 1\right) dx = -\cos\left(x^2 + 1\right) + C$

(iv) $\int \sin^2 (2x + 5) dx$ $\cos 2x = 1 - 2\sin^2 x$

$\therefore \sin^2 x = \dfrac{1 - \cos x}{2}$

$\int \sin^2 (2x+5) dx = \int \dfrac{1 - \cos 2(2x+5)}{2} dx$

$= \dfrac{1}{2} \int dx - \dfrac{1}{2} \int \cos\left(4x + 10\right) dx$

$= \dfrac{1}{2} x - \dfrac{1}{2} \int \cos\left(4x + 10\right) dx$

For $\int \cos(4x+10)\,dx$ Putting $4x+10=t$

$4dx = dt$

$dx = \frac{1}{4}dt$

$$\therefore \int \cos(4x+10)\,dx = \frac{1}{4}\int \cos t\,dt$$

$$= \frac{1}{4}\sin t$$

$$= \frac{1}{4}\sin t(4x+10)$$

$$\therefore \int \sin^2(2x+5)dx = \frac{1}{2}x - \frac{1}{8}\sin(4x+10)+C$$

$$\text{or } \int \sin^2(2x+5)dx = \frac{1}{2}x - \frac{1}{8}\sin 2(2x+5)+C$$

(v) $\int \sin^3(2x+1)\,dx$

$$\therefore \int \sin^3(2x+1)\,dx = \int \frac{3\sin(2x+1)-\sin 3(2x+1)dx}{4}$$

$$= \frac{3}{4}\int \sin(2x+1) - \frac{1}{4}\int \sin(6x+3)\,dx$$

$\because \sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

For

$\int \sin(2x+1)\,dx$ Putting $2x+1=t$

$$\therefore \frac{3}{4}\int \sin(2x+1)\,dx = \frac{3}{8}\int \sin t\,dt$$ $2\,dx = dt$

$$= -\frac{3}{8}\cos t$$ $dx = \frac{1}{2}dt$

$$= -\frac{3}{8}\cos t(2x+1)+C$$

And for $\int \sin(6x+3)\,dx$ Putting $6x+3=u$

$6\,dx = du$

$dx = \frac{1}{6}du$

$$\therefore \frac{1}{4}\int \sin(6x+3)\,dx = \frac{1}{24}\int \sin u\,du$$

$$= -\frac{1}{24}\cos u + C_2$$

$$= -\frac{1}{24}\cos(6x+3) + C_2$$

$$\therefore \int \sin^3(2x+1)\,dx = -\frac{3}{8}\cos(2x+1) + C_1 - \frac{1}{24}\cos(6x+3) + C_2$$

$$= -\frac{3}{8}\cos(2x+1) - \frac{1}{24}\cos(6x+3) + C$$

$$= -\frac{3}{8}\cos(2x+1) - \frac{1}{24}\cos 3(2x+1) + C$$

Where $C_1 + C_2 = C$

$$\therefore \int \sin^3(2x+1)\,dx = -\frac{3}{8}\cos(2x+1) - \frac{1}{24}\cos 3(2x+1) + C$$

(vi) $\int \cos^4 2x\,dx = \int \left(\cos^2 2x\right)^2 dx$

$$= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx \qquad \because \cos^2 x = \frac{1+\cos^2 x}{2}$$

$$= \int \left(\frac{1+\cos^2 4x + 2\cos 4x}{4}\right) dx$$

$$= \frac{1}{4}\int dx + \frac{2}{4}\int \cos 4x + \frac{1}{4}\int \frac{1+\cos 8x}{2}\cdot dx$$

$$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos 4x\,dx + \frac{1}{4}\int \frac{1+\cos 8x}{2}\cdot dx$$

$$= \frac{3}{8}\int dx + \frac{1}{2}\int \cos 4x\,dx + \frac{1}{8}\int \cos 8x\cdot dx$$

$$= \frac{3}{8}x + \frac{1}{8}\sin 4x + \frac{1}{8\times 8}\sin 8x + C$$

$$= \frac{3}{8}x + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C$$

$$\therefore \int \cos^4 2x = \frac{3}{8}x + \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$$

(vii) $\int \sin^4 x\,dx = \int \left(\sin^2 x\right)^2 dx \qquad \because \sin^2 x = \left(\frac{1-\cos 2x}{2}\right)$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1+\cos^2 2x - 2\cos 2x\right) dx$$

$$= \frac{1}{4}\int dx - \frac{2}{4}\int \cos 2x + \frac{1}{4}\int \cos^2 2x\, dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x\, dx$$

$$= \frac{3}{8}\int dx - \frac{1}{2}\int \cos 2x dx + \frac{1}{8}\int \cos 4x dx$$

$$= \frac{3}{8}x - \frac{1}{2}\times\frac{\sin 2x}{2} + \frac{1}{8}\frac{\sin 4x}{4} + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \sin\frac{1}{32}\sin 4x + C$$

$$\therefore \int \sin^4 x\, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

(viii) $\int \tan^4 x dx = \int \tan^2 x \tan^2 x dx$

$$= \int \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int \sec^2 x \tan^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \sec^2 x \tan^2 x\, dx - \int \left(\sec^2 x - 1\right) dx$$

$$= \int \sec^2 x \tan^2 x\, dx - \int \sec^2 x dx + \int dx$$

Now for, $\int \sec^2 x \tan^2 x dx$ Putting $\tan x = t$

$\therefore \sec^2 x dx = dt$

Thus $\int \sec^2 x \tan^2 x dx = t^2 dt$

$$= t^3/3$$

$$= \tan^3 x/3$$

$$\therefore \int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

$$\int \tan^4 x dx = x - \tan x + \frac{1}{3}\tan^3 x + C$$

(ix) $\int \cos 2x \cos 4x \cos 6x = \frac{1}{2}\int \left(\cos 6x + \cos 2x\right)\cos 6x dx$

$$\left(\because \cos A\cos B=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]\right)$$

$$=\frac{1}{2}\int\cos^2 6x dx+\frac{1}{2}\int\cos 2x\cos 6x dx$$

$$=\frac{1}{2}\int\frac{(1+\cos 12x)}{2}dx+\frac{1}{2}\int\frac{(\cos 8x+\cos 4x)}{2}dx$$

$$=\frac{1}{4}\int dx+\frac{1}{4}\cos 12x dx+\frac{1}{4}\int\cos 8x dx+\frac{1}{4}\int\cos 4x dx$$

$$=\frac{x}{4}+\frac{1}{48}\sin 12x+\frac{1}{32}\sin 8x+\frac{1}{16}\sin 4x+C$$

$$\therefore \int\cos 2x\cos 4x\cos 6x=\frac{1}{4}\left[x+\frac{1}{12}\sin 12x+\frac{1}{8}\sin 8x+\frac{1}{4}\sin 4x\right]+C$$

(x) $\int\frac{1-\cos x}{1+\cos x}dx=\int\frac{2\sin^2 x/2}{2\cos^2 x/2}dx$ $\because \cos 2x=2\cos^2 x-1$

$=1-2\sin^2 x$

$$=\int\tan^2 x/2\,dx$$

$$=\int\left(\sec^2 x/2-1\right)dx$$

$$=\int\sec^2 x/2dx-\int dx$$

$$=\int 2\tan\frac{x}{2}-x+C$$

For $\int\sec^2\frac{x}{2}dx$ Putting $\frac{x}{2}=t$

$dx=2dt$

$$\therefore \int\sec^2\frac{x}{2}dx=2\int\sec^2 t\,dt$$

$$=2\tan t$$

$$=2\tan\frac{x}{2}$$

$$\therefore \int\frac{1-\cos x}{1+\cos x}dx=2\tan\frac{x}{2}-x+C$$

EXERCISE 3.2

Integrate the following functions :

Q. 1.: (i) $\frac{4x^3}{1+x^8}$ (ii) $\frac{x^2}{\left(2+3x^3\right)^3}$ (iii) $\frac{1}{(x+x\log x)}$ (iv) $\frac{\sin x}{1+\cos x}$

(v) $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ (vi) $\frac{1}{x(\log x)^n}, x>0$ (vii) $\frac{x}{9+3x^2}$

Q. 2.: (i) $\sin(ax+b)\cos(ax+b)$ (ii) $\frac{1}{\sin(x-a)\sin(x-b)}$

(iii) $\frac{\sin x}{\sin(x-a)}$ (iv) $\frac{(x+1)(x+\log x)}{x}$ (v) $\frac{\cos x}{\sin^3 x}$

Q. 3.: $\frac{1}{\cos^2 x(1-\tan x)^2}$ **Q. 4.:** $\frac{\sqrt{\tan x}}{\sin x\cos x}$ **Q. 5.:** $\frac{1}{1-\tan x}$

Q. 6.: $\frac{1}{1+\cos x}$ **Q. 7.:** $\frac{1}{1-\cos x}$ **Q. 8.:** $\frac{x^3\sin x\tan^{-1}x^4}{1+x^8}$

Q. 9.: $\frac{x^3\tan x^4}{1+x^8}$ **Q. 10.:** $\frac{\tan^4\sqrt{x}\cdot\sec^2\sqrt{x}}{\sqrt{x}}$ **Q. 11.:** $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$

Q. 12.: $\sqrt{\sin 2x}\cos 2x$ **Q. 13.:** $\frac{\cos x}{1+\sin x}$ **Q. 14.:** $\frac{1}{1+\cos x}$

Q. 15.: $\frac{\cos x}{\sqrt{1+\sin x}}$ **Q. 16.:** $\frac{e^x-1}{e^x+1}$ **Q. 17.:** $\frac{\sin x}{1+\cos^2 x}$

Q. 18.: $\frac{\sin x\cos x}{\sqrt{1+\sin^2 x}}$ **Q. 19.:** $\frac{\sin^{-1}\sqrt{x}}{\sqrt{x}\cdot\sqrt{1-x}}$ **Q. 20.:** $\frac{1}{\sin x+\cos x}$

Q. 21.: $\frac{\cos x}{\cos(x-b)}$ **Q. 22.:** $\frac{e^x}{1+e^{2x}}$ **Q. 23.:** $\frac{e^x-\sin x}{e^x+\cos x}$

Q. 24.: $\int\frac{x^4}{x^2+1}$ **Q. 25.:** $\frac{1}{e^x-1}$ **Q. 26.:** $\frac{e^x(1+x)}{\cos^2\left(xe^x\right)}$

Q. 27.: $\frac{x^2\tan^{-1}x^3}{1+x^6}$ **Q. 28.:** $\sin^{-1}(\cos x)$ **Q. 29.:** $\cos^{-1}(\sin x)$ **Q. 30.:** $\frac{\cos^4 x}{\sin x}$

ANSWERS

(1) (i) $\frac{(4x+3)^{5/2}}{40}-\frac{(4x+3)^{3/2}}{8}+C$ (ii) $-\frac{1}{18\left(2+3x^3\right)^2}+C$

(iii) $\log(1+\log x)+C$ (iv) $\log\frac{1}{1+\cos x}+C$

(v) $\frac{1}{2}\log\left(e^{2x}+e^{-2x}\right)+C$ (vi) $\frac{(\log x)^{1-n}}{1-n}+C$ (vii) $\frac{1}{6}\log\left(4+3x^2\right)+C$

(2) (i) $-\frac{1}{4a}\cos(2ax+\sec x2b)+C$ (ii) $\frac{1}{\sin(a-b)}\log\left\{\frac{\sin(x-a)}{\sin(x-b)}\right\}+C$

(iii) $x\cos a+\sin a\log(x-a)+C$ (iv) $\frac{1}{3}(x+\log x)^3+C$ (v) $-\frac{1}{2}\operatorname{cosec}^2 x+C$

(3) $\frac{1}{(1-\tan x)}+C$ (4) $2\sqrt{\tan x}+C$ (5) $\frac{x}{2}-\frac{1}{2}\log(\cos x-\sin x)+C$

(6) $-\cot x+\operatorname{cosec} x+C$ (7) $-\cot x-\operatorname{cosec} x+C$ (8) $-\frac{1}{4}\cos\left(\tan^{-1}x^4\right)+C$

(9) $\frac{1}{8}\left(\tan^{-1}x\right)^2+C$ (10) $\frac{2}{5}\left(\tan\sqrt{x}\right)^5+C$ (11) $\frac{1}{2}\log(3\cos x+2\sin)+C$

(12) $\frac{1}{3}(\sin 2x)^{3/2}+C$ (13) $\log(1+\sin x)+C$ (14) $\frac{1}{2}x-\frac{1}{2}\log(\cos x+\sin x)+C$

(15) $2\sqrt{1+\sin x}+C$ (16) $\log\left(e^x+e^{-x}\right)+C$ (17) $-\tan^{-1}(\cos x)+C$

(18) $2\sqrt{1+\sin^2 x}+C$ (19) $\left(\sin^{-1}\sqrt{x}\right)^2+C$

(20) $\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$ or $\frac{1}{\sqrt{2}}\log(\operatorname{cosec}(x+\pi/4))-\cot(x+\pi/4)+C$

(21) $x\cos b+\sin b\log\cos(x-b)+C$ (22) $\tan^{-1}\left(e^x\right)+C$ (23) $\log\left(e^x+\cos x\right)+C$

(24) $\frac{x^3}{3}-x+\tan^{-1}x+C$ (25) $\log\left(1-e^x\right)+C$ (26) $\tan\left(xe^x\right)+C$

(27) $\frac{1}{6}\left(\tan^{-1}x^3\right)+C$ (28) $\frac{\pi}{2}x+\frac{x^2}{2}+C$ or $\frac{\pi}{2}x-\frac{x^2}{2}+C$

(29) $\frac{\pi}{2}x-\frac{x^2}{2}+C$ (30) $\log(\sin x)+\frac{1}{8}\sin^8 x-\frac{2}{3}\sin^6 x+\frac{3}{2}\sin^4 x-2\sin^2 x$

3.4 SOME SPECIAL INTEGRALS

The following formulae of integrals can be directly applied for integrating various functions.

(1) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+c}\right|+C$

(2) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

(3) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(4) $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sin^{-1} \frac{x}{a} + C$

(5) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log x + \sqrt{x^2 + a^2} + C$ or $\sinh^{-1} \frac{x}{a} + C$

(6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$ or $\cosh^{-1} \frac{x}{a}$

(7) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C$

(8) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2}\, \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

Now we shall prove the above results :

(1) $\int \frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a} \left\{ \frac{(x+a)(x-a)}{(x-a)(x+a)} \right\}$$

$$= \frac{1}{2a} \left\{ \frac{1}{(x-a)} - \frac{1}{(x+a)} \right\}$$

$$\therefore \int \frac{1}{x^2 - a^2} = \frac{1}{2a} \left[\int \frac{dx}{(x-a)} - \int \frac{dx}{(x+a)} \right]$$

$$= \frac{1}{2a} \log [|x-a| - \log |x+a|] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) We have

$$\frac{1}{a^2 - x^2} = -\frac{1}{x^2 - a^2}$$

$$\int \frac{dx}{a^2 - x^2} = -\int \frac{dx}{x^2 - a^2}$$

From result (1) we have,

$$\int \frac{dx}{a^2 - x^2} = -\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

or $$= \frac{1}{2a}\log\left|\frac{x+a}{x-a}\right| + C$$

(3) $\int \frac{dx}{x^2 + a^2}$ Putting $x = a\tan\theta$

$dx = a\sec^2\theta d\theta$

$$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2}$$

$$= \frac{1}{a}\int \frac{\sec^2\theta d\theta}{\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C$$

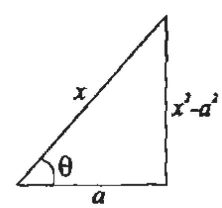
$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) $\int \frac{dx}{\sqrt{a^2 + x^2}}$ Putting $x = a\sec\theta$

$dx = a\sec\theta\tan\theta d\theta$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta}{a\sqrt{\sec^2\theta - 1}} d\theta$$



$$= \int \frac{\cancel{a}\sec\theta\cancel{\tan\theta}}{\cancel{a}\cancel{\tan\theta}} d\theta$$

$$= \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C$$ Where $C = C_1 - \log a$

(5) $\int \frac{dx}{\sqrt{a^2 - x^2}}$ Putting $x = a\sin\theta$, $dx = a\cos\theta d\theta$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta d\theta}{a\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{a\cos\theta d\theta}{a\sqrt{\cos^2\theta}} = \int \frac{\cancel{a}\cancel{\cos\theta} d\theta}{\cancel{a}\cancel{\cos\theta}}$$

$$= \int d\theta$$

$$= \theta + C$$

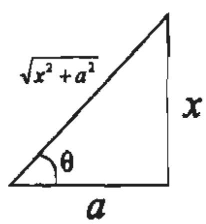
$$= \sin^{-1}\frac{x}{a} + C$$

(6) $\int \frac{dx}{\sqrt{x^2 + a^2}}$ Putting $x = a\tan\theta$, $dx = a\sec^2\theta d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$$

$$= \int \frac{a\sec^2\theta d\theta}{a\sqrt{\tan^2\theta + 1}} = \int \frac{a\sec^2\theta d\theta}{a\sqrt{\sec^2\theta}}$$

$$= \int \frac{\cancel{a}\sec^2\theta d\theta}{\cancel{a}\cancel{\sec\theta}}$$



$$= \int \sec\theta\, d\theta$$

$$= \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 + a^2}\right| + C$$ Where $C = a - \log a$

(7) $\int \frac{dx}{x\sqrt{x^2 - a^2}}$ Putting $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{\cancel{a}\cancel{\sec\theta}\tan\theta d\theta}{\cancel{a}\cancel{\sec\theta}\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{\tan\theta d\theta}{a\sqrt{\sec^2\theta - 1}}$$

$$= \frac{1}{a}\int \frac{\tan\theta d\theta}{\sqrt{\tan^2\theta}}$$

$$= \frac{1}{a}\int \frac{\tan\theta d\theta}{\tan\theta}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C_1$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

(8) $\int \sqrt{a^2 - x^2}dx$ Putting $x = a\sin\theta$, $dx = a\cos\theta d\theta$

$$\therefore \int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta d\theta$$

$$= \int a^2\sqrt{1 - \sin^2\theta} \cdot \cos\theta d\theta$$

$$= a^2\int \sqrt{\sin^2\theta} \cdot \cos\theta d\theta$$

$$= a^2\int \cos^2\theta d\theta$$

$$= a^2\int \frac{1 + \cos 2\theta}{2}d\theta \qquad \because \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{a^2}{2}\left[\int d\theta + \int \cos 2\theta d\theta\right]$$

$$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{a^2}{2}\left[\frac{\sin^{-1}x}{a} + \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}\right] + C$$

$$= \frac{a^2}{2} \cdot \frac{\sin^{-1}x}{a} + \frac{x}{2} \cdot \sqrt{a^2 - x^2} + C$$

$$= \frac{x}{2} \cdot \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \frac{\sin^{-1}x}{a} + C$$

3.5 INTEGRATION BY PARTS

This method of integration is more useful for integration of products of two functions. Any one of which may be algebraic, exponential, logarithmic, trigonometric function.

If u and v be two differential functions of independent variable x, then by product rule of differentiation of two functions, we have –

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x, we get,

$$uv = \int u\frac{dv}{dx}\cdot dx + \int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}\cdot dx = uv - \int v\frac{du}{dx}dx \qquad \ldots(1)$$

Let $u = f(x)$; *and* $\frac{dv}{dx} = g(x)$ *then*

$$\frac{du}{dx} = f'(x) \quad \text{and} \quad v = \int g(x)dx$$

Using above expression (1) can be rewritten as –

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left[\left(\int g(x)dx\right)f'(x)\right]dx$$

or $$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left[f'(x)\int g(x)dx\right]dx$$

Now, taking *f* and *g* separately first and second function, then the above formulae may be stated as :

The integral of product of two functions
= The first function × integral of second
– integral of differential coefficient of the first function × integral of second function.

3.5.1 SELECTION OF FIRST AND SECOND FUNCTION

We take the integral, whose integral is known as second function. If the integral of both functions are known then we must be more careful for the selection of first and second function.

(i) If out of two functions one contains the integral power of the variable *x*, and other function is either trigonometric or exponential, then we take the function involving integral power of *x* as first and other as second function.

For example in $\int x^2 \sin x\, dx$, we take x^2 as first function and sin x as second where as in $\int x^3 e^x\, dx$, we take x^3 as first function and e^x as second function.

(ii) If one of the function contains integral power of *x* and other is either logarithmic or inverse circular function, then in this case logarithmic or inverse circular function is taken as first function and the function involving integral powre of *x* as second function. For example – in $\int x^4 \log x\, dx$, $\log x$ is taken as first function as x^4 as second. Similarly, in $\int x^3 \sin^{-1} x\, dx$, $\sin^{-1} x$ is taken as first function and x^3 as second function.

(iii) Sometimes, it becomes difficult to integrate a single function. So we take unity (i. e. 1) as second function. For example –we can not evaluate $\int \log x\, dx$, so we take 1 as second function and write the integral as $\int \log x \cdot 1\, dx$, similarly, to evaluate $\int \sin^{-1} x\, dx$ we take $\sin^{-1}$x as first function and 1 as second function and we write $\int \sin^{-1} x \cdot 1 \cdot dx$.

(iv) If out of the two functions one is trigonometric and other is exponential then to integrate such functions we choose first or second function according to our convenience. For example in $\int e^x \cdot \sin x$, we can take any of the functions e^x or $\sin x$ as first or second function.

Sometimes, after certain stage, the original integral occurs, in the process of integration, then we transfer it to the left hand side.

SOLVED EXAMPLES

(1) $\int x^2 \sin x\, dx$

Let $I = \int x^2 \sin x\, dx$

Taking x^2 as first function and $\sin x$ as second function.

Using integration by parts, we get,

$$I = \int x^2 \sin x\, dx = x^2 \int [\sin x\, dx] - \left[\int \frac{d}{dx} x^2 \int \sin x\, dx\right] dx$$

or $$I = \int x^2 \sin x\, dx = -x^2 \cos x - \int [2x(-\cos x)]\, dx$$

or $$I = \int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx \qquad ...(1)$$

Let $I_1 = \int x \cos x\, dx$

Taking x as first function and sin x as second, and integrating by parts we get,

$$I_1 = x\int \cos x\, dx - \int \frac{d}{dx}(x) \int [\cos x dx]\, dx$$

$$= x(\sin x) - \int 1 \cdot \sin x dx$$

$$= -x \sin x + \cos x + C$$

$$= -x \cos x + \sin x + C_1$$

Putting two values in (1) we get,

$$I = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

(2) $\int x^2 e^x dx$

Let $I = \int x^2 e^x dx$

Taking x^2 as first function and e^x as second, and integrating by parts we get,

$$I = x^2 \int e^x dx - \int \left[\frac{d}{dx} x^2 \int e^x dx \right] dx$$

$$= x^2 e^x - \int 2x \cdot e^x dx$$

$$= x^2 e^x - 2\left[x \cdot e^x - \int \frac{d}{dx}(x) \cdot \int e^x dx \right] dx$$

$$= x^2 e^x - 2x \cdot e^x - 2\int e^x dx$$

$$= x^2 - 2x e^x - 2e^x + C$$

(3) $\int x^2 \log x\, dx$

Let $I = \int x^2 \log x$

Taking $\log x$ as first function and x^2 as second, and integrating by parts we get,

$$I = \log x \int x^2 dx - \int \left[\frac{d}{dx}(\log x) \int x^2 dx \right] dx$$

$$= \log x \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \cdot \frac{x^3}{3} + C$$

$$= \frac{1}{3} x^3 \log x - \frac{1}{9} x^3 + C$$

(4) $\int x^2 \tan^{-1} x dx$

Let $I = \int x^2 \tan^{-1} x dx$

Taking $\tan^{-1} x$ as first function and x^2 as second, and integrating by parts we get,

$$I = \tan^{-1} x \int x^2 dx - \int \left[\frac{d}{dx} \tan^{-1} x \int x^2 dx \right] dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \left[\int x dx - \int \frac{x}{1+x^2} dx \right]$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \left[\frac{x^2}{2} - \frac{1}{2} \log\left(1+x^2\right) \right] + C$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \cdot \log\left(1+x^2\right) + C$$

(5) $\int \log x dx$

Let $\quad I = \int \log x \, dx$

We can not evaluate the given integral directly, so, we rewrite it as :

$I = \int (\log x \cdot 1) dx.$ We have taken 1 as the second function and by applying integration by parts, we get

$$I = \log x \int 1 \cdot dx - \int \left[\frac{d}{dx} \log x \, f \, i \, dx \right] dx$$

$$= \log x \cdot xn - \int \frac{1}{x} \cdot x \, dx$$

$$= \log x - x + C$$

$$I = x(\log x - 1) + C$$

(6) $\int \sin^{-1} x \, dx$

Let $\qquad I = \int \sin^{-1} x \, dx$

Rewriting above integral and using integration by part taking 1 as second function we get,

$$I = \int \left(\sin^{-1} x \cdot 1 \right) dx$$

$$= \sin^{-1} x \int 1 \, dx - \int \left[\frac{d}{dx} \sin^{-1} x \int 1 \cdot dx \right] dx$$

$$= x\sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} dx$$

$$= x\sin^{-1} - \sqrt{1-x^2} + C$$

(7) $\int e^x \sin x \, dx$

Let $\qquad I = \int e^x \sin x \, dx$

Taking e^x as first and sin x as second function and applying integration by parts we get,

$$I = e^x \int \sin x \, dx - \int \left[\frac{d}{dx} e^x \int \sin x \cdot dx \right] dx$$

$$= e^x(-\cos x) - \int e^x(-\cos x) dx$$

$$= -\cos x \, e^x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + e^x \int \cos x - \int \left[\frac{d}{dx} e^x \int (dx) dx \right] dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$= -e^x \cos x + e^x \sin x - I$$

$$\therefore \; I + I = -e^x \cos x + e^x \sin x + C_1$$

$$2I = -e^x \cos x + e^x \sin x + C_1$$

or $\qquad I = e^x(\sin x - \cos x) + C$

(8) $\int e^{ax} \sin bx \cdot dx$

Let $\qquad I = \int e^{ax} \sin bx \cdot dx$

Taking e^{ax} as first and sin bx as second function and applying integration by parts we get,

$$I = e^{ax} \int \sin bx - \int \left[\frac{d}{dx} e^{ax} \int \sin bx \cdot dx \right] dx$$

$$= e^{ax} \left(\frac{-\cos bx}{b} \right) - \int a e^{ax} \left(\frac{-\cos bx}{b} \right) dx$$

$$= -\frac{\cos bx \, e^{ax}}{b} + \frac{a}{b} \int e^{ax} \cos bx \, dx$$

$$= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} I_1, \qquad \ldots(1)$$

Where, $I_1 = \int e^{ax} \cos bx\, dx$

Taking e^{ax} as first and $\cos bx$ as second function in I_1 and integration by parts we get,

$$I_1 = e^x \int \cos bx dx - \int \left[\frac{d}{dx} e^x \int (\cos bx dx)\, dx \right] dx$$

$$= e^x \frac{\sin bx}{b} - \int a e^{ax} - \left(\frac{\sin bx}{b} \right) dx$$

$$= \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \cdot \int a e^{ax} \sin bx\, dx$$

$$= \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \cdot I$$

Putting I_1 in (i) from above, we get

$$I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \left[\frac{e^{ax} \sin bx}{b} - \frac{a}{b} \cdot I \right]$$

$$= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \cdot I$$

or $$I + \frac{a^2}{b^2} \cdot I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx$$

$$\left(\frac{b^2 + a^2}{b^2} \right) \cdot I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx$$

Hence, $$I = \frac{e^{ax}}{(a^2 + b^2)} - (a \sin bx - b \cos bx) + C$$

$$= \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$

Similarly. $$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{(a^2 + b^2)} \cdot (a \cos bx - b \sin bx) + C$$

(9) $\int \sec^3 x \cdot dx$

Let $I = \int \sec^3 x \cdot dx$

Taking $\sec x$ as first and $\sec^2 x$ as second function and applying integration by parts we get,

$$I = \sec x \int \sec^2 x\,dx - \int \left[\frac{d}{dx} \sec x \int \sec^2 x\,dx \right] dx$$

$$= \sec x \cdot \tan x - \int \sec x \tan x \tan x\,dx$$

$$= \sec x \cdot \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \cdot \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \cdot \tan x - \int \sec x \cdot \sec^2 x\,dx + \int \sec x\,dx$$

$$= \sec x \cdot \tan x - I + \int \sec x\,dx$$

or $$2I = \sec x \cdot \tan x + \log\left(\sec x + \tan x\right) + C$$

$$\therefore\ I = \frac{1}{2} \sec x \cdot \tan x + \frac{1}{2} \log\left(\sec x + \tan x\right) + C$$

or $$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log\left(\sec x + \tan x\right) + C$$

(10) $\int \frac{x - \sin x}{1 - \cos x} \cdot dx$

Let $$I = \int \frac{x - \sin x}{1 - \cos x} \cdot dx$$

$$= \int \frac{x - 2\sin x/2 \cos x/2}{2\sin^2 x/2} \cdot dx$$

$$= \frac{1}{2} \int x \operatorname{cosec}^2 x/2\,dx - \int \cot x/2\,dx$$

$$= \frac{1}{2}\left[x \cdot \left(-2\cot x/2\right) - \int 1 \cdot \left(-2\cot x/2\right) dx \right] - \int \cot x/2\,dx$$

$$= -x\cot\frac{x}{2} + \int \cancel{\cot x/2\,dx} - \int \cancel{\cot x/2\,dx}$$

$$\therefore\ \int \frac{x - \sin x}{1 - \cos x} \cdot dx = -x\cot\frac{x}{2} + C$$

(11) $\int \frac{x e^x}{(1+x)^2} \cdot dx$

Let $$I = \int \frac{x e^x}{(1+x)^2} \cdot dx$$

$$= \int \frac{(x+1-1)e^x}{(1+x)^2} \cdot dx$$

$$= \int \frac{(x+1)e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} \cdot dx$$

$$= \int \frac{e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} \cdot dx$$

$$= \frac{1}{(1+x)^2} \int e^x - \int \left[\frac{d}{dx} \frac{1}{(1+x)} x \int e^x \, dx \right] dx - \int \frac{e^x}{(1+x)^2} dx$$

$$= \frac{e^x}{(1+x)} - \int \left(-\frac{1}{(1+x)^2} e^x \, dx \right) - \int \frac{e^x}{(1+x)^2} dx$$

$$= \frac{e^x}{(1+x)} + \int \frac{1}{(1+x)^2} e^x \, dx - \int \frac{e^x}{(1+x)^2} dx$$

$$= \frac{e^x}{(1+x)^2} + C$$

(12) $\int e^x (\sin x + \cos x) \cdot dx$

Let $I = \int e^x (\sin x + \cos x) \cdot dx$

$$= \int e^x \sin x \, dx + \int e^x \cos x \, dx$$

$$= \int e^x \sin x \, dx + e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x + C$$

$$\therefore \int e^x (\sin x + \cos x) \cdot dx = e^x \sin x + C$$

(13) $\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) \cdot dx$

Let $I = \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) \cdot dx$

$$= \int e^x \tan^{-1} x \cdot dx + e^x \frac{1}{1+x^2} \cdot dx$$

$$= \int e^x \tan^{-1} x \cdot dx + e^x \tan^{-1} x - \int e^x \tan^{-1} x \cdot dx$$

$$= e^x \tan^{-1} x + C$$

$$\therefore \int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)\cdot dx = e^x\tan^{-1}x + C$$

(14) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)\cdot dx$

Let $I=\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)\cdot dx$

$$=\int e^x\frac{1}{x}dx-\int e^x\frac{1}{x^2}\cdot dx$$

$$=\frac{1}{x}\cdot e^x+\int\frac{1}{x^2}e^x\cdot dx-\int e^x\frac{1}{x^2}\cdot dx$$

$$=\frac{e^x}{x}+C$$

(15) $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)\cdot dx$

Let $I=\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)\cdot dx$

$$=\int e^x\left[\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right]\cdot dx$$

$$=\int e^x\frac{1}{1+\cos x}\cdot dx+\int e^x\frac{\sin x}{1+\cos x}\cdot dx$$

$$=\int\frac{e^x}{2\cos^2 x/2}\cdot dx+\int e^x\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}\cdot dx$$

$$=\frac{1}{2}\int e^x\sec^2 x/2\cdot dx+\int e^x\tan x/2\cdot dx$$

$$=\frac{1}{2}e^x\times 2\tan x/2-\frac{1}{2}\int e^x 2\tan x/2\cdot dx+\int e^x\tan x/2$$

$$=e^x\tan x/2+C$$

$$\therefore \int e^x\left(\frac{1+\sin x}{1+\cos x}\right)\cdot dx=e^x 2\tan x/2+C$$

(16) $\int\frac{x}{1+\cos x}\cdot dx$

Let $I=\int\frac{x}{1+\cos x}\cdot dx$

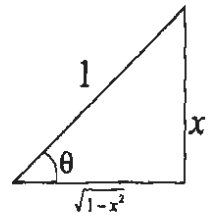
$$= \int \frac{x}{2\cos^2 \frac{x}{2}} \cdot dx$$

$$= \frac{1}{2} \int x \sec^2 \frac{x}{2} \cdot dx$$

$$= \frac{1}{2} \left[x \cdot 2\tan\frac{x}{2} - \int 1 \cdot 2\tan\frac{x}{2}\, dx \right]$$

$$= x\tan\frac{x}{2} - 2\log\sec\frac{x}{2} + C$$

$$\therefore \int \frac{x}{1+\cos x} \cdot dx = x\tan\frac{x}{2} - 2\log\sec\frac{x}{2} + C$$



(17) $\int \frac{\sin^{-1} x}{\left(1-x^2\right)^{3/2}} \cdot dx$

Putting $x = \sin\theta$
$dx = \cos\theta d\theta$

Let $I = \int \frac{\theta \cdot \cos\theta d\theta}{\cos^3 \theta d\theta} \cdot dx$

$$= \int \theta \cdot \sec^2 \theta d\theta = \theta \tan\theta - \int 1 \cdot \tan\theta d\theta$$

$$= \theta \cdot \tan\theta - \log\sec\theta + C$$

$$= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} - \log\frac{1}{\sqrt{1-x^2}} + C$$

$$= \frac{x}{\sqrt{1-x^2}} \sin^{-1} x + \frac{1}{2}\log\left(1-x^2\right) + C$$

$$\therefore \int \frac{\sin^{-1} x}{\left(1-x^2\right)^{3/2}} \cdot dx = \frac{x}{\sqrt{1-x^2}} \sin^{-1} x + \frac{1}{2}\log\left(1-x^2\right) + C$$

EXERCISE 5.1

Evaluate the following integrals :

Q.1.: $x^2 \sin x$ **Q.2.:** $\tan^{-1} x$ **Q.3.:** $x \sin 3x$ **Q.4:** $e^{2x} \sin x$

Q.5.: $\left(x^2+1\right)\log x$ **Q.6.:** $\left(\sin^{-1} x\right)^2$ **Q.7.:** $\frac{(x-3)e^x}{(x-1)^3}$ **Q.8:** $\sin^{-1} \frac{2x}{\left(1+x^2\right)}$

Q.9.: $\sin^{-1} \sqrt{\frac{x}{a+x}}$ **Q.10.:** $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ **Q.11.:** $\frac{e^{\sqrt{x}} \cos\sqrt{x}}{\sqrt{x}}$ **Q.12.:** $\frac{1}{\log x} - \frac{1}{(\log x)^2}$

Q.13.: $\frac{x}{1+\sin x}$ **Q.14.:** $\frac{\log(\log x)}{x}$ **Q.15.:** $e^x \sin x \cos x$ **Q.16.:** $\frac{x \tan^{-1} x}{\left(1+x^2\right)^{3/2}}$

Q. 17.: $\dfrac{x^2e^x}{(x+2)^2}$ **Q. 18.:** $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$ **Q. 19.:** $\dfrac{e^{m\tan^{-1}x}}{\left(1+x^2\right)^{3/2}}$ **Q. 20.:** $\dfrac{e^x(1-\sin x)}{(1-\cos x)}$

Q. 21.: $\dfrac{\log x}{x^3}$ **Q. 22.:** $\dfrac{e^2(x+1)}{(1+x)^2}$ **Q. 23.:** $e^x\left\{1+\sqrt{1-x^2}\sin^{-1}x\right\}\sqrt{1-x^2}$

Q. 24.: $\dfrac{e^x\left(x^2+3x+3\right)}{(x+2)^3}$ **Q. 25.:** $\left(2x^2+1\right)e^{x^2}$

ANSWERS

(1) $-x^2\cdot\cos x+2x\sin x+2\cos x+C$ (2) $x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)+C$

(3) $-\frac{x}{3}\cos 3x+\frac{1}{9}\sin 3x+C$ (4) $\frac{e^{2x}}{5}(2\sin x-\cos x)+C$

(5) $\left(\frac{x^3}{3}+x\right)\log x-\frac{x^3}{9}-x+C$ (6) $x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$

(7) $\dfrac{e^x}{(x-1)^2}+C$ (8) $2x\tan^{-1}x-\log\left(1+x^2\right)+C$ (9) $(a+x)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+C$

(10) $2x\tan^{-1}x-\log\left(1+x^2\right)+C$ (11) $e^{\sqrt{x}}\left(\cos\sqrt{x}+\sin\sqrt{x}\right)+C$ (12) $\dfrac{x}{\log x}+C$

(13) $x\tan x-\log\sec x-x\sec x+\log(\sec x+\tan x)+C$

(14) $\log x\left[\log(\log x)-1\right]+C$ (15) $\frac{e^x}{10}(\sin 2x-2\cos 2x)+C$

(16) $\dfrac{x-\tan^{-1}x}{\sqrt{1+x^2}}+C$ (17) $\dfrac{(x-2)}{(x+2)}e^x+C$ (18) $-\left[\cos^{-1}x\sqrt{1-x^2}+x\right]+C$

(19) $e^{m\tan^{-1}x}\left[\dfrac{m+x}{\left(1+m^2\right)\sqrt{1+x^2}}\right]+C$ (20) $-e^x\cot\frac{x}{2}+C$

(21) $\dfrac{-\left(\frac{1}{4}+\frac{1}{2}\log x\right)}{x^2}+C$ (22) $e^x(x-1)(x+1)+C$ (23) $e^x\sin^{-1}x+C$

(24) $e^x(x+1)(x+2)+C$ (25) $xe^{x^2}+C$

3.6 INTEGRATION BY PARTIAL FRACTION

If a fraction is in the form of an improper fraction or composite fraction then to find the integration of such functions we resolve then into proper fraction and partial fraction. Now we integrate each fraction differently and find the algebraic sum to get the integral of the given function.

This method is used in finding the integration of rational algebraic fractions as well as the integration of product of some trigonometrical functions.

SOLVED EXAMPLES

1. Evaluate $\int \frac{xdx}{(x+1)(x+2)}$

To integrate the above function, we have to breack it into partial fractions.

Let, $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

Where A and B are real numbers to be determined. These real numbers can be determined in many ways. Here, we shall discuss two methods only.

Method - 1. (Comapring the Coefficient of like Terms)

We have, $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$ ...(1)

or $x = A(x+2) + B(x+1)$

$x = (A+B)x + 2A + B$ ...(2)

Comparing the coefficient of x and the constant terms we get –

$A + B = 1$

$2A + B = 0$

Solving these equations, we get $A = -1,\ B = 2$

Method - 2. (Choosing the value of x)

Choosing x = 0 and x = 1, in (2), we get

$2A + B = 0$ *and* $3A + 2B = 1$

Solving these equations we get, $A = -1,\ B = 2$

$\therefore$ the integrand is given by

$$\frac{x}{(x+1)(x+2)} = -\frac{1}{(x+1)} + \frac{2}{(x+2)}$$

$$\therefore \int \frac{x}{(x+1)(x+2)}dx = -\int \frac{1}{(x+1)}dx + 2\int \frac{1}{(x+2)}dx$$

$$= -\log(x+1) + 2\log(x+2) + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

2. Evaluate $\int \frac{x^2+1}{\left(x^2+5x+6\right)}dx$

Here, the integrand $\frac{x^2+1}{\left(x^2+5x+6\right)}$ is not proper rational function. So we have to divide the numerator by the denominator,

$$\therefore \quad \frac{x^2+1}{\left(x^2+5x+6\right)} = 1 - \frac{5x+5}{x^2+5x+6}$$

$$= 1 - \frac{5x+5}{(x+2)(x+3)}$$

Let, $\frac{5x+5}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$$5x+5 = A(x+3) + B(x+2)$$

$$5x+5 = (A+B)x + (3A+2B)$$

Equating the coefficient of x and constant we get,

$$A + B = 5$$

$$3A + 2B = 5$$

Solving these equation we get,

$$A = -5, \quad B = 10$$

$$\frac{5x+5}{(x+2)(x+3)} = -\frac{5}{x+2} + \frac{10}{x+3}$$

$$\int \frac{x^2+1}{\left(x^2+5x+6\right)}dx = \int 1dx - \left[\int\left(\frac{-5}{x+2} + \frac{10}{x+3}\right)dx\right]$$

$$= \int dx + 5\int \frac{1}{x+2}dx - 10\int \frac{1}{x+3}dx$$

$$= x + 5\log(x+2) - 10\log(x+3) + C$$

3. Evaluate $\int \frac{xdx}{\left(x^2+1\right)(x+1)}$

Here, the itegrand is a proper fraction, so we decompose it into partial fraction. Now,

$$\frac{x}{\left(x^2+1\right)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$x = A\left(x^2+1\right) + (Bx+C)(x+1)$$

$$x = (A+B)x^2 + (B+C)x + 1(A+C)$$

Equating the coefficient of x, x^2 and constant on both sides, we get,

$$A + B = 5$$
$$B + C = 1$$
$$A + C = 0$$

Solving these equation we get,

$$A = \frac{1}{2}, \quad B = \frac{1}{2} \quad and \quad C = -\frac{1}{2}$$

$$\frac{x}{(x+1)\left(x^2+1\right)} = \frac{1}{2(x+1)} + \frac{1}{2}\frac{x-1}{x^2+1}$$

$$\therefore \int \frac{xdx}{\left(x^2+1\right)(x+1)} = \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{1+x^2}dx$$

$$= \frac{1}{2}\log(x+1) + \frac{1}{4}\log\left(x^2+1\right) - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{2}\log(x+1) + \frac{1}{4}\log\left(x^2+1\right) - \frac{1}{2}\tan^{-1}x + C$$

4. $\int \frac{3x+5}{x^3+x^2+x+1}dx$

$$\int \frac{3x+5}{x^3+x^2+x+1} = \frac{3x+5}{(x+1)(x-1)^2}$$

$$\frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$3x+5 = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$
$$= A\left(x^2-2x+1\right) + B\left(x^2-1\right) + C(x+1)$$

Factoring the denominator, we have

$$x^3+x^2+x+1$$
$$= x^3+1-x^2-x$$
$$= (x+1)\left(x^2-x+1\right) - x(x+1)$$
$$= (x+1)\left(x^2-x+1-x\right)$$
$$= (x+1)\left(x^2-2x+1\right)$$

$$3x+5 = (A+B)x^2 + (-2A+C)x + (A+B+C)$$

Equating the coefficient of x, x^2 and constant terms on both sides, we get,

$$A + B = 0$$
$$-2A + C = 3$$
$$A + B + C = 5$$

On solving these equation we get,

$$A=\frac{1}{2},\quad B=-\frac{1}{2}\quad and\quad C=4$$

$$\therefore\ \int\frac{3x+5}{x^3+x^2-x+1}dx=\frac{1}{2}\int\frac{1}{x+1}dx-\frac{1}{2}\int\frac{1}{x-1}dx+4\int\frac{1}{(1-x)^2}dx$$

$$=\frac{1}{2}\log(x+1)-\frac{1}{2}\log(x-1)-\frac{4}{x-1}+C$$

$$=\frac{1}{2}\log\frac{(x+1)}{(x-1)}-\frac{4}{(x-1)}+C$$

5. $\int\frac{\cos x}{(1-\sin x)(2+\sin x)}dx$

Let $\quad I=\int\frac{\cos x}{(1-\sin x)(2+\sin x)}dx$

Putting $\sin x = t$, then $\cos x dx = dt$

$$\therefore\ I=\int\frac{dt}{(1-t)(2+t)}dx$$

Now,

$$\frac{1}{(1-t)(2+t)}=\frac{A}{(1-t)}+\frac{B}{(2+t)}$$

or
$$1=A(2+t)+B(1-t)$$
$$1=(2A+B)+(A-B)t$$

Equating the coefficient of t and constant terms on both sides, we get,

$$A-B=0\quad and\quad 2A+B=1$$

on solving these equation we get,

$$A=\frac{1}{3},\quad B=\frac{1}{3}$$

$$\int\frac{dt}{(1-t)(2+t)}=\frac{1}{3}\int\frac{dt}{(1-t)}+\frac{1}{3}\int\frac{1}{2+t}dt$$

$$=-\frac{1}{3}\log(1-t)+\frac{1}{3}\log(2+t)+C$$

Hence, $\int\frac{\cos x}{(1-\sin x)(2+\sin x)}=-\frac{1}{3}\log(1-\sin x)+\frac{1}{3}\log(2+\sin x)+C$

6. $\int\frac{2x}{(x^2+1)(x^2+3)}dx$

Let $I=\int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx$

Putting $x^2=t$, so that $2\,xdx=dt$

$$\therefore\ I=\int \frac{dt}{(t+1)(t+3)}$$

Now,

$$\frac{1}{(t+1)(t+3)}=\frac{A}{(t+1)}+\frac{B}{(t+3)}$$

or

$$1=A(t+3)+B(t+1)$$

$$1=(A+B)t+3A+B$$

Equating the coefficient of t and constant terms on both sides, we get,

$$A+B=0 \quad and \quad 3A+B=1$$

on solving these equation we get,

$$A=\frac{1}{2},\quad B=-\frac{1}{2}$$

$$\int \frac{dt}{(t+1)(t+3)}=\frac{1}{2}\int \frac{dt}{(t+1)}-\frac{1}{2}\int \frac{1}{(t+3)}$$

$$=\frac{1}{2}\log(t+1)-\frac{1}{2}\log(t+3)+C$$

Hence, $\int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}=\frac{1}{2}\log\left(x^2+1\right)-\frac{1}{2}\log\left(x^2+3\right)+C$

7. $\int \frac{dx}{x^2+2x+10}=\int \frac{\sin x\,dx}{\cos^2 x-5\cos x+4}$

Let $I=\int \frac{\sin x\,dx}{\cos^2 x-5\cos x+4}dx$

Putting $\cos x=t$, so that $-\sin x dx=dt$

$$\therefore\quad I=-\int \frac{dt}{t^2-5t+4}$$

$$=-\int \frac{dt}{(t-4)(t-1)}$$

Now,

$$\frac{1}{(t-4)(t-1)}=\frac{A}{(t-1)}+\frac{B}{(t-4)}$$

or $$1 = A(t-4)+B(t-1)$$

$$1=(A+B)t-(4A-B)$$

Equating the coefficient of t and constant terms on both the sides, we get,

$$A+B=0 \quad and \quad -4A-B=1$$

On solving these equation we get,

$$A=-\frac{1}{3}, \quad B=+\frac{1}{3}$$

$$\therefore \int\frac{dt}{(t-4)(t-1)} = -\left[-\frac{1}{3}\int\frac{1dt}{(t-1)}+\frac{1}{3}\int\frac{1dt}{(t-4)}\right]$$

$$=\frac{1}{3}\int\frac{dt}{(t-1)}-\frac{1}{3}\int\frac{dt}{(t-4)}$$

$$=\frac{1}{3}\log(t-1)-\frac{1}{3}\log(t-4)+C$$

Hence, $$\int\frac{\sin x\,dx}{\cos^2 x-5\cos x+4}=\frac{1}{3}\log(\cos x-1)-\frac{1}{3}\log(\cos x-4)+C$$

8. $\int\frac{dx}{\sin x+\sin 2x}$

Let $$I=\int\frac{dx}{\sin x+\sin 2x}=\int\frac{dx}{\sin x+2\sin x\cos x}$$

$$=\int\frac{1}{\sin x(1+2\cos x)}dx=\int\frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

$$=\int\frac{\sin x\,dx}{\left(1-\cos^2 x\right)(1+2\cos x)}$$

Putting $\cos x = t$, so that $-\sin x dx = dt$

$$\therefore\ I=-\int\frac{dt}{\left(1-t^2\right)(1+2t)}$$

Now,

$$\frac{1}{\left(1-t^2\right)(1+2t)}=\frac{A}{(1+2t)}+\frac{B}{(1-t)}+\frac{C}{(1+t)}$$

or $$1=A\left(1-t^2\right)+B(1+2t)(1+t)+C(1-t)(1+2t)$$

or $$1=(A+B+C)+(3B+C)t+(-A+2B-2C)t^2$$

Equating the coefficient of t^2, t and constant terms on both the sides, we get,

$$-A+2B-2C=0, \quad 3B+C=0 \quad and \quad A+B+C=1$$

on solving these equation we get,

$$A=\frac{4}{3}, \quad B=\frac{1}{6}, \quad C=-\frac{1}{2}$$

$$I=\int\frac{dt}{\left(1-t^2\right)\left(1+2t\right)}=-\left[\frac{4}{3}\int\frac{dt}{\left(1+2t\right)}+\frac{1}{6}\int\frac{1dt}{\left(1-t\right)}-\frac{1}{2}\int\frac{1dt}{\left(1+t\right)}\right]$$

$$=-\frac{4}{3}\int\frac{dt}{\left(1+2t\right)}+\frac{1}{6}\int\frac{1dt}{\left(1-t\right)}+\frac{1}{2}\int\frac{1dt}{\left(1+t\right)}$$

$$=-\frac{4^2}{3}\times\frac{1}{2}\log\left(1+2t\right)+\frac{1}{6}\log\left(1-t\right)+\frac{1}{2}\left(1+t\right)+C$$

$$=-\frac{2}{3}\log\left(1+2t\right)+\frac{1}{6}\log\left(1-t\right)+\frac{1}{2}\left(1+t\right)+C$$

Hence, $\int\frac{dx}{\sin x+\sin 2x}=-\frac{2}{3}\log\left(1+2\cos x\right)+\frac{1}{6}\log\left(1-\cos x\right)+\frac{1}{2}\left(1+\cos x\right)+C$

9. $\int\frac{\cos x}{4-\sin^2 x}dx$

Let $I=\int\frac{\cos x}{4-\sin^2 x}dx$

Putting $\sin x=t$, so that $\cos x dx=dt$

$$\therefore\ I=\int\frac{dt}{4-t^2}$$

Now,

$$\frac{1}{4-t^2}=\frac{A}{(2-t)}+\frac{B}{(2+t)}$$

or $\quad 1=A\left(2+t\right)+B\left(2-t\right)$

or $\quad 1=\left(A-B\right)t+\left(2A+2B\right)$

Equating the coefficient of t and constant terms on both the sides, we get,

$$A-B=0, \quad 2A+2B=1$$

on solving these equation we get,

$$A=\frac{1}{4}, \quad B=\frac{1}{4}$$

$$\therefore\int\frac{dt}{4-t^2}=\frac{1}{4}\int\frac{dt}{2-t}+\frac{1}{4}\int\frac{dt}{(2+t)}$$

$$= -\frac{1}{4}\log(2-t) + \frac{1}{4}\log(2+t) + C$$

Hence, $\int \frac{\cos x}{4-\sin^2 x}\,dx = -\frac{1}{4}\log(2-\sin x) + \frac{1}{4}\log(2+\sin x) + C$

or $\int \frac{\cos x}{4-\sin^2 x} = \frac{1}{4}\log\left(\frac{2+\sin x}{2-\sin x}\right) + C$

10. $\int \frac{\sec x}{1+\operatorname{cosec} x}\,dx$

Let $I = \int \frac{\sec x}{1+\operatorname{cosec} x}\,dx = \int \frac{\sin x}{\cos x(\sin x+1)}\,dx$

$$= \int \frac{\sin x \cos x}{\cos^2 x(1+\sin x)} - dx$$

Putting $\sin x = t$, so that $\cos x dx = dt$

$$\therefore\ I = \int \frac{tdt}{(1-t^2)(1+t)} = \int \frac{tdt}{(1-t)(1+t)^2}$$

Now,

$$\frac{1}{(1-t)(1+t)^2} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2}$$

or $t = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$

or $t = A(1+2t+t^2) + B(1-t^2) + C(1-t)$

$$t = (A+B+C)(2A-C)t + (A-B)t^2$$

Equating the coefficient of t^2, t and constant terms on both the sides, we get,

$$A - B = 0, \quad 2A - C = 1, \quad A + B + C = 0$$

on solving these equation we get,

$$A = \frac{1}{4}, \quad B = \frac{1}{4}, \quad C = -\frac{1}{2}$$

$$\therefore \int \frac{dt}{(1-t)(1+t)^2} = \int \frac{1}{4}\frac{dt}{(1-t)} + \frac{1}{4}\int \frac{dt}{(1+t)} - \int \frac{1}{2}\frac{dt}{(1+t)^2}$$

$$= -\frac{1}{4}\log(1-t) + \frac{1}{4}\log(1+t) + \frac{1}{2(1+t)} + C$$

$$= \frac{1}{4}\log\frac{(1+t)}{(1-t)} + \frac{1}{2(1+t)} + C$$

$$= \frac{1}{4}\log\frac{(1+\sin x)}{(1-\sin x)} + \frac{1}{2(1+\sin x)} + C$$

$$\therefore \int \frac{\sec x}{1+\operatorname{cosec} x}\,dx = \frac{1}{4}\log\frac{(1+\sin x)}{(1-\sin x)} + \frac{1}{2(1+\sin x)} + C$$

EXERCISE 6.1

Evaluate the following :

Q. 1.: $\int \frac{dx}{x^2+2x+10}$ **Q. 2.:** $\int \frac{dx}{(x-1)(x^2-4)}$ **Q. 3. :** $\int \frac{dx}{1-x-x^2}$

Q. 4 : $\int \frac{(x^3+2)\,dx}{(x-2)^2(x-1)}$ **Q. 5.:** $\int \frac{(3x-1)\,dx}{(x-1)(x-2)(x-3)}$ **Q. 6.:** $\int \frac{(x^2+x+1)\,dx}{x^2(x+2)}$

Q. 7. : $\int \frac{x^4}{(x^2+1)}\,dx$ **Q. 8 :** $\int \frac{x+1}{x(x^2+1)}\,dx$ **Q. 9.:** $\int \frac{1}{x(x^4-1)}\,dx$

Q. 10.: $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$ **Q. 11.:** $\int \frac{1}{x(x^4+1)}\,dx$ **Q. 12.:** $\int \frac{5x}{(x+1)(x^2-4)}\,dx$

Q. 13.: $\int \frac{e^x\,dx}{e^x(e^x-1)}$ **Q. 14.:** $\int \frac{e^x\,dx}{\sqrt{a^2+e^{2x}}}$ **Q. 15.:** $\int \frac{dx}{1+3e^x+2e^{2x}}$

Q. 16.: $\int \frac{dx}{1+\cos^2 x}$ **Q. 17.:** $\int \frac{\cos x\,dx}{(1-\sin x)^3(2+\sin x)}$ **Q. 18.:** $\int \frac{\cos x\,dx}{\sqrt{4-\sin^2 x}}$

Q. 19.: $\int \frac{\cos x}{\cos 3x}\,dx$ **Q. 20.:** $\int \frac{dx}{2-3\cos 2x}$

ANSWERS

(1) $\frac{1}{3}\tan^{-1}\cdot\frac{x+1}{3}+C$ **(2)** $\frac{1}{3}\log(x-1)+\frac{1}{4}\log(x-2)+\frac{1}{12}\log(x+2)+C$

(3) $\frac{1}{\sqrt{5}}\log\frac{\sqrt{5}+2x+1}{\sqrt{5}-2x-1}+C$ **(4)** $\log\frac{(x-2)^4}{(x-1)^3}+\frac{2}{x-2}-\frac{5}{(x-2)^2}+C$

(5) $\log(x-1)-5\log(x-2)+4\log(x-3)+C$

(6) $\frac{3}{4}\log(x+2)+\frac{1}{4}\log x-\frac{1}{2x}+C$ **(7)** $\frac{x^3}{3}-x+\tan^{-1}x+C$

(8) $\log x-\frac{1}{2}\log(x^2+1)+\tan^{-1}x+C$ **(9)** $\frac{1}{4}\log\frac{x^4-1}{x^4}+C$

(10) $x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$ **(11)** $\frac{1}{4}\log\frac{x^4}{x^4+1}+C$

(12) $\frac{5}{3}\log(x+1)-\frac{5}{2}\log(x+2)+\frac{5}{60}+\frac{5}{6}\log(x-2)+C$ (13) $\log\left(\frac{e^x-1}{e^x}\right)+C$

(14) $\log\left\{e^x+\sqrt{e^{2x}+a^2}\right\}+C$ (15) $\log\left[\frac{e^x(1+e^x)}{(1+2e^x)^2}\right]+C$ (16) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+C$

(17) $-\frac{1}{27}\log(\sin x-1)-\frac{1}{9(\sin x-1)}+\frac{1}{6(\sin x-1)^2}+\frac{1}{27}\log(\sin x+2)+C$

(18) $\sin^{-1}\left(\frac{\sin x}{2}\right)+C$ (19) $\frac{1}{2\sqrt{3}}\log\left(\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right)+C$ (20) $\frac{1}{2\sqrt{5}}\log\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}+C$

3.7 INTEGRATION OF RATIONAL AND IRRATIONAL ALGEBRAIC FUNCTIONS

(a) Integration of rational algebraic functions. :

In earlier sections we have integrated some rational algebraic functions by using partial fractions. However, here some more important integration of rational algebraic functions have been illustrated –

(1) $\int \frac{1}{ax^2+bx+c}dx$

To evaluate the above integral we first write ax^2+bx+c in the form of standard integrals.

Now, $\int(ax^2+bx+c)\,dx = ax^2+bx+\frac{b^2}{4a}+c-\frac{b^2}{4a}$

$$= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+\frac{4ac-b^2}{4a}$$

$$= a\left(x^2+\frac{b}{2a}x+\frac{b^2}{4a^2}\right)+\frac{4ac-b^2}{4a}$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}\right]$$

$$= a\left[\left(x+\frac{1}{2a}\right)^2+\left\{\sqrt{\frac{4ac-b^2}{4a}}\right\}^2\right]$$

Case I : When $4ac > b^2$, *then*

Then, $\int \frac{1}{ax^2+bx+c}dx = \frac{1}{a}\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\sqrt{\frac{4ac-b^2}{4a^2}}\right)^2}$

$$= \frac{1}{a}\int \frac{dx}{X^2 + A^2}$$

$$= \frac{1}{a} \times \frac{1}{A}\tan^{-1}\frac{X}{A} \qquad \text{Where} \quad X = x + \frac{b}{2a}, \quad A = \sqrt{\frac{4ac - b^2}{4a^2}}$$

$$= \frac{1}{a} \times \frac{2a}{\sqrt{4ac-b^2}}\tan^{-1}\frac{\left(x+\frac{b}{2a}\right)}{\sqrt{4ac-b^2}/2a} + C$$

$$= \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} + C$$

Case II : When $b^2 > 4ac$, then

$$\int \frac{1}{ax^2+bx+c}dx = \frac{1}{a}\frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2}$$

$$= \frac{1}{a}\int \frac{dx}{X^2 - A^2} \qquad \text{Where} \quad X = x + \frac{b}{2a}, \quad A = \sqrt{\frac{b^2+4ac}{4a^2}}$$

$$= \frac{1}{a} \times \frac{1}{2A}\log\frac{X-A}{X+A} + C$$

$$= \frac{1}{a} \times \frac{2a}{2 \times \sqrt{b^2-4ac}}\log\frac{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}}{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}} + C$$

$$= \frac{1}{\sqrt{b^2-4ac}}\log\frac{2x+b-\sqrt{b^2-4ac}}{2x+b+\sqrt{b^2-4ac}} + C$$

(2) $\int \frac{x^3+x}{x^4-16}dx$

Let, $I = \int \frac{x^3+x}{x^4-16}dx$

$$= \int \frac{x^3}{x^4-16}dx + \int \frac{x}{x^4-16}dx$$

$I_1 + I_2 \quad (say)$

We have, $I_1 = \int \frac{x^3}{x^4+16} dx$

Putting $x^4 - 16 = t$

$4x^3 dx = dt$

or $x^3 dx = \frac{1}{4} dt$

$$\therefore I_1 = \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log t + C_1$$

$$= \frac{1}{4}\log\left(x^4 - 16\right) + C_1$$

and $I_2 = \int \frac{x}{x^4 - 16}$

Putting $x^2 = t$

$2x\, dx = dt$

$x\, dx = \frac{1}{2} dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{x}{t^2 - 16}$$

$$= \frac{1}{2} \times \frac{1}{2\times 4}\log\frac{t-4}{t+4} + C_2$$

$$= \frac{1}{16}\log\frac{x^2-4}{x^2+4} + C_2$$

$$\therefore I = I_1 + I_2 = \frac{1}{4}\log\left(x^2 - 16\right) + \frac{1}{16}\log\frac{x^2-4}{x^2+4} + C$$

$C = C_1 + C_2$

$$I = \int \frac{x^3 + x}{x^4+16} dx = \frac{1}{4}\log\left(x^2 - 16\right) + \frac{1}{16}\log\frac{x^2-4}{x^2+4} + C$$

(3) $\int \frac{x^2 dx}{x^4 + x^2 + 1} dx$

Let, $I = \int \frac{x^2 dx}{x^4 + x^2 + 1}$

$$= \int \frac{dx}{x^2 + 1 + \frac{1}{x^2}}$$

$$= \frac{1}{2}\int \frac{2\, dx}{x^2 + 1 + \frac{1}{x^2}}$$

$$= \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2} + 1 + \frac{1}{x^2}\right)dx}{x^2 + \frac{1}{x^2} + 1}$$

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1} + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}+1}dx$$

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1} + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3}$$

$$I = \frac{1}{2}I_1 + \frac{1}{2}I_2 \qquad \ldots(1)$$

Now, $I_1 = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$ *Putting* $x+\frac{1}{x}=t$

$\left(1-\frac{1}{x^2}\right)dx = dt$

$$I_1 = \int \frac{dt}{t^2-I} = \frac{1}{2}\log\frac{(t-1)}{(t+1)} = \frac{1}{2}\log\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1} + C_1 = \frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1} + C_1$$

and $I_2 = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3}$ *Putting* $x-\frac{1}{x}=t$

$\left(x+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+\left(\sqrt{3}\right)^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C_2$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{3}} + C_2$$

$$= \frac{1}{3}\tan^{-1}\frac{x^2-1}{\sqrt{3}x} + C_2$$

Substituting I_1 *and* I_2 in (1) we get,

$$I = \frac{1}{2}I_1 + \frac{1}{2}I_2 = \frac{1}{4}\log\frac{x^2-x+1}{x^2+x+1} + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x^2-1}{\sqrt{3}x} + \frac{1}{2}C_1 + \frac{1}{2}C_2$$

$$I = \int \frac{x^2dx}{x^4+x^2+1} = \frac{1}{4}\log\frac{x^2-x+1}{x^2+x+1} + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + C$$

$$\left(\textit{Where } C = \frac{1}{2}C_1 + \frac{1}{2}C_2\right)$$

(4) $\int \frac{1}{x^4+1}\,dx$

$$Let\ I = \int \frac{1}{x^4+1}\,dx = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}-1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\left[\frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}} - \int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}}\right]$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2}$$

$$= \frac{1}{2}I_1 - \frac{1}{2}I_2 \quad (say) \qquad (1)$$

Now, $I_1 = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{2}\right)^2}$

Putting $x-\frac{1}{x}=t$

$\left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}} + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) + C_1$$

or $$I_2 = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2 - \left(\sqrt{2}\right)^2}$$

Putting $x + \frac{1}{x} = t$

$\left(1-\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2 - \left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}} \log \frac{t-\sqrt{2}}{t^2 - \left(\sqrt{2}\right)^2} + C_2 = \frac{1}{2\sqrt{2}} \log \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{\sqrt{2}}+\left(\sqrt{2}\right)^2} + C_2$$

Substituting I_1 and I_2 in (1) we get,

$$I = \int \frac{1}{x^4+1} dx = \frac{1}{2}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)\right] + C_1$$

$$-\frac{1}{2}\left[\frac{1}{2\sqrt{2}} \log \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right] - \frac{1}{2}C_2$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log \frac{x^2-\sqrt{2x+1}}{x^2+\sqrt{2x+1}} + C$$

$\left(\textit{Where } C = \frac{1}{2}C_1 - \frac{1}{2}C_2\right)$

(5) $$\int \frac{1}{x^4+x^2+1} dx$$

Let $$I = \int \frac{dx}{x^4+x^2+1}$$

$$= \int \frac{\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}+1} dx$$

$$= \frac{1}{2} \frac{\left(1+\frac{1}{x^2}-1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}+1} dx$$

$$=\frac{1}{2}\int\frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}-\frac{1}{2}\int\frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}$$

$$=\frac{1}{2}\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3}-\frac{1}{2}\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$$

$$=\frac{1}{2}I_1-\frac{1}{2}I_2 \qquad \ldots(1)$$

Now, $$I_1=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3}$$

Putting $x-\frac{1}{x}=t$

$\therefore \left(1+\frac{1}{x^2}\right)dx=dt$

$$=\int\frac{dt}{t^2+\left(\sqrt{3}\right)^2}$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}+C_1$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)+C_1$$

or $$I_2=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$$

Putting $x+\frac{1}{x}=t$

$\therefore \left(1-\frac{1}{x^2}\right)dx=dt$

$$=\int\frac{dt}{t^2-1}$$

$$=\frac{1}{2}\log\frac{t-1}{t+1}+C_2$$

$$=\frac{1}{2}\log\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}+C_2=\frac{1}{2}\log\frac{x^2-x+1}{x^2-x+1}+C_2$$

Substituting I_1 and I_2 in (1) we get,

$$I = \int \frac{1}{x^4 + x^2 + 1} dx = \frac{1}{2}\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right)\right] + \frac{1}{2} C_1$$

$$- \frac{1}{2}\left[\frac{1}{2} \log \frac{x^2 - x + 1}{x^2 + x + 1}\right] - \frac{1}{2} C_2$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x^2 - 1}{\sqrt{3}\,x} - \frac{1}{4} \log \frac{x^2 - x + 1}{x^2 + x + 1} + C$$

$$\left(\textit{Where } C = \frac{1}{2} C_1 - \frac{1}{2} C_2\right)$$

(6) $\int \frac{x}{x^4 + x^2 + 1} dx$

Putting $x^2 = t$

$2xdx = dx$

$xdx = \frac{dt}{2}$

Let $I = \frac{1}{2} \int \frac{dt}{t^2 + t + 1}$

$$= \frac{1}{2} \int \frac{dt}{t^2 + 2\,t/2 + 1 + \frac{1}{4} - \frac{1}{4}}$$

$$= \frac{1}{2} \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{2} \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{2t + 1}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x^2 + 1}{\sqrt{3}} + C$$

(7) $\int \frac{3x^2}{x^6 + 1} dx$

Let $I = \int \frac{3x^2}{(x^3)^2 + 1} dx$ 　　*Putting* $x^3 = t$, $3x^2 dx = dt$

$$= \int \frac{dt}{t^2 + 1}$$

$$= \tan^{-1} + C$$

$$= \tan^{-1} x^3 + C$$

(8) $\int \frac{dx}{2x^2 + 3x + 5}$

Let $I = \frac{1}{2} \int \frac{dx}{x^2 + \frac{3}{2}x + \frac{5}{2}}$

$$= \frac{1}{2} \int \frac{dx}{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{31}}{4}\right)^2}$$

Putting $x + \frac{3}{4} = t$ $\therefore\ dx = dt$

$$= \frac{1}{2} \times \frac{4}{\sqrt{31}} \tan^{-1} \frac{1}{\left(\sqrt{31/4}\right)^2} + C$$

$$= \frac{2}{\sqrt{31}} \tan^{-1} \frac{4(x + 3/4)}{\sqrt{31}} + C$$

$$= \frac{2}{\sqrt{31}} \tan^{-1} \frac{4x + 3}{\sqrt{31}} + C$$

(9) $\int \frac{5x - 2}{1 + 2x + 3x^2}$

Let $I = \int \frac{(5x - 2)}{1 + 2x + 3x^2} dx$

$$= 5 \int \frac{(x - 2/5)\, dx}{1 + 2x + 3x^2}$$

$$= \frac{5}{6} \int \frac{\left(6x - \frac{12}{5}\right)}{1 + 2x + 3x^2} dx$$

$$= \frac{5}{6} \int \frac{\left(6x + 2 - 2 - \frac{12}{5}\right)}{1 + 2x + 3x^2} dx$$

or, $$I=\frac{5}{6}\int\frac{(6x+2)dx}{1+2x+3x^2}-\frac{5}{6}\int-\frac{\frac{22}{5}}{1+2x+3x^2}dx$$
$$=\frac{5}{6}\int\frac{(6x+2)}{1+2x+3x^2}dx-\frac{11}{3}\int-\frac{dx}{3x^2+2x+1}$$
$$=\frac{5}{6}I_1-\frac{11}{3}I_2 \qquad \ldots(1)$$

Now, $$I_1=\int\frac{(6x+2)}{1+2x+3x^2}dx$$ Putting $1+2x+3x^2=t$

$(2+6x)dx=dt$

$$=\int\frac{dt}{t}$$
$$=\log t+C_1$$
$$I_1=\log\left(1+2x+3x^2\right)+C_1$$

And $$I_2=\int\frac{dx}{3x^2+2x+1}=\frac{1}{3}\int\frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}=\frac{1}{3}\int\frac{dx}{x^2+2\times x\times\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+\frac{1}{3}}$$
$$=\frac{1}{3}\int\frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{1}{9}}=\frac{1}{3}\int\frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$$
$$=\frac{1}{3}\times\frac{3}{\sqrt{2}}\tan^{-1}\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}+C_2$$
$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}}+C_2$$

Substituting I_1 and I_2 in (1) we get,

$$I=\int\frac{(5x-2)}{1+2x+3x^2}dx=\frac{5}{6}\log\left(1+2x+3x^2\right)+\frac{5}{6}C_1-\frac{11}{3}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}}\right)-\frac{11}{3}C_2$$
$$=\frac{5}{6}\log\left(1+2x+3x^2\right)-\frac{11}{3\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}}+C$$
$$\left(\textit{Where } C=\frac{5}{6}C_1-\frac{11}{3}C_2\right)$$

(10) $\int \frac{dx}{x^{1/3} - x^{1/2}}$

Solution : Let $x = t^6$

$\therefore\ dx = 6t^5 dt$

$$\therefore \int \frac{dx}{x^{1/3} - x^{1/2}} = \int \frac{6t^5}{t^2(1-t)} dt = \int \frac{6t^5}{(1-t)} dt$$

$$= \int 6\left(-t^2 - t - 1 + \frac{1}{1-t}\right) dt$$

$$= -\frac{6t^2}{3} - \frac{6t^2}{2} - 6\log(1-t) + C$$

$$= -2x^{1/2} - 3x^{1/3} - 6\log\left(1 - x^{1/6}\right) + C$$

(11) $\int \frac{dx}{2+\cos x}$

Solution : $\int \frac{dx}{2+\cos x} = \int \frac{dx}{2 + \cos^2 x/2 - \sin^2 x/2}$

$$= \int \frac{dx}{(2+1)\cos^2 x/2 + (2-1)\sin^2 x/2}$$

$$= \int \frac{dx}{3\cos^2 x/2 + \sin^2 x/2} = \int \frac{\sec^2 x/2\, dx}{3 + \tan^2 x/2}$$

$$= 2\int \frac{du}{3+u^2}$$

Putting $\tan\frac{x}{2} = u$

$\frac{1}{2}\sec^2 x/2\, dx = du$

$\left(\sec^2 \frac{1}{2}x\right)dx = 2du$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$$

(12) $\int \frac{x^2}{1-x^6} dx$

Let $I = \int \frac{x^2}{1-(x^3)^2} dx$

Putting $x^3 = t$

$3x^2 dx = dt$

$x^2 dx = \frac{1}{3} dt$

$$= \frac{1}{3}\int \frac{dt}{1-t^2}$$

$$= \frac{1}{3} \times \frac{1}{2} \log \frac{1-t}{1+t} + C$$

$$= \frac{1}{6} \log \frac{1-x^3}{1+x^3} + C$$

(b) Integration of irrational algebraic functions :

In this sections we will find the integration of different forms of irrational algebraic functions. Before doing this, it is most essential to find the integrtion of some special functions, to get formulae which can be applied directly to integrate many irrational algebraic functions.

(1) $\int \sqrt{x^2 + a^2}\, dx$a

Let $I = \int \sqrt{x^2 + a^2}\, dx$

Putting $x = a\tan\theta$

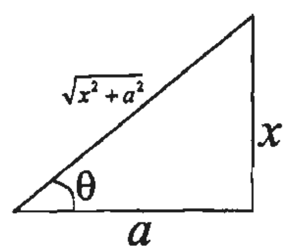
$dx = a\sec^2\theta d\theta$

$$\therefore\ I = \int \sqrt{x^2 + a^2}\, dx$$

$$= \int \sqrt{a^2 \tan^2\theta + a^2 \cdot a\sec^2\theta d\theta}$$

$$= a^2 \int \sqrt{\sec^2\theta}\, \sec^2\theta d\theta \text{ a}$$

$$= a^2 \int \sec\theta \sec^2\theta d\theta$$



Now taking $\sec^2\theta$ as second function and integrating by parts we get,

$$I = a^2[\sec\theta \cdot \tan\theta - \int \sec\theta \tan\theta \tan\theta d\theta]$$

$$= a^2[\sec\theta \cdot \tan\theta - \int \sec\theta \tan^2\theta d\theta]$$

$$= a^2[\sec\theta \cdot \tan\theta - \int \sec\theta(\sec^2\theta - 1) d\theta]$$

$$= a^2[\sec\theta \cdot \tan\theta - a^2\int \sec\theta \sec^2\theta d\theta + a^2 \int \sec\theta d\theta]$$

$$= a^2 \sec\theta \cdot \tan\theta - I + a^2 log|\sec\theta + \tan\theta| + C_1$$

or $$2I = a^2 \frac{\sqrt{x^2+a^2}}{a} \cdot \frac{x}{a} + a^2 log\left|\frac{\sqrt{x^2+a^2}}{a} \cdot \frac{x}{a}\right| + C_1$$

$$= x\sqrt{x^2+a^2} + a^2 log\left|x\sqrt{x^2+a^2}\right| + C$$

$$I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \cdot \log\left|x + \sqrt{x^2+a^2}\right| + C$$

Where $C = \left(\frac{C_1 - a^2 \log a}{2}\right)$

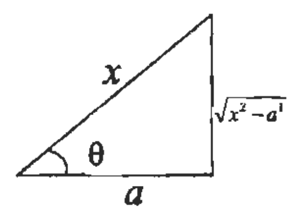
(2) $\int \sqrt{x^2 - a^2}\, dx$

Let $I = \int \sqrt{x^2 - a^2}\, dx$

Putting $x = a\sec\theta$

$dx = a\sec\theta\tan\theta d\theta$

$$= \int \sqrt{a^2 + \sec^2\theta - a^2} \cdot a\sec\theta\tan\theta d\theta$$



$$= a^2 \int \sqrt{\tan^2\theta}\, \sec\theta\tan\theta d\theta$$

$$= a^2 \int \sec\theta \tan^2\theta d\theta$$

$$= a^2 \int \sec\theta\left(\sec^2 - 1\right) d\theta$$

$$I = a^2 \int \sec\theta\sec^2\theta\, d\theta 0 - a^2 \int \sec\theta d\theta$$

$$I = a^2 I_1 - a^2 \log|\sec\theta + \tan\theta| \qquad \ldots(1)$$

Where $I_1 = \int \sec\theta\sec^2\theta\, d\theta$

Taking $\sec^2\theta$ as second function and $\sec\theta$ as first and integrating by parts we get,

$$I_1 = \sec\theta \cdot \tan\theta - \int \sec\theta\tan\theta d\theta$$

$$= \sec\theta \cdot \tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right) d\theta$$

$$= \sec\theta \cdot \tan\theta - \int \sec\theta\sec^2\theta + \int \sec\theta d\theta$$

$$= \sec\theta \cdot \tan\theta - I_1 + \log|\sec\theta + \tan\theta|$$

$$\therefore \quad 2I = \sec\theta \cdot \tan\theta + \log|\sec\theta + \tan\theta|$$

$$I_1 = \frac{1}{2}\sec\theta \cdot \tan\theta + \frac{1}{2} log|\sec\theta + \tan\theta| \qquad \ldots(2)$$

From equation (1) and (2) we get,

$$I = \frac{a^2}{2}\sec\theta \cdot \tan\theta + \frac{a^2}{2}\log|\sec\theta + \tan\theta| - a^2 \log|\sec\theta + \tan\theta| + a^2 C_1$$

$$= \frac{a^2}{2}\sec\theta\cdot\tan\theta - \frac{a^2}{2}\log|\sec\theta+\tan\theta| + a^2C_1$$

$$= \frac{a^2}{2}\cdot\frac{\sqrt{x^2-a^2}}{a}\cdot\frac{x}{a} - \frac{a^2}{2}\log\left|\frac{\sqrt{x^2-a^2}}{a}+\frac{x}{a}\right| + a^2C_1$$

$$= \frac{x}{2}\cdot\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|\sqrt{x^2-a^2}+x\right| - \frac{a^2}{2}\log a + a^2C_1$$

$$= \frac{x}{2}\cdot\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|\sqrt{x^2-a^2}+x\right| + C$$

Where $C = a^2C_1 - \frac{a^2}{2}\log a$

(3) $\int\sqrt{a^2-x^2}\,dx$

Let $I = \int\sqrt{a^2-x^2}\,dx$

Putting $x = a\sin\theta$
$dx = a\cos\theta d\theta$

$$\therefore\ I = \int\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta)\,d\theta$$

or $$I = \int a\sqrt{1-\sin^2\theta}\cdot a\cos\theta d\theta$$

$$I = \int a\sqrt{\cos^2\theta}\cdot a\cdot\cos\theta d\theta$$

$$= a^2\int\sin\theta\cos\theta\cdot d\theta \qquad \ldots(1)$$

$$= \frac{a^2}{2}\int 2\sin\theta\cos\theta\,d\theta$$

$$= \frac{a^2}{2}\int\sin 2\theta\ d\theta$$

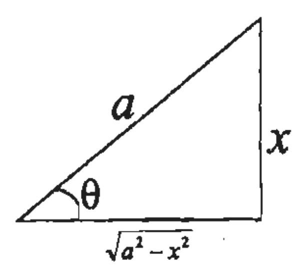
$$= a^2\int\cos^2\theta\ d\theta$$

$$= a^2\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int\cos 2\theta d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{\sin 2\theta}{2}$$

$$= \frac{a^2}{4}\theta\cdot 2\sin\cos\theta + \frac{a^2}{2}\theta$$



$$= \frac{a^2}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$\boldsymbol{I = \frac{x}{2} \cdot \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}}$$

(4) $\int \frac{1}{\sqrt{x^2 + a^2}} dx$

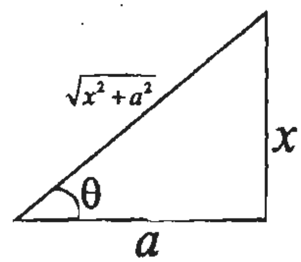
Let $I = \int \frac{1}{\sqrt{x^2 + a^2}} dx$

Putting $x = a \tan\theta$

$dx = a \sec^2 \theta d\theta$

$$= \int \frac{a \sec^2 \theta d\theta}{\sqrt{a^2 \tan^2 \theta + a^2}}$$

$$= \int \frac{a \sec^2 \theta d\theta}{a\sqrt{\tan^2 \theta + 1}}$$



$$= \int \frac{\sec^2 \theta d\theta}{\sec\theta}$$

$$= \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right) + C_1$$

$$= \log\left(x + \sqrt{x^2 + a^2}\right) + C$$

(Where $C = C_1 - \log a$)

(5) $\int \frac{1}{\sqrt{x^2 - a^2}} dx$

Let $I = \int \frac{1}{\sqrt{x^2 - a^2}} dx$

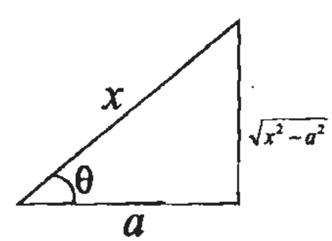
Putting $x = a \tan\theta$

$dx = a \sec\theta \tan\theta d\theta$

$$= \int \frac{a \sec\theta \tan\theta d\theta}{\sqrt{a^2 \sec^2 \theta - a^2}}$$

$$= \int \frac{a \sec\theta \tan\theta d\theta}{a\sqrt{\sec^2 \theta - 1}}$$

$$= \int \frac{\sec\theta \tan\theta d\theta}{\tan\theta}$$



$$= \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C_1$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C \qquad (\text{Where } C = C_1 - \log a)$$

(6) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$

Let $I = \frac{1}{\sqrt{a^2 - x^2}}\,dx$ *Putting* $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

$$= \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta\,d\theta}{a\sqrt{1 - \sin^2\theta}}$$

$$= \int \frac{\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$= \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

Note : On applying these standard formulae we can obtain some more formulae which are useful and can be applied to evaluate other integrals.

(7) To find the integral of $\int \sqrt{ax^2 + bx + c}$

We have $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)\right]$$

Here two cases are shown below :

Case 1 : If $4ac - b^2 > 0$

Then $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)\right]$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\sqrt{\frac{4ac - b^2}{4a^2}}\right)^2\right]$$

Case 2 : If $b^2 > 4ac$

Then $$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{b^2 - 4ac}{4a^2}\right)\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2\right]$$

By substituting $x + \dfrac{b}{2a} = t$, so that $dx = dt$, then the above integral is reduced to the forms – $a\int\sqrt{t^2 + k^2}\,dt$ Where $k^2 = \dfrac{4ac - b^2}{4a^2}$

and $a\int\sqrt{t^2 + A^2}\,dt$ Where $A^2 = \dfrac{b^2 - 4ac}{4a^2}$

These integrals can be obtained by the formulae directly mentioned earlier in this section.

(8) $\displaystyle\int\frac{1}{\sqrt{ax^2 + bx + c}}dx$

From the example (7) the given integral can be written as :

$$\int\frac{1}{a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)}dx \qquad \text{if } 4ac > b^2$$

and $$\int\frac{1}{a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{b^2 - 4ac}{4a^2}\right)}dx \qquad \text{if } b^2 > 4ac$$

On putting $x + \dfrac{b}{2a} = t$ and $dx = dt$

then these integrals can be reduced to the form

$$\frac{1}{a}\int\frac{dt}{\sqrt{t^2 + k^2}} \qquad \text{Where } k^2 = \frac{4ac - b^2}{4a^2}$$

and $$\frac{1}{a}\int\frac{dt}{\sqrt{t^2 - A^2}} \qquad \text{Where } A^2 = \frac{b^2 - 4ac}{4a^2}$$

These integrals can be obtained by the formulae mentioned earlier in this section.

SOLVED EXAMPLES

(1) $\displaystyle\int\sqrt{5 + 4x + x^2}\,dx$

Let $\quad I = \int \sqrt{5+4x+x^2}\, dx$

$$= \int \sqrt{4+4x+x^2+1}\;\; dx$$

$$= \int \sqrt{(x+2)^2+1}\;\; dx$$

$$= \frac{(x+2)}{2}\sqrt{(x+2)^2+1} + \frac{1}{2}\log\left|x+2+\sqrt{(x+2)^2+1}\right| + C$$

$$= \frac{(x+2)}{2}\sqrt{x^2+4x+5} + \frac{1}{2}\log\left|x+2+\sqrt{x^2+4x+5}\right| + C$$

(2) $\int \sqrt{2+4x+x^2}\, dx$

Let $\quad I = \int \sqrt{2+4x+x^2}\, dx$

$$= \int \sqrt{4+4x+x^2-2}\;\; dx$$

$$= \int \sqrt{(x+2)^2-\left(\sqrt{2}\right)^2}\, dx$$

$$= \frac{(x+2)}{2}\sqrt{(x+2)^2-\left(\sqrt{2}\right)^2} - \frac{2}{2}\log\left|x+2+\sqrt{(x+2)^2-\left(\sqrt{2}\right)^2}\right| + C$$

$$= \frac{(x+2)^2}{2}\sqrt{2+4x+x^2} - \log\left|(x+2)+\sqrt{x^2+4x+2}\right| + C$$

(3) $\int \frac{dx}{\sqrt{3+2x+x^2}}$

Let $\quad I = \int \frac{dx}{\sqrt{x^2+2x+3}}$

$$= \int \frac{dx}{\sqrt{x^2+2x+1+2}}$$

$$= \int \frac{dx}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}$$

$$= \log\left((x+1)+\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}\right) + C$$

$$= \log\left(x+1+\sqrt{x^2+2x+3}\right) + C$$

(4) $\int \frac{dx}{\sqrt{x^2+4x+1}}$

Let $$I=\int\frac{dx}{\sqrt{x^2+4x+4-3}}$$

$$=\int\frac{dx}{\sqrt{(x+2)^2+(\sqrt{3})^2}}$$

$$=\log\left|x+2+\sqrt{(x+2)^2+(\sqrt{3})^2}\right|+C$$

$$=\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$$

(5) $\int\sqrt{2x-x^2}\,dx$

Let $$I=\int\sqrt{2x-x^2}\,dx$$

$$=\int\sqrt{1-1+2x-x^2}\,dx$$

$$=\int\sqrt{1-\left(x^2-2x+1\right)}\,dx$$

$$=\int\sqrt{1-(x-1)^2}\,dx$$

$$=\left(\frac{x-1}{2}\right)\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}\frac{x-1}{1}+C$$

$$=\left(\frac{x-1}{2}\right)\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+C$$

(6) $\int\frac{1}{\sqrt{4x-x^2}}\,dx$

Let $$I=\int\frac{1}{\sqrt{4x-x^2}}\,dx$$

$$=\int\frac{1}{\sqrt{4-4+4x-x^2}}\,dx$$

$$=\int\frac{1}{\sqrt{4-\left(x^2-4x+4\right)}}\,dx$$

$$=\int\frac{1}{\sqrt{4-(x-2)^2}}\,dx$$

$$=\sin^{-1}\frac{x-2}{2}+C$$

3.7.1 INTEGRATION OF $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Let $\quad I=\int (px+q)\sqrt{ax^2+bx+c}\,dx$

We choose constant A and B, such that,

$$(px+q)=A\frac{d}{dx}\left(ax^2+bx+c\right)+B$$
$$=A(2ax+b)+B$$

Comparing the coefficient of x and the constant term on both sides, we get –

$$2aA=p \qquad and \qquad Ab+B=q$$

Solving these we get the values of A and B.

Thus, the given integral to takes the form

$$I=A\int(2ax+b)\sqrt{ax^2+bx+c}\,dx+B\sqrt{ax^2+bx+c}\,dx$$
$$=AI_1+BI_2$$

Now, $\quad I_1=\int(2ax+b)\sqrt{ax^2+bx+c}\;\;dx$

Putting $ax^2+bx+c=t,\;\;(2ax+b)\,dx=dt$

$$\therefore\;\; I_1=\frac{2}{3}\int\left(ax^2+bx+c\right)^{3/2}+C_1$$

Similarly, $\quad I_2=\int\sqrt{ax^2+bx+c}\;\;dx$ can be found.

SOLVED EXAMPLE

Example 1 : $\int x\sqrt{1+x-x^2}\;dx$

Let, $\quad x=A\left[\frac{d}{dx}\left(1+x-x^2\right)\right]+B$

$$x=A(1-2x)+B$$

Equating the coefficient of x and constant term on both sides, we get

$$-2A=1, \qquad A+B=0$$

On solving these we get, $A=-\frac{1}{2}$ *and* $B=\frac{1}{2}$ thus the given integral takes the form

$$I=\int x\sqrt{1+x-x^2}\,dx=-\frac{1}{2}\int(1-2x)\sqrt{1+x-x^2}\,dx+\frac{1}{2}\sqrt{1+x-x^2}\,dx$$

$$=-\frac{1}{2}I_1+\frac{1}{2}I_2 \qquad \ldots(1)$$

Now, $I_1 = \int (1-2x)\sqrt{1+x-x^2}\,dx$ Putting $1+x-x^2 = t$

$(1-2x)dx = dt$

$$\therefore\ I_1 = \int (t)^{1/2}\,dt$$

or $$I_1 = \frac{t^{3/2}}{3/2} + C_1$$

$$I_1 = \frac{3}{2}\left(1+x-x^2\right)^{3/2} + C_1$$

and $$I_2 = \int \sqrt{1+x-x^2}\,dx = \int \sqrt{1+2\times\frac{x}{2}+\frac{1}{4}-\frac{1}{4}-x^2}$$

$$= \int \sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}\ dx$$ Putting $x-\frac{1}{2}=t$

so that $dx = dt$

$$= \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\ \ dt$$

$$= \frac{1}{2}t\sqrt{\frac{5}{4}-t^2} + \frac{1}{2}\cdot\frac{5}{4}\sin^{-1}\frac{2t}{\sqrt{5}} + C_2$$

$$= \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\frac{2\left(x-\frac{1}{2}\right)}{\sqrt{5}} + C_2$$

$$= \frac{1}{4}(2x-1)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\frac{(2x-1)}{\sqrt{5}} + C_2$$

Putting these values in (1), we get –

$$\int x\left(1+x-x^2\right)\,dx = \frac{1}{3}\left(1+x-x^2\right)^{3/2} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2} + \frac{5}{16}\sin^{-1}\frac{(2x-1)}{\sqrt{5}} + C$$

Where $C = \dfrac{C_1+C_2}{2}$

3.7.2 INTEGRATION OF $\int \frac{px+q}{ax^2+bx+c}$

Let $$I = \int \frac{px+q}{ax^2+bx+c}\,dx$$

We choose constants A and B such that

$$px+q=A\frac{d}{dx}\left(ax^2+bx+c\right)+B$$

$$px+q=A(2ax+b)+B$$

Comparing the coefficient of x and constant terms on both sides, we get

$$p=2Aa, \qquad Ab+B=q$$

On solving these equation we get,

$$A=\frac{p}{2a}, \quad and \quad B=\left(q+\frac{pb}{2a}\right)$$

Putting these values the given integral reduces to –

$$I=\frac{p}{2a}\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}+\int\frac{q-bp/2a}{\sqrt{ax^2+bx+c}}\,dx$$

$$I=\frac{p}{2a}\cdot\sqrt{ax^2+bx+c}+q-\frac{bp}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

The integral of second term can be easily found by the methods discussed earlier in this section.

SOLVED EXAMPLE

Example 1 : $\int\frac{x+1}{\sqrt{x^2-x+1}}\,dx$

Let, $I=\int\frac{x+1}{\sqrt{x^2-x+1}}\,dx$

Choosing constant A and B such that,

$$x+1=A\frac{d}{dx}\left(x^2-x+1\right)+B$$

$$x+1=A(2x-1)+B$$

Comparing the coefficient of x and constant terms on both sides, we get

$$2A=1, \qquad -A+B=1$$

Solving these we get $A=\frac{1}{2}, \quad B=\frac{3}{2}$

$\therefore$ the given integral takes the form

$$I=\frac{1}{2}\int\frac{2x-1}{\sqrt{x^2-x+1}}\,dx+\frac{3}{2}\int\frac{dx}{\sqrt{x^2-x+1}}$$

$$=\sqrt{x^2-x+1}+\frac{3}{2}I_1 \qquad \text{...(1)}$$

Now for, $I_1=\int\frac{dx}{\sqrt{x^2-x+1}}$

$$= \int \frac{dx}{\sqrt{x^2 - 2\times\frac{1}{2}\times x + \frac{1}{4} - \frac{1}{4} + 1}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \log\left[\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \log\left[(2x-1) + \sqrt{x^2 - x + 1}\right] + C_1 - \log 2$$

Putting these values of I_1 in (1) we get,

$$\therefore\ I = \frac{1}{2}\sqrt{x^2 - x + 1} + \frac{3}{2}\log\left[(2x-1) + \sqrt{x^2 - x + 1}\right] + \frac{3}{2}C_1 - \frac{3}{2}\log 2$$

$$= \frac{1}{2}\sqrt{x^2 - x + 1} + \frac{3}{2}\log\left[(2x-1) + \sqrt{x^2 - x + 1}\right] + C$$

Where $C = \frac{3}{2}(C_1 - \log)$

3.7.3 INTEGRATION OF $\frac{\left(a_0x^n + a_1x^{n-1} + \ldots + a_n\right)}{\sqrt{ax^2 + bx + C}}dx$

Let $\quad I = \int \frac{\left(a_0x^n + a_1x^{n-1} + \ldots + a_n\right)}{\sqrt{ax^2 + bx + c}}dx$

$$= \left(C_0x^{n-1} + C_1x^{n-2} + \ldots + C_{n-1}\right)\sqrt{ax^2 + bx + c} + C_n\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

When $C_0, C_1 \ldots C_n$ are constants to be determined,

Differentiating both sides and multiplying by $\sqrt{ax^2 + bx + c}$, we get

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} \ldots + a_n = \left[(n-1)C_0x^{n-2} + (n-2)C_1x^{n-3} + \ldots + C_{n-2}\right]$$

$$\left(ax^2 + bx + c\right) + \left(C_0x^{n-1} + C_1x^{n-2} + \ldots C_{n-1}\right)\left(\frac{ax + b}{2}\right) + C_n$$

Equating the coefficient of like powers of x, and constant terms, we can easily find the values of constants $C_0, C_1 \ldots C_n$ and thus the integral can be evaluated as earlier.

SOLVED EXAMPLE

Example 1: $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

Let $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = (C_0x+C_1)\sqrt{x^2+x+1} + C_2\int \frac{dx}{\sqrt{x^2+x+1}}$ ...(1)

Differentiating both sides with respect to x, we get–

$$\frac{x^2+2x+3}{\sqrt{x^2+x+1}} = C_0\sqrt{x^2+x+1} + (C_0x+C_1)\frac{1}{2}\frac{(2x+1)}{\sqrt{x^2+x+1}} + \frac{C_2}{\sqrt{x^2+x+1}}$$

Multiplying above equation by $\sqrt{x^2+x+1}$, we get

$$x^2+2x+3 = C_0\left(x^2+x+1\right) + \frac{1}{2}(C_0x+C_1)(2x-1) + C_2$$

$$= C_0x^2 + C_0x + C_0 + C_0x^2 + \frac{1}{2}C_0x + \frac{1}{2}C_1x + \frac{1}{2}C_1 + C_2$$

$$= 2C_0x^2 + \frac{3}{2}C_0x + C_1x + C_0 + \frac{1}{2}C_1 + C_2$$

$$x^2+2x+3 = 2C_0x^2 + \left(\frac{3}{2}C_0 + C_1\right)x + C_0 + \frac{1}{2}C_1 + C_2$$

Equating the coefficient of like powers of x and constants terms on both sides we get,

$$2C_0 = 1,\quad \frac{3}{2}C_0 + C_1 = 2,\quad C_0 + \frac{1}{2}C_1 + C_2 = 3$$

On solving these equations we get

$$C_0 = \frac{1}{2},\quad C_1 = \frac{5}{4},\quad C_2 = \frac{15}{8}$$

Putting the values of C_0, C_1 and C_2 in (1) we get,

$$\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \left(\frac{1}{2}x+\frac{5}{4}\right)\sqrt{x^2+x+1} + \frac{15}{8}\int \frac{dx}{\sqrt{x^2+x+1}} \quad ...(1)$$

$$= \frac{1}{4}(2x+5)\sqrt{x^2+x+1} + \frac{15}{8}I_1 \quad ...(2)$$

Now,

$$I_1 = \int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{dx}{\sqrt{x^2+2\times\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}}$$

$$= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2}}$$

$$= \log\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} + C_1$$

$$= \log\left\{(2x+1) + \sqrt{x^2+x+1}\right\} + C_2 \qquad C_2 = C_1 - \log 2$$

Putting this in (2) we get,

$$\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \frac{1}{4}(2x+5)\sqrt{x^2+x+1}$$

$$+\frac{15}{8}\log\left|(2x+1)+\sqrt{x^2+x+1}\right| + \frac{15}{8}C_2$$

$$= \frac{1}{4}(2x+5)\sqrt{x^2+x+1} + \frac{15}{8}\log\left|(2x+1)+\sqrt{x^2+x+1}\right| + C,$$

Where $C = \frac{15}{8}C_2$

3.7.4 INTEGRATION OF $\frac{1}{(px+q)^r\sqrt{ax^2+bx+c}}$, $a \neq 0,\ p \neq 0$

Let $I = \int \frac{dx}{(px+q)^r\sqrt{ax^2+bx+c}}$

Putting $px+q = \frac{1}{z}$, $px+q>0$, so that $pdx = -\frac{1}{z^2}dz$ and $x = \left(\frac{1}{z}-q\right)p$

$$\therefore\ I = -\frac{1}{p}\cdot\int \frac{dz}{z^2\cdot z^{-r}\sqrt{a\left(\frac{1}{z}-q\right)^2\Big/p^2 + b\left(\frac{1}{2}-q\right)\Big/p + c}}$$

Which after simplification takes the form

$$I = -\int \frac{z^{r-1}\,dz}{Az^2+Bz+C}$$ which can be eaily evaluated.

SOLVED EXAMPLE

Example 1: $\int \frac{1}{(x-1)^2\sqrt{x^2+1}}dx$

Let $I = \int \frac{dx}{(x-1)^2\sqrt{x^2+1}}$ ...(1)

Putting $x-1 = \frac{1}{z}$, so that $dx = -\frac{1}{z^2}dz$ and $x = \left(\frac{1}{z}+1\right) = \left(\frac{1+z}{z}\right)$

$$\therefore\ I = -\int \frac{1}{\left(\frac{1}{z}\right)^2 \cdot \sqrt{\left(\frac{1+z}{z}\right)^2 + 1}} \left(\frac{dz}{z^2}\right)$$

$$= -\int \frac{z\ dz}{\sqrt{2z^2+2z+1}}$$

Choosing constant A and B, such that

$$-z = A\frac{d}{dz}\left(2z^2+2z+1\right) + B$$

or $$-z = A(4z+2) + B$$

$$-z = 4Az + 2A + B$$

Comparing the coefficient of z and the constant terms on both sides, we get

$$4A = -1, \qquad 2A + B = 0$$

On solving these equation, we get $A = -\frac{1}{4}$ and $B = -\frac{1}{2}$

$$\therefore -\int \frac{z\ dz}{\sqrt{2z^2+2z+1}} = -\frac{1}{4}\int \frac{(4z+2)dz}{\sqrt{2z^2+2z+1}} - \frac{1}{2}\int \frac{dz}{\sqrt{2z^2+2z+1}}$$

$$= -\frac{2}{4}\cdot\sqrt{2z^2+2z+1} - \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{z^2+z+\frac{1}{2}}}$$

$$= -\frac{1}{2}\cdot\sqrt{2z^2+2z+1} + \frac{1}{2\sqrt{2}}I_1 \qquad \ldots(1)$$

Now, $$I_1 = \int \frac{dz}{\sqrt{z^2+z+\frac{1}{2}}}$$

$$= \int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)^2 + \frac{1}{4}}}$$

$$= \log\left|\left(z+\frac{1}{2}\right) + \sqrt{\left(z+\frac{1}{2}\right)^2 + \frac{1}{4}}\right| + C_1$$

$$= \log\left|\left(\frac{2z+1}{2}\right) + \sqrt{\left(z^2+z+\frac{1}{2}\right)}\right| + C_1$$

$$= \log\left|\frac{2z+1+2\sqrt{\left(z^2+z+\frac{1}{2}\right)}}{2}\right| + C_1$$

$$= \log\left|(2z+1)+2\sqrt{\left(z^2+z+\frac{1}{2}\right)}\right| + C_2$$

Where $C_2 = (C_1 - \log 2)$

Putting these value of I_1 in (2), we get

$$-\int \frac{z\ dz}{\sqrt{2z^2+2z+1}} = -\frac{1}{2}\cdot\sqrt{2z^2+2z+1}$$

$$+\frac{1}{2\sqrt{2}}\log\left|(2z+1)+2\sqrt{\left(z^2+z+\frac{1}{2}\right)}\right| + \frac{1}{2\sqrt{2}}\times C_2$$

or

$$-\int \frac{z\ dz}{\sqrt{2z^2+2z+1}} = -\frac{1}{2}\cdot\sqrt{2z^2+2z+1}$$

$$+\frac{1}{2\sqrt{2}}\log\left|(2z+1)+2\sqrt{\left(z^2+z+\frac{1}{2}\right)}\right| + C$$

Where $C = \frac{1}{2\sqrt{2}}C_2$

Replacing z by $\frac{1}{(x-1)}$; we get

$$\int \frac{dx}{(x-1)^2\sqrt{x^2+1}} = -\frac{1}{2}\sqrt{2\left(\frac{1}{x-1}\right)^2+2\left(\frac{1}{x-1}\right)+1}$$

$$+\frac{1}{2\sqrt{2}}\log\left|2\left(\frac{1}{x-1}+1\right)+2\sqrt{\left(\frac{1}{x-1}\right)^2+\frac{1}{x-1}+\frac{1}{2}}\right| + C$$

$$= -\frac{1}{2}\frac{\sqrt{2+2(x-1)+(x-1)^2}}{(x-1)}$$

$$+\frac{1}{2\sqrt{2}}\log\left|\frac{2+2(x-1)}{(x-1)}+2\frac{\sqrt{1+(x-1)+\frac{1}{2}(x-1)^2}}{(x-1)}\right| + C$$

$$= -\frac{1}{2}\frac{\sqrt{x^2+1}}{(x-1)} + \frac{1}{2\sqrt{2}}\log\left|2x+\sqrt{2}\sqrt{x^2+1}\right| + \frac{1}{2\sqrt{2}}\log\left|(x-1)\right| + C$$

$$\therefore \int \frac{dx}{(x-1)^2\sqrt{x^2+1}} = -\frac{1}{2}\frac{\sqrt{x^2+1}}{(x-1)} + \frac{1}{2\sqrt{2}}\log\left|2x+\sqrt{2}\sqrt{x^2+1}\right| - \frac{1}{2\sqrt{2}}\log\left|(x-1)\right| + C$$

3.7.5 INTEGRATION OF $\int \frac{1}{(ax^2+b)\sqrt{cx^2+e}}$

Let $$I = \int \frac{dx}{(ax^2+b)\sqrt{cx^2+e}}$$ Putting $x = \frac{1}{t}$,

So that $dx = -\frac{1}{t^2}dt$

$$I = \int \frac{(-1/t^2)\,dt}{(a/t^2+b)\sqrt{(c/t^2+e)}}$$

$$= -\int \frac{t\,dt}{(a+bt^2)\sqrt{c+et^2}}$$

Putting $c + et^2 = z^2$,

$\therefore\ e\,t\,dt = 2z\,dz$

$\therefore\ t\,dt = \frac{z}{e}dz$

$$I = -\frac{1}{e}\int \frac{z\,dz}{\left[a+\left(\frac{z^2-c}{e}\right)\right]\cdot z}$$

$$= -\frac{1}{e}\int \frac{e}{ae-bc+bz^2}dz$$

$$= -\frac{1}{e}\int \frac{dz}{bz^2-ae+bc}dz$$

$$= \int \frac{dz}{z^2+\left(\frac{ae-bc}{b}\right)}$$

$$= -\frac{1}{b}\int \frac{dz}{z^2+k^2}$$ Where $k^2 = \left(\frac{ae-bc}{b}\right)$

$$= -\frac{1}{b}\times\frac{1}{k}\tan^{-1}\frac{z}{k}+C$$ If $ae > bc$

$$= -\frac{1}{b}\times\frac{1}{2k}\log\frac{z-k}{z+k}+C$$ If $ae < bc$

Now replacing z by t and t by x, we get the required integral.

Example 1.: $\int \frac{1}{(x^2-1)\sqrt{x^2-1}}\,dx$

Let $I = \int \frac{dx}{(x^2-1)\sqrt{x^2-1}}$

Putting $x = \frac{1}{t}$

$dx = -\frac{1}{t^2}\,dt$

$$\therefore\ I = -\int \frac{(1/t^2)\,dt}{(1/t^2-1)\sqrt{1/t^2+1}}$$

$$= -\int \frac{t\,dt}{(1-t^2)\sqrt{1+t^2}}$$

Again putting $1+t^2 = z^2$

$2t\,dt = 2z\,dz$

$\therefore\ t\,dt = z\,dz$

$$= -\int \frac{\not{z}\,dz}{1-(z^2-1)\times\not{z}}$$

$$= -\int \frac{dz}{1-z^2+1}$$

$$= \int \frac{dz}{z^2-2}$$

$$= \frac{1}{2\sqrt{2}}\log\frac{z-\sqrt{2}}{z+\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}}\log\frac{\sqrt{1+t^2}-\sqrt{2}}{\sqrt{1+t^2}+\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}}\log\frac{\sqrt{1+\frac{1}{x^2}}-\sqrt{2}}{\sqrt{1+\frac{1}{x^2}}+\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}}\log\frac{\sqrt{x^2+1}-\sqrt{2}\,x}{\sqrt{x^2+1}+\sqrt{2}\,x} + C$$

$$= \frac{\sqrt{2}}{4}\log\frac{\sqrt{x^2+1}-\sqrt{2}x}{\sqrt{x^2+1}+\sqrt{2}x}+C$$

3.7.6 INTEGRATION OF $x^m(a+bx^n)^p$

Where m, n and p are not necessarily integers.

Let $\qquad x^m\left(a+bx^n\right)^p dx$

Case I : If p is a possitive integer then applying binomial theorem for $\left(a+bx^n\right)^p$ and this integration can be found.

Case II : If $(m+n)/n$ is an integer and $p = r/s$. In this case we take $a+bx^n = t^s$, so

$$I = \frac{s}{bn}\int\left(\frac{t^s-a}{b}\right)^k \cdot t^{ps+s-1}dt \qquad \text{Where } k = \frac{m}{n}$$

Now expanding $\left(\frac{t^s-a}{b}\right)^k$ by binomial theorem, the integral can be obtained.

Case III : If $p+(m+1)/n$ is an integer and p is not an integer. In this case we put $x = 1/t$, then the integral is reduces to the case II and consequently the required integral is obtained.

SOLVED EXAMPLES

Example 1 : $\int x^{\frac{1}{2}}\left(1+x^{\frac{1}{4}}\right)^4 dx$

Let $\qquad I = \int x^{\frac{1}{2}}\left(1+x^{\frac{1}{4}}\right)^4 dx$

Expanding $\left(1+x^{1/4}\right)^4$ by binomial theorem, we get –

$$I = \int x^{\frac{1}{2}}\left(x+4x^{\frac{3}{4}}+6x^{\frac{1}{2}}+4x^{\frac{1}{4}}+1\right)dx$$

$$= \int\left(x^{\frac{3}{2}}+4x^{\frac{5}{4}}+6x+4x^{\frac{3}{4}}+x^{\frac{1}{2}}\right)dx$$

$$= \frac{x^{5/2}}{5/2} + \frac{4 \cdot x^{9/4}}{9/4} + 6 \cdot \frac{x^2}{2} + 4 \cdot \frac{x^{7/4}}{7/4} + \frac{x^{3/2}}{3/2} + C$$

$$= \frac{2}{5}x^{5/2} + \frac{16}{9}x^{9/4} + 3 \cdot x^2 + \frac{16}{7} \cdot x^{7/4} + \frac{2}{3}x^{3/2} + C$$

Example 2 : $\int x^7 \left(1 + x^8\right)^{2/3} dx$

Let $I = \int x^7 \left(1 + x^8\right)^{2/3} dx$

Hence, $m = 7$, $n = 8$, $p = 2/3$ and $\frac{m+1}{n} = \frac{7+1}{8} = 2$, which is an integer.

$\therefore$ Putting $1 + x^8 = u^3$

$$8x^7 dx = 3u^7 du$$

$$\therefore\ I = \frac{3}{8}\int u^2 \left(u^3\right)^{2/3} du$$

$$= \frac{3}{8}\int u^4 \cdot du$$

$$= \frac{3}{8} \cdot \frac{u^5}{5} + C$$

$$= \frac{3}{40} \cdot u^5 + C$$

$$\therefore\ I = \frac{3}{40}\left(1 + x^8\right)^{5/3} + C$$

$$\therefore\ \int x^7 \left(1 + x^8\right)^{2/3} dx = \frac{3}{40}\left(1 + x^8\right)^{5/3} + C$$

Example 3 : $\int x^{-6} \left(1 + x^2\right)^{1/2} dx$

Let $I = \int x^{-6} \left(1 + x^2\right)^{1/2} dx$

Here, $m = -6$, $n = 2$, $p = 1/2$ and $\frac{m+1}{2} + p = \frac{-6+1}{2} + \frac{1}{2} = -2$, which is an integer then

$\therefore$ Putting $x = -1/t$ so that $dx = \left(-1/t^2\right)dt$

$$\therefore\ I = \int t^6 \cdot \left(1 + 1/t^2\right)^{1/2} \left(-1/t^2\right) dt$$

$$= -\int t^6 \cdot \frac{(t^2+1)^{1/2}}{t \cdot t^2} \cdot dt$$

$$= -\int t^3 \cdot (t^2+1)^{1/2} \cdot dt$$

$$= -\int t^2 \cdot t(t^2+1)^{1/2} \cdot dt$$

Putting $t^2 + 1 = z^2$

$2t\,dt = 2\,z dz$

$$\therefore\ I = -\int (z^2-1)\,(z^2)^{1/2} \cdot z\,dz$$

$$= -\int (z^2-1)\,(z^2) \cdot dz$$

$$= \int (z^2 - z^4) \cdot dz$$

$$= \frac{z^3}{3} - \frac{z^5}{5} + C$$

$$= \frac{(t^2+1)^{3/2}}{3} - \frac{(t^2+1)^{5/2}}{5} + C$$

$$= \frac{1}{3}\frac{\left(\frac{1}{x^2}+1\right)^{3/2}}{3} - \frac{\left(\frac{1}{x^2}+1\right)^{5/2}}{5} + C$$

$$= \frac{1}{3}\frac{\left(\frac{1}{x^2}+1\right)^{3/2}}{x^3} - \frac{1}{5}\frac{\left(\frac{1}{x^2}+1\right)^{5/2}}{x^5} + C$$

3.7.7 INTEGRATION OF $f\left\{x,(ax+b)^{1/m},(ax+b)^{1/n}\right\}$

To integrate a function of this type, we make the substitute $ax + b = t^p$, where p is the lowest common multiple of m and n. In this way the integrand is reduced to a rational function.

Example 1 : $\displaystyle\int \frac{x}{(1+x)^{1/2} - (1+x)^{1/3}}\,dx$

Let $\displaystyle I = \int \frac{x\,dx}{(1+x)^{1/2} - (1+x)^{1/3}}$ $\quad\therefore\ l.c.m.\ of\ 2\ and\ 3 = 6$

Putting $1+x=t^6$

$dx = 6t^5 dt$

$$I=\int\frac{\left(t^6-1\right)\times 6t^5 dt}{\left(t^3-t^2\right)}$$

$$=6\int\frac{\left(t^6-1\right)t^3}{(t-1)}\,dt$$

$$=\int\left(t^8+t^7+t^6+t^5+t^4+t^3\right)dt$$

$$=\frac{t^9}{9}+\frac{t^8}{8}+\frac{t^7}{7}+\frac{t^6}{6}+\frac{t^5}{5}+\frac{t^4}{4}+C$$

$$=\frac{1}{9}(1+x)^{3/2}+\frac{1}{8}(1+x)^{4/3}+\frac{1}{7}(1+x)^{7/6}+\frac{1}{6}(1+x)+\frac{1}{5}(1+x)^{5/6}$$

$$+\frac{1}{4}(1+x)^{3/2}+C$$

3.7.8 INTEGRATION BY RATIONALISATION

Integration of $\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Let $\quad I=\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Multiplying the numerator and denominator by $\sqrt{x+a}-\sqrt{x+b}$, we have

$$I=\int\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}+\sqrt{x+b}}\,dx$$

$$=\int\frac{\sqrt{x+a}-\sqrt{x+b}}{(a-b)}\,dx$$

$$=\frac{1}{(a-b)}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left(\frac{(x+a)^{3/2}}{3/2}-\frac{(x+b)^{3/2}}{3/2}\right)+C$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+C$$

Example 1: $\int \frac{1\ dx}{\sqrt{x}-\sqrt{1+x}}$

Let $I = \frac{1}{\sqrt{x}-\sqrt{1+x}}dx$

Multiplying the numerator and denominator by $\left(\sqrt{x}+\sqrt{1+x}\right)$, we get

$$I = \int \frac{\sqrt{x}+\sqrt{1+x}}{x-(1+x)}dx$$

$$= \int \sqrt{x}+\sqrt{1+x}\,dx$$

$$= \left(\frac{x^{3/2}}{3/2} - \frac{(1+x)^{3/2}}{3/2}\right) + C$$

$$= -\frac{2}{3}\left[x^{3/2}+(1+x)^{3/2}\right]+C$$

EXERCISE 7.2

Integrate the following functions :

Q. 1.: $\sqrt{4-3x-2x^2}$ **Q. 2.:** $\sqrt{x^2-2x-5}$ **Q. 3.:** $\sqrt{9-x^2}$

Q. 4.: $\sqrt{1-4x^2}$ **Q. 5.:** $\sqrt{x^2+3x}$ **Q. 6.:** $x\sqrt{1+x-x^2}$

Q. 7.: $(x+3)\sqrt{3-4x-x^2}$ **Q. 8.:** $\frac{1}{\sqrt{3-4x-2x^2}}$ **Q. 9.:** $\frac{1}{\sqrt{2+x-3x^2}}$

Q. 10.: $\frac{1}{\sqrt{(x-\alpha)(x-\beta)}}$ **Q. 11.:** $\frac{2x+5}{\sqrt{x^2+3x+1}}$ **Q. 12.:** $\frac{1+x}{\sqrt{x^2-x+1}}$

Q. 13.: $\frac{2x^2+3}{\sqrt{3-2x-x^2}}$ **Q. 14.:** $\frac{1}{(x-1)\sqrt{(1-x^2)}}$ **Q. 15.:** $\frac{1}{(1+x^2)\sqrt{(x^2-1)}}$

Q. 16.: $x^2(1+x^3)^{1/3}$ **Q. 17.:** $x(1+x^3)^{1/3}$ **Q. 18.:** $\frac{1}{(x-3)\sqrt{(x+1)}}$

Q. 19.: $\frac{x}{(1+x)^{1/3}-(1+x)^{1/2}}$ **Q. 20.:** $x\sqrt{\frac{(1+x)}{(1-x)}}$

ANSWERS

(1) $\frac{1}{8}(4x+3)\cdot\left(4-3x-2x^2\right)+\frac{41\sqrt{2}}{92}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+C$

(2) $\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

(3) $\frac{1}{2}x\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}+C$

(4) $\frac{1}{4}\sin^{-1}2x+\frac{1}{2}x\sqrt{1-4x^2}+C$

(5) $\frac{2x+3}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right|+C$

(6) $-\frac{1}{3}\left(1+x-x^2\right)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\frac{(2x-1)}{\sqrt{5}}+C$

(7) $-\frac{1}{3}\left(3-4x-x^2\right)^{1/2}+\frac{7}{2}\sin^{-1}\frac{(x+2)}{\sqrt{7}}+(x+2)\frac{\sqrt{3-4x-x^2}}{2}+C$

(8) $-\frac{1}{\sqrt{2}}\sin^{-1}\frac{(x-1)\sqrt{2}}{5}+C$

(9) $-\frac{1}{\sqrt{3}}\sin^{-1}\frac{(6x-1)}{5}+C$ (10) $2\log\left|\sqrt{x-\alpha}+\sqrt{x-\beta}\right|+C$

(11) $2\sqrt{x^2+3x+1}+2\log\left|(2x+3)-\sqrt{(2x+3)^2-5}\right|+C$

(12) $\sqrt{x^2-x+1}+\frac{3}{2}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x+1}\right|+C$

(13) $(3-x)\sqrt{3-2x-x^2}+\sin^{-1}\left(\frac{x+1}{2}\right)+C$ (14) $3\left(\frac{1-x}{1+x}\right)^{3/2}+C$

(15) $\frac{1}{2\sqrt{2}}\log\frac{x\sqrt{2}+\sqrt{x^2-1}}{x\sqrt{2}-\sqrt{x^2-1}}+C$ (16) $\frac{1}{4}\left(1+x^3\right)^{4/3}+C$

(17) $\frac{1}{3\left(t^3-1\right)}+\frac{1}{9}\log\frac{\sqrt{t^2+t+1}}{t-1}+\frac{\sqrt{3}}{4}\tan^{-1}\frac{2t-1}{\sqrt{3}}+C,$ Where $t=\frac{\left(1+x^3\right)^{1/3}}{x}$

(18) $\frac{2}{3}\left\{(1+x)^{3/2}+x^{3/2}\right\}+C$

(19) $-\left[\frac{2}{3}(1+x)^{3/2}+\frac{3}{4}(1+x)^{4/3}+\frac{6}{7}(1+x)^{7/6}+\frac{6}{5}(1+x)^{5/6}+\frac{3}{2}(1+x)^{2/3}\right]+C$

(20) $-\sqrt{\left(1-x^2\right)}-\frac{1}{2}x\sqrt{\left(1-x^2\right)}+\frac{1}{2}\sin^{-1}x+C$

3.8 INTEGRATION OF TRANSCENDENTAL FUNCTION

The functions other than algebraic are called transcendental function. These include, trigonometric, functions defined in special ways, we will study the integration of these functions one by one.

3.8.1 INTEGRATION OF TRIGONOMETRIC FUNCTION

(a) $\int \frac{1}{a+b\cos x}dx$ $\quad (a \neq \pm b)$

Let $I = \int \frac{dx}{a+b\cos x}$

$$= \int \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2 \; dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

Here, two case arise

Case I : If $a > b$, then putting $\sqrt{a-b}\tan\frac{x}{2} = u$

$$\frac{1}{2}\sqrt{a-b}\sec^2\frac{x}{2}dx = du$$

So that the given integral takes the form,

$$I = \frac{1}{\sqrt{a-b}}\int \frac{2du}{(a+b)+u^2}$$

$$= \frac{2}{\sqrt{a-b}} \times \frac{1}{a+b}\tan^{-1}\left(\frac{u}{\sqrt{a+b}}\right)$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

Case II : If $a < b$, then we have

$$I = \frac{1}{\sqrt{a-b}} \int \frac{\sec^2 x/2 \, dx}{\tan^2 x/2 - \left(\frac{b+a}{b-a}\right)^2}$$

Putting $\tan\frac{x}{2} = u$

$\therefore \frac{1}{2}\sec^2\frac{x}{2}dx = du$

$$= -\frac{2}{(b-a)} \times \frac{1}{2\sqrt{\frac{b+a}{b-a}}} \log\left[\frac{u - \sqrt{\frac{b+a}{b-a}}}{u + \sqrt{\frac{b+a}{b-a}}}\right]$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log\left[\frac{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}\right]$$

(b) $\int \frac{1}{a + b\sin x} dx \qquad (a \neq \pm b)$

Let $I = \int \frac{dx}{a + b\sin x}$

$$= \int \frac{dx}{a + b\left(\frac{2\tan x/2}{1 + \tan^2\frac{x}{2}}\right)}$$

$\therefore \sin 2x = \frac{2\tan x}{1 + \tan^2 x}$

$$= \int \frac{\left(1 + \tan^2 x/2\right)dx}{a\left(1 + \tan^2 x/2\right) + 2b\tan x/2}$$

$$= \int \frac{\sec^2 x/2 \, dx}{a + a\tan^2 x/2 + 2b\tan x/2}$$

Putting $\tan x/2 = 4$

$\therefore \frac{1}{2}\sec^2 x/2 \, dx = du$

$\sec^2 x/2 \, dx = 2du$

$$= \int \frac{2du}{a + a u^2 + 2bu}$$

$$= \frac{2}{a}\int \frac{du}{u^2 + 2\frac{b}{a}u + 1}$$

$$= \frac{2}{a}\int \frac{du}{u^2 + 2\times\frac{b}{a}u + \frac{b^2}{a^2} - \frac{b^2}{a^2} + 1}$$

$$= \frac{2}{a}\int \frac{du}{\left(u + \frac{b}{a}\right) + \left(\frac{a^2 - b^2}{a^2}\right)}$$

Here, two case arise

Case I : If $a < b$, then putting $a^2 - b^2 < 0$ then we have,

$$I = \frac{2}{a}\int \frac{du}{\left(u + \frac{b}{a}\right)^2 - \left(\frac{b^2 - a^2}{a^2}\right)}$$

$$= \frac{2}{a} \times \frac{1}{2\sqrt{\frac{b^2 - a^2}{a^2}}} \log \frac{\left(u + b/a\right) - \sqrt{\frac{b^2 - a^2}{a^2}}}{\left(u + b/a\right) + \sqrt{\frac{b^2 - a^2}{a^2}}}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\left(\tan x/2 + b/a\right) - \frac{\sqrt{b^2 - a^2}}{a}}{\left(\tan x/2 + b/a\right) + \frac{\sqrt{b^2 - a^2}}{a}}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\left(a\tan x/2 + b\right) - \sqrt{b^2 - a^2}}{\left(a\tan x/2 + b/a\right) + \sqrt{b^2 - a^2}}$$

Case II : If $a > b$, then $a - b > 0$, we have

$$I = \frac{2}{a}\int \frac{du}{\left(u + \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}}$$

or

$$I = \frac{2}{a}\int \frac{du}{\left(u + \frac{b}{a}\right)^2 + \left(\frac{a^2 - b^2}{a}\right)^2}$$

$$= \frac{2}{a} \times \frac{1}{\sqrt{\frac{a^2 - b^2}{a^2}}} \tan^{-1} \frac{u + b/a}{\sqrt{\frac{a^2 - b^2}{a^2}}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{au + b/a}{\sqrt{\frac{a^2 - b^2}{a^2}}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\frac{a \tan^{-1} x/2 + b}{\sqrt{a^2 - b^2}} \right)$$

SOLVED EXAMPLES

(1) $\int \frac{dx}{5 + 4\cos x}$

Let $I = \int \frac{dx}{5 + 4\cos x}$

$$= \int \frac{dx}{5 + 4\left(\frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}\right)} \qquad \because \cos 2x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$$

$$= \int \frac{\left(1 + \tan^2 x/2\right) dx}{5\left(1 + \tan^2 x/2\right) + 4\left(1 - \tan^2 x/2\right)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{9 + \tan^2 x/2}$$

Putting $\tan x/2 = u$

$$\therefore \frac{1}{2} \sec^2 x/2 \, dx = du$$

$$= \frac{2\,du}{9 + u^2} = 2 \times \frac{1}{3} \tan^{-1} \frac{4}{3} + C$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{\tan x/2}{3} \right) + C$$

(2) $\int \frac{dx}{4 + 5\sin x}$

Let $I = \int \frac{dx}{4 + 5\sin x}$

$$= \int \frac{dx}{5 + 4\left(\frac{2\tan x/2}{1 + \tan^2 x/2}\right)} \qquad \because \sin 2x = \frac{2\tan x}{1 + \tan^2 x/2}$$

$$= \int \frac{\left(1 + \tan^2 x/2\right) dx}{4\left(1 + \tan^2 x/2\right) + 10\tan x/2}$$

$$= \int \frac{\sec^2 x/2 \, dx}{4 + 4\tan^2 x/2 + 10\tan x/2}$$

$$= \frac{1}{4}\int \frac{\sec^2 x/2 \, dx}{\tan^2 x/2 + 10/4 \tan x/2 + 1}$$

Putting $\tan x/2 = t$

$$\therefore \frac{1}{2}\sec^2 x/2 \, dx = dt$$

$$= \frac{1}{4}\int \frac{2\,dt}{t^2 + 10/4\, t + 1}$$

$$= \frac{1}{2}\int \frac{dt}{t^2 + 2 \times 5/4\, t + 1}$$

$$= \frac{1}{2}\int \frac{dt}{t^2 + 2 \times 5/4\, t + \frac{25}{16} - \frac{25}{16} + 1}$$

$$= \frac{1}{2}\int \frac{dt}{\left(t + 5/4\right)^2 - \frac{9}{16}}$$

$$= \frac{1}{2} \times \frac{1}{2 \times 3/4} \log\left|\frac{t + 5/4 - 3/4}{t + 5/4 - 3/4}\right| + C_1$$

$$= \frac{1}{3}\log\left[\frac{4t + 5 - 3}{4t + 5 + 3}\right] + C_1 - \frac{1}{3}\log 4$$

$$= \frac{1}{3}\log\left[\frac{4\tan x/2 + 2}{4\tan x/2 + 8}\right] + C, \qquad \text{Where } C = C_1 + \frac{1}{3}\log 4$$

(3) $\int \frac{1}{4\cos x - 1} dx$

Let $I = \int \frac{dx}{4\cos x - 1}$

$$= \int \frac{dx}{5 + 4\left(\frac{1 - \tan^x/_2}{1 + \tan^2 x/_2}\right) - 1}$$

$$= \int \frac{\left(1 + \tan^2 x/_2\right) dx}{4\left(1 - \tan^2 x/_2\right) - \left(1 - \tan^2 x/_2\right)}$$

$$= \int \frac{\sec^2 x/_2 \, dx}{3 - 5\tan^2 x/_2}$$

$$= \frac{1}{5}\int \frac{\sec^2 x/_2 \, dx}{3/_5 - \tan^2 x/_2}$$ Putting $\tan x/_2 = u$,

$\therefore \frac{1}{2}\sec^2 x/_2 \, dx = du$

$$= \frac{1}{5} \times \frac{1}{2\sqrt{3/_5}} \log\left|\frac{\sqrt{3/_5} + u}{\sqrt{3/_5} - u}\right| + u$$

$$= \frac{1}{5} \times \frac{\sqrt{5}}{2} \log\left\{\frac{\sqrt{3/_5} + u}{\sqrt{3/_5} - u}\right\} + C_1$$

$$= \frac{1}{\sqrt{15}} \log\left\{\frac{\sqrt{3} + \sqrt{5}\,u}{\sqrt{3} - \sqrt{5}\,u}\right\} + C_1 - \frac{1}{\sqrt{15}\log}\sqrt{5}$$

$$= \frac{1}{\sqrt{15}} \log\left\{\frac{\sqrt{3} + \sqrt{5}\tan x/_2}{\sqrt{3} - \sqrt{5}\tan x/_2}\right\} + C,$$ Where $C = C_1 + \frac{1}{\sqrt{15}}\log\sqrt{5}$

(4) $\int \frac{1}{1 - 2\sin x} dx$

Let $I = \int \frac{dx}{1 - 2\sin x}$ $\because \sin 2x = \frac{2\tan x/_2}{1 + \tan^2 x/_2}$

$$= \int \frac{dx}{1 - 2\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) - 1}$$

$$= \int \frac{\sec^2 x/2 \, dx}{\tan^2 x/2 - 4\tan x/2 + 1}$$

$$= \int \frac{\sec^2 x/2 \, dx}{\tan^2 x/2 - 2\times 2\tan x/2 + 4 - 4 + 1}$$

$$= \int \frac{\sec^2 x/2 \, dx}{\left(\tan x/2 - 2\right)^2 - 3}$$

Putting $\tan x/2 = u$,

$$\therefore \frac{1}{2}\sec^2 x/2 \, dx = du$$

$$= \int \frac{2\,du}{(u-2)^2 - 3}$$

$$= 2 \times \frac{1}{2\sqrt{3}} \log \frac{(u-2) - \sqrt{3}}{(u-2) + \sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \log \frac{\left(\tan x/2 - 2\right) - \sqrt{3}}{\tan x/2 - 2 + \sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \log \frac{\tan x/2 - \left(2+\sqrt{3}\right)}{\tan x/2 - \left(2-\sqrt{3}\right)} + C$$

(5) $\int \sqrt{\tan x}\, dx$

Putting $\tan x = t^2$

Let $I = \int \sqrt{\tan x}\, dx$

$$\therefore \sec^2 x dx = 2t dt$$

$$dx = \frac{2t dt}{\sec^2 x}$$

$$= \frac{2t dt}{1+\tan^2 x}$$

$$= \frac{2t dt}{1+t^4}$$

$$= \int t \cdot \frac{2\,t dt}{1+t^4}$$

$$= \int \frac{2t^2}{1+t^4}\, dt$$

$$= \int \frac{(t^2+1)+(t^2-1)}{t^4+1}\, dt$$

$$= \int \frac{t^2+1}{t^4+1}\, dt + \int \frac{t^2-1}{t^4+1}\, dt$$

$$= \int \frac{1+1/t^2}{t^2+1/t^2}\, dt + \int \frac{1-1/t^2}{t^2+1/t^2}\, dt$$

$$= I_1 + I_2$$

Consider $$I_1 = \int \frac{\left(1-1/t^2\right)}{\left(t^2+1/t^2\right)}\, dt$$

$$= \int \frac{\left(1-1/t^2\right)dt}{t^2+1/t^2+2-2}$$

$$= \int \frac{\left(1-1/t^2\right)dt}{\left(t+1/t\right)^2+\left(\sqrt{2}\right)^2}$$

Putting $t + 1/t = u$,

$$\left(1-\frac{1}{t^2}\right)dt = du$$

$$= \int \frac{du}{u^2-\left(\sqrt{2}\right)}$$

$$= \frac{1}{2\sqrt{2}} \log\left\{\frac{u-\sqrt{2}}{u+\sqrt{2}}\right\} + C_1$$

$$= \frac{1}{2\sqrt{2}} \log\left[\frac{t+1/t-\sqrt{2}}{t+1/t+\sqrt{2}}\right] + C_1$$

$$= \frac{1}{2\sqrt{2}} \log\left[\frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}}\right] + C_1$$

$$= \frac{1}{2\sqrt{2}} \log\left[\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right] + C_1$$

Now consider $\qquad I_2 = \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{t^2 + \frac{1}{t^2}}$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{t^2 + \frac{1}{t^2} - 2 + 2}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2}$$

Putting $t - \frac{1}{t} = v$,

$$\left(1 + \frac{1}{t^2}\right) dt = dv$$

$$= \int \frac{dv}{v^2 + \left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{v}{\sqrt{2}} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t^2 - 1}{\sqrt{2}\, t} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\tan x - 1}{\sqrt{2\tan x}}\right] + C_2$$

$$\therefore \int \sqrt{\tan x} = \frac{1}{2\sqrt{2}} \log\left[\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right] \frac{1}{2} \tan + \frac{1}{2} \tan^{-1}\left[\frac{\tan x - 1}{\sqrt{2\tan x}}\right] + C$$

Where $\qquad C = C_1 + C_2$

(c) $\int \frac{1}{a\cos x + b\sin x}\,dx$

Let $I = \int \frac{dx}{a\cos x + b\sin x}$ $\quad \because \sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

and $\cos x = \frac{1-\sin^2 x/2}{1+\tan^2 x/2}$

$$I = \int \frac{dx}{a\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + b\frac{2\tan x/2}{\left(1+\tan^2 x/2\right)}}$$

$$= \int \frac{\left(1+\tan^2 x/2\right)dx}{a - a\tan^2 x/2 + 2b\tan x/2}$$

Putting $\tan x/2 = t,$

$\frac{1}{2}\sec^2 x/2\,dx = dt$

$$= \int \frac{\sec^2 x/2\,dx}{a - a\tan^2 x/2 + 2b\tan x/2}$$

$$= \int \frac{2\,dt}{a - at^2 + 2bt}$$

$$= \frac{2}{a}\int \frac{dt}{1-\left(t^2 - \frac{2b}{a}t + \frac{b^2}{a^2} - \frac{b^2}{a^2}\right)}$$

$$= \frac{2}{a}\int \frac{dt}{1-\left(t-\frac{b}{a}\right)^2 + \frac{b^2}{a^2}}$$

$$= \frac{2}{a}\int \frac{dt}{\left(\frac{a^2+b^2}{a^2}\right)^2 - \left(t-\frac{b}{a}\right)^2}$$

$$= \frac{2}{a}\times 2\,\frac{1}{\sqrt{\frac{a^2+b^2}{a^2}}}\log\left[\frac{\frac{\sqrt{a^2+b^2}}{a} + \left(t - b/a\right)}{\frac{\sqrt{a^2+b^2}}{a} - \left(t - b/a\right)}\right]$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log\left[\frac{\sqrt{a^2+b^2}+(at-b)}{\sqrt{a^2+b^2}-(at-b)}\right]$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log\left[\frac{\sqrt{a^2+b^2}+\left(a\tan\frac{x}{2}-b\right)}{\sqrt{a^2+b^2}-\left(a\tan\frac{x}{2}-b\right)}\right]$$

Example 6 : $\int\frac{dx}{3\cos x+4\sin x}$

Let $I=\int\frac{dx}{3\cos x+4\sin x}$

$$I=\int\frac{dx}{3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{3-3\tan^2\frac{x}{2}+8\tan\frac{x}{2}}$$

$$=\frac{1}{3}\int\frac{\left(\sec^2\frac{x}{2}\right)dx}{1-\tan^2\frac{x}{2}+2\times\frac{4}{3}\tan\frac{x}{2}}$$

Putting $\tan\frac{x}{2}=t$,

$\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

$$=\frac{2}{3}\int\frac{dt}{1-t^2+2\times\frac{4}{3}t}$$

$$=\frac{2}{3}\int\frac{dt}{1-\left(t^2-2\times\frac{4}{3}t+\frac{16}{9}-\frac{16}{9}\right)}$$

$$=\frac{2}{3}\int\frac{dt}{1+\frac{16}{9}-\left(t-\frac{4}{3}\right)^2}$$

$$=\frac{2}{3}\int\frac{dt}{\frac{25}{9}-\left(t-\frac{4}{3}\right)^2}$$

$$= \frac{2}{3} \times \frac{3}{2 \times 5} \log\left[\frac{5/3 + \left(t - 4/3\right)}{5/3 - \left(t - 4/3\right)}\right] + C_1$$

$$= \frac{1}{5} \log\left[\frac{1 + 3\tan x/2}{9 - 3\tan x/2}\right] - \frac{1}{5}\log 3 + C_1$$

$$= \frac{1}{5} \log\left[\frac{3\tan x/2 + 1}{3 - \tan x/2}\right] + C \qquad C = C_1 - \frac{1}{5}\log 3$$

(d) $\int \frac{1}{a\cos x + b\sin x + c}$

Let $I = \int \frac{1}{a\cos x + b\sin x + c}$

$$= \int \frac{dx}{a\left(\frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}\right) + b\left(\frac{2\tan x/2}{1 + \tan^2 x/2}\right) + c}$$

$$= \int \frac{\left(1 + \tan^2 x/2\right) dx}{a - a\tan^2 x/2 + 2b\tan^2 x/2 + C\left(1 + \tan^2 x/2\right)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{(a + c) + (c + a)\sec^2 x/2 + 2b\tan x/2}$$

$$= \frac{1}{c - a} \int \frac{\sec^2 x/2 \, dx}{\frac{c + a}{c - a} + \tan^2 x/2 + \frac{2b}{c - a}\tan x/2}$$

Here two cases arise –

Case I : If $c > a$, then

$$I = \frac{1}{c - a} \int \frac{\sec^2 x/2 \, dx}{\frac{c + a}{c - a} + \left(\tan x/2 + \frac{b}{c - a}\right)^2 - \frac{b^2}{(c - a)^2}}$$

$$= \frac{1}{c - a} \int \frac{\sec^2 x/2 \, dx}{\frac{c^2 - a^2 - b^2}{(c - a)^2} + \left(\tan x/2 + \frac{b}{c - a}\right)^2}$$

Putting $\tan \frac{x}{2} + \frac{b}{c-a} = t,$

$\frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt$

$$= \frac{1}{c-a}\int \frac{2\,dt}{\frac{c^2-a^2-b^2}{(c-a)^2} + t^2}$$

$$= \frac{2}{(c-a)} \times \frac{(c-a)}{\sqrt{c^2-a^2-b^2}} \tan^{-1} \frac{t(c-a)}{\sqrt{c^2-a^2-b^2}}$$

$$= \frac{2}{\sqrt{c^2-a^2-b^2}} \tan^{-1} \frac{(c-a)\left(\tan \frac{x}{2} + \frac{b}{c-a}\right)}{\sqrt{c^2-a^2-b^2}}$$

$$= \frac{2}{\sqrt{c^2-a^2-b^2}} \tan^{-1}\left[\frac{(c-a)\left(\tan \frac{x}{2} + b\right)}{\sqrt{c^2-a^2-b^2}}\right]$$

Case II : If $c < a$, then

$$I = \frac{1}{a-c}\int \frac{\sec^2 \frac{x}{2}\, dx}{\frac{a+c}{a-c} - \tan^2 \frac{x}{2} - \frac{2b}{a-c}\tan \frac{x}{2}}$$

Putting $\tan \frac{x}{2} + \frac{b}{a-c} = t,$

$\frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt$

$$= \frac{1}{(a-c)}\int \frac{\sec^2 \frac{x}{2}\, dx}{\left(\frac{a+c}{a-c}\right) - \left\{\tan^2 \frac{x}{2} + 2 \times \frac{b}{a-c}\tan \frac{x}{2} + \frac{b^2}{(a-c)^2}\right\} + \frac{b^2}{(a-c)^2}}$$

$$= \frac{1}{(a-c)}\int \frac{\sec^2 \frac{x}{2}\, dx}{\left(\frac{a+c}{a-c}\right) + \frac{b^2}{(a-c)^2} - \left(\tan \frac{x}{2} + \frac{b}{a-c}\right)^2}$$

$$= \frac{1}{(a-c)}\int \frac{\sec^2 \frac{x}{2}\, dx}{\frac{a^2+b^2-c^2}{(a-c)^2} - \left(\tan \frac{x}{2} + \frac{b}{a-c}\right)^2}$$

$$= \frac{1}{a-c}\int \frac{2\,dt}{\frac{a^2+b^2-c^2}{(c-a)^2} - t^2}$$

$$= \frac{1}{(a-c)} \times \frac{(a-c)}{2\sqrt{a^2+b^2-c^2}} \log\left[\frac{\frac{\sqrt{a^2+b^2-c^2}}{a-c}+t}{\frac{\sqrt{a^2+b^2-c^2}}{a-c}-t}\right]$$

$$= \frac{1}{2\sqrt{a^2+b^2-c^2}} \log\left|\frac{\sqrt{a^2+b^2-c^2}+(a-c)t}{\sqrt{a^2+b^2-c^2}-(a-c)t}\right|$$

$$-\frac{1}{2\sqrt{a^2+b^2+c^2}} \log(a-c)+C_1$$

$$= \frac{1}{2\sqrt{a^2+b^2-c^2}} \log\left|\frac{\sqrt{a^2+b^2-c^2}+\left\{(a-c)\tan x/2+b\right\}}{\sqrt{a^2+b^2-c^2}-\left\{(a-c)\tan x/2+b\right\}}\right|+C$$

$$\text{Where } C = -\frac{1}{2\sqrt{a^2+b^2+c^2}} \log(a-c)+C_1$$

Remark : The above integral may also be evaluated by putting $a = r\cos\alpha$, $b = r\sin\alpha$, so that $r^2 = a^2 + b^2$.

Example 7 : $\int \frac{dx}{\cos x + \sin x + 3}$

Let $I = \int \frac{dx}{\cos x + \sin x + 3}$

$$= \int \frac{dx}{\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)+\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)+3}$$

$$= \int \frac{\left(1+\tan^2 x/2\right)dx}{2\left(1-\tan^2 x/2\right)+2\tan x/2+3+3\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2\,dx}{\tan^2 x/2+2\tan x/2+5}$$

$$= \int \frac{\sec^2 x/2\,dx}{\left(\tan x/2+1\right)^2+4}$$

Putting $\tan x/2 + 1 = t,$

$$\frac{1}{2}\sec^2 x/_2\, dx = dt$$

$$= \int \frac{2\,dt}{t^2+4} + C$$

$$= 2 \times \frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$= \tan^{-1}\frac{\left(\tan x/_2 + 1\right)}{2}$$

(e) $\int \frac{a + b\cos x + c\sin x}{l + m\cos x + n\sin x}dx$

Let $I = \int \frac{a + b\cos x + c\sin x}{l + m\cos x + n\sin x}dx$

Let us choose the real numbers, A, B and C, such that

$$a + b\cos x + c\sin x = A\left(l + m\cos x + n\sin x\right) + B\frac{d}{dx}\left(l + m\cos x + n\sin x\right)$$

$$a + b\cos x + c\sin x = A\left(l + m\cos x + n\sin x\right) + B\frac{d}{dx}\left(l + m\cos x + n\sin x\right) + C$$

$$a + b\cos x + c\sin x = A\left(l + m\cos x + n\sin x\right) + B\left(-m\sin x + n\cos x\right) + C$$

$$a + b\cos x + c\sin x = \left(Am + Bn\right)\cos x + \left(An - Bm\right)\sin x + C + Al$$

Equating the coefficient of cos x, sin x and constant terms on both sides, we get

$$Am + Bn = b, \quad An - Bm = c, \quad C + Al = a$$

On solving these equation we get the value of A, B and C, as

$$A = \frac{bm + cn}{m^2 + n^2}, \qquad B = \frac{bn - cm}{m^2 + n^2}, \qquad C = \frac{a\left(m^2 + n^2\right) - l\left(bm + cn\right)}{m^2 + n^2}$$

Now, the given integral can be written as

$$\int \frac{a + b\cos x + c\sin x}{l + m\cos x + n\sin x}dx = \left(\frac{bm + cn}{m^2 + n^2}\right)\int dx + \left(\frac{bn - cm}{m^2 + n^2}\right)\int \frac{\left(-m\sin x + n\cos x\right)dx}{l + m\cos x + n\sin x}$$

$$+ \left\{\frac{a\left(m^2 + n^2\right) - l\left(bm + cn\right)}{m^2 + n^2}\right\}\int \frac{dx}{l + m\cos x + n\sin x}$$

$$= \left(\frac{bm + cn}{m^2 + n^2}\right)x + \log\left(l + m\cos x + n\sin x\right)$$

$$+ \left\{\frac{a\left(m^2 + n^2\right) - l\left(bm + cn\right)}{m^2 + n^2}\right\}\int \frac{dx}{l + m\cos x + n\sin x}$$

The integral in the third term can be evaluated by the method given in (d).

Example 8 : $\int \frac{2+4\cos x+3\sin x}{1+3\sin x+2\cos x}dx$

Let $I=\int \frac{2+4\cos x+3\sin x}{1+3\sin x+2\cos x}dx$

Let us choose the real numbers, A, B and C, such that

$$2+4\cos x+3\sin x=A(1+3\sin x+2\cos x)+B\frac{d}{dx}(1+3\sin x+2\cos x)+C$$

$$2+4\cos x+3\sin x=A(1+3\sin x+2\cos x)+B(3\cos x-2\sin x)+C$$

$$2+4\cos x+3\sin x=(3A-2B)\sin x+(2A+3B)\cos x+A+C$$

Equating the coefficient of cos x, sin x and the constant terms on both sides we get,

$$3A+3B=4,\qquad 3A-2B=3,\quad \text{and}\quad A+C=2$$

On solving these equation we get,

$$A=\frac{17}{13},\qquad B=\frac{6}{13}\qquad \text{and}\qquad C=\frac{9}{13}$$

Thus the given integral can be written as –

$$I=\int \frac{2+4\cos x+3\sin x}{1+3\sin x+2\cos x}dx$$

$$=\frac{17}{13}\int(1+3\sin x+2\cos x)dx+\frac{6}{13}\int\frac{(3\cos x-2\sin x)dx}{1+3\sin x+2\cos x}+\frac{9}{13}\int\frac{dx}{1+3\sin x+2\cos x}$$

$$I=\frac{17}{13}-\frac{51}{13}\cos x+\frac{34}{13}\sin x+\frac{6}{13}\log(1+3\sin x+2\cos x)+\frac{9}{13}\times I_1 \qquad \ldots(1)$$

Where $I_1=\int\frac{dx}{1+3\sin x+2\cos x}$

$$=\int\frac{dx}{1+3\times\frac{2\tan x/2}{1+\tan^2 x/2}+2\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$$

$$=\int\frac{(1+\tan^2 x/2)dx}{1+\tan^2 x/2+6\tan x/2+2-2\tan^2 x/2}$$

$$=\int\frac{\sec^2 x/2\,dx}{3-\tan^2 x/2+6\tan x/2}$$

$$=\frac{1}{3}\int\frac{\sec^2 x/_2\,dx}{1-\left(\tan^2 x/_2-2\tan x/_2+1-1\right)}$$

$$=\frac{1}{3}\int\frac{\sec^2 x/_2\,dx}{2-\left(\tan x/_2-1\right)^2}$$

Putting $\tan x/_2-1=t,$

$$\frac{1}{2}\sec^2 x/_2\,dx=dt$$

$$I=\frac{1}{3}\int\frac{2dt}{2-t^2}$$

$$=\frac{2}{3}\times\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|$$

$$=\frac{1}{3\sqrt{2}}\log\left|\frac{\sqrt{2}+\left(\tan x/_2-1\right)}{\sqrt{2}-\left(\tan x/_2-1\right)}\right|$$

$$=\frac{1}{3\sqrt{2}}\log\left[\frac{\sqrt{2}-1+\tan x/_2}{\sqrt{2}+1-\tan x/_2}\right]$$

Putting this value of I_1, in (1) we get–

$$I=\frac{17}{13}x-\frac{51}{13}\cos x+\frac{34}{13}\sin x+\frac{6}{13}\log(1+3\sin x+2\cos x)$$

$$+\frac{9}{13}\times\frac{1}{3\sqrt{2}}\log\left[\frac{\sqrt{2}-1+\tan x/_2}{\sqrt{2}+1-\tan x/_2}\right]$$

or $$I=\frac{1}{13}\Bigg[17x-51x+34\sin x+\frac{6}{13}\log(1+3\sin x+2\cos x)$$

$$+\frac{3}{\sqrt{2}}\log\left|\frac{\sqrt{2}-1+\tan x/_2}{\sqrt{2}+1-\tan x/_2}\right|\Bigg]+C$$

(f) $$\int\frac{p\cos x+q\sin x}{a\cos x+b\sin x}dx$$

Let $$I=\int\frac{p\cos x+q\sin x}{a\cos x+b\sin x}dx$$

Now, we choose the real numbers, A, and B, such that

$$p\cos x + q\sin x = A(a\cos x + b\sin x) + B\frac{d}{dx}(a\cos x + b\sin x)$$

or $$p\cos x + q\sin x = (Aa + Bb)\cos x + (Ab - Ba)\sin x$$

Equating the coefficient of cos x, sin x and the constant terms on both sides we get,

$$Aa + Bb = p, \quad \text{and} \quad Ab - Ba = q$$

On solving these equation we get,

$$A = \frac{ap + bq}{a^2 + b^2}, \quad \text{and} \quad B = \frac{bp + aq}{a^2 + b^2},$$

Thus the given integral can be written as –

$$I = \int \frac{(p\cos x + q\sin x)}{(a\cos x + b\sin x)}\,dx$$

$$= \frac{(ap + bq)}{(a^2 + b^2)}\int (a\cos x + b\sin x)\,dx + \left(\frac{bp + aq}{a^2 + b^2}\right)\int \frac{(-a\sin x + b\cos x)}{(a\cos x + b\sin x)}\,dx$$

$$= \frac{(ap + bq)}{a^2 + b^2}[-a\sin x + b\cos x] + \frac{(bp + aq)}{a^2 + b^2}\log(a\cos x + b\sin x) + C$$

Example 9 : $\int \frac{3\cos x + 4\sin x}{2\cos x + 3\sin x}\,dx$

Let $I = \int \frac{3\cos x + 4\sin x}{2\cos x + 3\sin x}\,dx$

Choosing the real numbers, A, and B, such that

$$3\cos x + 4\sin x = A(2\cos x + 3\sin x) + B\frac{d}{dx}(2\cos x + 3\sin x)$$

$$3\cos x + 4\sin x = A(2\cos x + 3\sin x) + B(-2\sin x + 3\cos x)$$

or $$3\cos x + 4\sin x = (2A + 3B)\cos x + (3A - 2B)\sin x$$

Equating the coefficient of cos x, and sin x on both sides we get,

$$2A + 3B = 3, \quad \text{and} \quad 3A - 2B = 4$$

On solving these equation we get,

$$A = \frac{18}{13}, \quad \text{and} \quad B = \frac{1}{13}$$

Thus the given integral can be written as –

$$I = \int \frac{3\cos x + 4\sin x}{2\cos x + 3\sin x}\,dx = \frac{18}{13}\int (2\cos x + 3\sin x)\,dx + \frac{1}{13}\int \frac{(-2\sin x + 3\cos x)}{(2\cos x + 3\sin x)}\,dx$$

$$=\frac{18}{13}(-2\sin x+3\cos x)+\frac{1}{13}\log|2\cos x+3\sin x|$$

$$\therefore\ I=\int\frac{3\cos x+4\sin x}{2\cos x+3\sin x}dx=\frac{1}{13}\left[54\cos x-36\sin x+\log|2\cos x+3\sin x|\right]+C$$

Example 10 : $\int\frac{x^2dx}{(x\sin x+\cos x)^2}$

Let $I=\int\frac{x^2dx}{(x\sin x+\cos x)^2}$

$$=\int\frac{x\cos x\,dx}{(x\sin x+\cos x)^2}\cdot\frac{x}{\cos x}dx$$

Taking $\frac{x}{\cos x}$ as first and $\frac{x\cos x}{(x\sin x+\cos x)^2}$ as the second function and integrating by parts, we get

$$I=\frac{x}{\cos x}\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx-\int\left[\frac{d}{dx}\left(\frac{x}{\cos x}\right)\int\frac{x\cos x dx}{(x\sin x+\cos x)^2}\right]dx$$

Let $I_1=\int\frac{x\cos x}{(x\sin x+\cos x)^2}$

Putting $x\sin x+\cos x=t$,

$(x\cos x+\sin x-\sin x)dx=dt$

$x\cos x\,dx=dt$

$$\therefore\ I_1=\int\frac{dt}{t^2}$$

$$=-\frac{1}{t}=\frac{1}{(x\sin x+\cos x)}$$

$$\therefore\ I=\frac{x}{\cos x}\left[-\frac{1}{x\sin x+\cos x}\right]-\int\frac{(\cos x+x\sin x)}{\cos^2x}\left(-\frac{1}{(x\sin x\cos x)}\right)dx$$

$$=\frac{-x}{\cos x(x\sin x+\cos x)}+\tan x$$

$$=\frac{-x}{\cos x(x\sin x+\cos x)}+\frac{\sin x}{\cos x}$$

$$=\frac{1}{\cos x}\left[\frac{-x+x\sin^2x+\sin x\cos x}{(x\sin x+\cos x)}\right]$$

$$= \frac{1}{\cos x}\left[\frac{\sin x\cos x - \left(1-\sin^2 x\right)x}{(x\sin x + \cos x)}\right]$$

$$= \frac{1}{\cos x}\left[\frac{\sin x\cos x - x\cos^2 x}{(x\sin x + \cos x)x\cos^2 x}\right]$$

$$= \frac{\sin x - x\cos x}{x\sin x + \cos x}$$

$$\therefore \int \frac{x^2}{(x\sin x + \cos x)^2}\,dx = \frac{\sin x - x\cos x}{x\sin x + \cos x} + C$$

(g) $$\int \frac{1}{a + b\tan x}\,dx$$

Let $$I = \int \frac{dx}{a + b\tan x}$$

$$= \int \frac{\cos x\,dx}{a\cos x + b\sin x}$$

Now, taking the real numbers A and B such that

$$\cos x = A(a\cos x + b\sin x) + B\frac{d}{dx}(a\cos x + b\sin x)$$

$$= A(a\cos x + b\sin x) + B(b\cos x - a\sin x)$$

$$\cos x = (Aa + Bb)\cos x + (Ab - Ba)\sin x$$

Equating the coefficient of cos x, and sin x on both sides we get

$$Aa + Bb = 1 \quad and \quad Ab - Ba = 0$$

On solving these equation we get,

$$A = \frac{a}{a^2 + b^2} \quad and \quad B = \frac{b}{a^2 + b^2}$$

$$\therefore \frac{\cos x}{a\cos + b\sin x} = \frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}\frac{(b\cos x - a\sin x)}{a\cos x + b\sin x}$$

∴ The given integral takes the form

$$I = \int \frac{\cos x}{a\cos + b\sin x}\,dx = \frac{a}{a^2 + b^2}\int dx + \frac{b}{a^2 + b^2}\int \frac{(b\cos x - a\sin x)}{a\cos x + b\sin x}dx$$

$$= \frac{ax}{a^2 + b^2} + \frac{b}{a^2 + b^2}\log|a\cos x + b\sin x| + C$$

Example 11 : $$\int \frac{dx}{3 + 2\tan x}$$

Let $I = \int \frac{dx}{3+2\tan x}$

$= \int \frac{\cos x dx}{3\cos x + 2\sin x}$

Choosing real number A and B such that,

$$\cos x = A(3\cos x + 2\sin x) + B\frac{d}{dx}(3\cos x + 2\sin x)$$

$$= A(3\cos x + 2\sin x) + B(-3\sin x + 2\cos x)$$

$$\cos x = (3A + 2B)\cos x + (2A - 3B)\sin x$$

Equating the coefficient of cos x and sin x on both sides, we get

$$3A + 2B = 1, \quad 2A - 3B = 0$$

On solving these equation we get,

$$A = \frac{3}{13} \quad \text{and} \quad B = \frac{2}{13}$$

Thus the given integral can be written as –

$$I = \int \frac{\cos x\, dx}{3\cos 2x + 2\sin x} = \frac{3}{13}\int dx + \frac{2}{13}\int \frac{(2\cos x - 3\sin x)}{(3\cos x + 2\sin x)}dx$$

$$\therefore I = \int \frac{\cos x\, dx}{3\cos 2x + 2\sin x} = \frac{3}{13}x + \frac{2}{13}\log|3\cos x + 2\sin x| + C$$

(h) $\int \sin^m x \cos^n dx$

If m and n are positive integers, the integral can be obtained by successive reduction or by expressing $\sin^m x \cos^n dx$ as –

$\sin^m x\cos^n = \sin^m x\cos^{n-1} x\cos^n x$ if m is odd and n is even

or $\sin^m x\cos^n = \sin^m x\cos^{n-1} x\cos x$ if m is even and n is odd

This will be more clear from the following example :

Example 12: $\int \sin^3 x\cos^4 dx$

Let $I = \int \sin^3 x\cos^4 dx$

Here m = 3 (odd) and n = 4 (even)

$$\therefore I = \int \sin x \sin^2 x \cos^4 x dx$$

$$= \int \sin x(1 - \cos^2 x)\cos^4 x dx$$

Putting $\cos x = t$ and $-\sin x = dt$

$$= -\int \left(1-t^2\right)t^4 dt$$

$$= -\int \left(t^4 - t^6\right)dt = \int \left(t^6 - t^4\right)dt$$

$$= \frac{t^7}{7} - \frac{t^5}{5}$$

$$= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

$$\therefore \int \sin^3 x \cos^4 dx = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

Example 13 : $\int \sin^4 x \cos^3 dx$

Let $I = \int \sin^4 x \cos^3 dx$

Here m = 4 (even) and n = 3 (odd)

$$\therefore I = \int \sin^4 x \cdot \cos^2 x \cos x dx$$

$$= \int \sin^4 x \left(1-\sin^2 x\right)\cos x dx$$

Putting sin x = t and cos x dx = dt

$$= \int t^4 \left(1-t^2\right) dt$$

$$= \int \left(t^4 - t^6\right)dt$$

$$= \frac{t^5}{5} - \frac{t^7}{7}$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

$$\therefore \int \sin^4 x \cos^3 x\, dx = \frac{1}{5}\sin^5 x - \frac{1}{7}\cos^7 x + C$$

(i) $\int \sin^m x \cos^n x\, dx$

If m or n is an odd positive integer or (m + n) is an even negative integer then the integral can be obtained more easily by substitution.

Case I : When m or n is an odd positive integer, then putting m = 2 r + 1, where r is an integer and $r \geq 0$ then by substituting cos x = t, we can find the integral for

$$\int \sin^m x \cos^n x\, dx = \int \sin^{2r+1} x \cos^n x\, dx$$

$$= \int \sin x \sin^{2r} x \cdot \cos^n x dx$$

Putting $\cos x = t$

$-\sin x = dt$

$$= \int \sin x\left(1-\cos^2 x\right)^r \cdot \cos^n x dx$$

$$= -\int \left(1-t^2\right)^r \cdot t^n dt$$

Now expanding $\left(1-t^2\right)^r$ by binomial theorem we can easily evaluate it.

Similarly, when n is an odd positive integer, we can evaluate it by following the same procedure as above.

Example 14 : $\int \sin^7 x \cos^2 x dx$

Let $I = \int \sin^7 x \cos^2 x dx$

Here m = 7 (odd) and n = 3 (even) the given integral can be written as

$$I = \int \sin x \sin^6 x \cos^2 x \, dx$$

$$= \int \sin x\left(1-\cos^2 x\right)^3 \cdot \cos^2 x dx$$

Putting cos x = t and – sin x dx = dt

$$= -\int \left(1-t^2\right)^3 t^2 dt$$

$$= \int \left(t^2-1\right)^3 t^2 dt$$

$$= \int \left(t^6 - 3t^4 \cdot t + 3t^2 - 1\right) t^2 dt$$

$$= \int \left(t^8 - 3t^6 + 3t^4 - t^2\right) dt$$

$$= \frac{t^9}{9} - \frac{3t^7}{7} + \frac{1t^5}{5} - \frac{t^3}{3} + C$$

$$= \frac{1}{9}(\cos x)^9 - \frac{3}{7}(\cos x)^7 + \frac{3}{5}(\cos x)^5 - \frac{1}{3}(\cos x)^3 + C$$

$$\therefore \int \sin^7 x \cos x dx = \frac{1}{9}\cos^9 x - \frac{3}{7}\cos^7 x + \frac{3}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

Example 15 : $\int \sin^6 x \cos^3 x dx$

Let $I = \int \sin^6 x \cos^3 x dx$

Here m = 6 (even) and n = 3 (odd)

$\therefore$ the given integral can be written as

$$I = \int \sin^6 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^6 x \left(1 - \sin^2 x\right)^r \cos x dx$$

Putting sin x = t and cos x dx = dt

$$= \int t^6 \left(1 - t^2\right) dt$$

$$= \int \left(t^6 - t^8\right) dt$$

$$= \frac{t^7}{7} - \frac{t^9}{9} + C$$

$$= \frac{(\sin x)^7}{7} - \frac{(\sin x)^9}{9} + C$$

$$\therefore \int \sin^6 x \cos^3 x dx = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$$

Case II : When (m + n) is an even negative integer, where m and n are not necessarily integers.

Let (m + n) = – 2r, where r is a positive integer, we can evaluate the integral of $\sin^m x \cos^n x$ by putting tan x = t.

$$\therefore \int \sin^m x \cos^n x dx = \int \frac{\sin^m x \cos^m x \cos^n x}{\cos^m x} dx$$

$$= \int \tan^m x \cos^{m+n} x \, dx$$

$$= \int \tan^m x \cos^{-2r} x \, dx \qquad \because m + n = -2r$$

$$= \int \tan^m x \sec^{2r} x \, dx$$

$$= \int \tan^m x \sec^{2r-2} x \sec^2 x \, dx$$

$$= \int \tan^m x (1 + \tan^2 x)^{r-1} \sec^2 x \, dx$$

Putting tan x = t and $\sec^2 x \, dx = dt$

$$= \int t^m (1 + t^2)^{r-1} dt$$

The integral can be found by expanding $(1 + t)^{r-1}$ using binomial theorem.

Example 16 : $\displaystyle\int \frac{1}{\sin^{5/2} x \cos^{3/2} x} dx$

Let $I = \int \sin^{-5/2} x \cos^{-3/2} x dx$

$\therefore \quad m + n = -\left(\frac{5}{2} + \frac{3}{2}\right) = -4$ (an even integer)

$\therefore$ the given integral can be written as

$$I = \int \frac{\sin^{-5/2} x}{\cos^{-5/2} x} \cdot \cos^{-3/2} x \cdot \cos^{-5/2} x \, dx$$

$$= \int \frac{\cos^{-4} x dx}{\tan^{5/2} x}$$

$$= \int \frac{\sec^4 x \, dx}{\tan^{5/2} x}$$

$$= \int \frac{\sec^2 x \times \sec^2 x \, dx}{\tan^{5/2} x}$$

Putting $\tan x = t$ and $\sec^2 x \, dx = dt$

$$= \int \frac{\left(1 + t^2\right) dt}{t^{5/2}}$$

$$= \int \left(t^{-5/2} + t^{-1/2}\right) dt$$

$$= \int \frac{t^{-5/2+1}}{-5/2+1} + \frac{t^{-1/2+1}}{-1/2+1} + C$$

$$= -\frac{2}{3} t^{-3/2} + 2t^{1/2} + C$$

$$\therefore \int \frac{1}{\sin^{5/2} x \cos^{3/2} x} dx = 2\sqrt{\tan x} + \frac{2}{3}\left(\sqrt{\cot x}\right)^3 + C$$

MISCELLANEOUS SOLVED EXAMPLES

(1) Evaluate $\int \sin^{3/4} x \cos^3 x \, dx$

Let $I = \int \sin^{3/4} x \cos^3 x \, dx$

$$= \int \sin^{3/4} x \cos^2 x \cos x \, dx$$

$$= \int \sin^{3/4} \left(1 - \sin^2 x\right) \cos x \, dx$$

Putting sin x = t and cos x dx = dt

$$= \int t^{3/4}\left(1-t^2\right)dt$$

$$= \int \left(t^{3/4} - t^{11/4}\right)dt$$

$$= \frac{t^{3/4+1}}{3/4+1} - \frac{t^{11/4+1}}{11/4+1} + C$$

$$= \frac{4}{7}t^{7/4} - \frac{4}{15}t^{15/4} + C$$

$$\therefore \int \sin^{3/4} x \cos^3 x\, dx = \frac{4}{7}\sin^{7/4} x - \frac{4}{15}\sin^{15/4} x + C$$

(2) Evaluate $\int \frac{dx}{1+8\cos^2 x}$

Let $I = \int \frac{dx}{1+8\cos^2 x}$

$$= \int \frac{\sec^2 x\, dx}{\sec^2 x + 8} \quad \text{(dividing Numerator and denominator by} \cos^2 x\text{)}$$

$$= \int \frac{\sec^2 x\, dx}{\tan^2 x + 9}$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$

$$= \int \frac{dt}{t^2+9}$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right) + C$$

(3) Evaluate $\int \frac{dx}{1+3\sin^2 x}$

Let $I = \int \frac{dx}{1+3\sin^2 x}$

$$= \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x} \quad \text{(dividing Numerator and denominator by} \cos^2 x\text{)}$$

$$= \int \frac{\sec^2 x\, dx}{1+4\tan^2 x}$$

$$= \frac{1}{4}\int \frac{\sec^2 x\, dx}{\tan^2 x + 1/4}$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$

$$= \frac{1}{4}\int \frac{dt}{t^2 + \frac{1}{4}}$$

$$= \frac{1}{4} \times \frac{1}{\frac{1}{2}} \tan^{-1} \frac{t}{\frac{1}{2}} + C$$

$$= \frac{2}{4} \tan^{-1}(2\tan x) + C$$

$$= \frac{1}{2} \tan^{-1}(2\tan x) + C$$

(4) Evaluate $\int \frac{\sqrt{\cos 2x}}{\sin x} dx$

Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x\, dx}}{\sin x}$$

$$= \int \sqrt{\cos^2 x - 1}\, dx$$

(By Multiplying numerator and denominator by $\sqrt{\cot^2 x - 1}$)

$$= \int \frac{(\cos ec^2 x)}{\sqrt{\cot^2 x - 1}} dx - 2\int \frac{\cos ec^2 x\, dx}{\cos ec^2 x \sqrt{\cot^2 x - 1}}$$

Putting $\cot x = t$ and $-\cos ec^2 x\, dx = dt$

$$= -\int \frac{dt}{\sqrt{t^2 - 1}} + 2 \frac{dt}{(1 + t^2)\sqrt{(t^2 - 1)}} + C$$

$$= I_1 + I_2$$

Now, $I_1 = -\int \frac{dt}{\sqrt{t^2 - 1}} = -\log\left|t + \sqrt{(t^2 - 1)}\right| + C_1$

$$= \log\left|t - \sqrt{(t^2 - 1)}\right| + C_1 = \log\left|\cos x - \sqrt{\cos^2 x - 1}\right| + C_1$$

And $I_2 = 2\int \frac{dt}{(t^2 + 1)\sqrt{t^2 - 1}}$ Putting $t = \frac{1}{2}$, $dt = -\frac{1}{z^2} dz$

$$= 2\int \frac{-z\, dz}{(1 + z^2)\sqrt{1 - z^2}}$$ Again putting $1 - z^2 = u$, $2z\, dz = 2u\, du$

$$= \int \frac{2u\,du}{(2-u^2)u} = 2\int \frac{du}{2-u^2}$$

$$= 2 \cdot \times \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+u}{\sqrt{2}-u} + C_2$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+\sqrt{1-z^2}}{\sqrt{2}-\sqrt{1-z^2}} + C_2$$

$$I_2 = \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}t+\sqrt{t^2-1}}{\sqrt{2}-\sqrt{t^2-1}} + C_2$$

$$\therefore I_2 = \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}\cot x+\sqrt{\cot^2-1}}{\sqrt{2}\cot x-\sqrt{\cot^2-1}} + C_2$$

$$I = \log\left|\frac{\cos x-\sqrt{\cos 2x}}{\sin x}\right| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}\,\cos x+\sqrt{\cos 2x}}{\sqrt{2}\,\cos x-\sqrt{\cos 2x}}\right| + C, \quad \text{Where } C_1 + C_2 = C$$

(6) Evaluate $\int\left(\sqrt{\tan x}+\sqrt{\cos x}\right)dx$

Let $I = \int\left(\sqrt{\tan x}+\sqrt{\cos x}\right)dx$

$$= \int \frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$

$$= \sqrt{2}\int \frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$

Multiplying numerator and denominator by $\sqrt{2}$

$$= \sqrt{2}\int \frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Putting sin x – cos x = t

$(\cos x+\sin x)dx = dt$

$$= \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1} t + C$$

$$= \sqrt{2}\sin^{-1}(\sin x-\cos x) + C,$$

(7) Evaluate $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Let $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$ Putting x = cos u

d x = –sin udu

$$= \int \tan^{-1}\sqrt{\frac{1-\cos u}{1+\cos u}}(-\sin u)\,du$$

$$= -\int \tan^{-1}\sqrt{\frac{2\sin^2 u/2}{2\cos^2 u/2}}(\sin u)\,du$$

$$= -\int \tan^{-1}\left(\tan u/2\right)\sin u\,du$$

$$= -\int \frac{u}{2}\sin u\,du$$

$$= -\frac{1}{2}\left[u\times(-\cos u)-1\int -\cos u\,du\right]$$

$$= \frac{u}{2}\cos u + \frac{1}{2}\int \cos u\,du$$

$$= \frac{u}{2}\cos u - \frac{1}{2}\sin u + C$$

$$= \frac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + C$$

(8) **Evaluate** $\int \sin^5 x\,dx$

Let $I = \int \sin^5 x\,dx$

$$= \int \sin^4 x \sin x\,dx$$

$$= \int \left(1-\cos^2 x\right)^2 \sin x\,dx$$

Putting cos x = t

–sin x dx = dt

$$= -\int \left(1-t^2\right)^2 dt$$

$$= -\int \left(1+t^4-2t^2\right) dt$$

$$= \int \left(2t^2-t^4-1\right) dt$$

$$=2\frac{t^3}{3}-\frac{t^5}{5}-t+C$$

$$=\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x-\cos x+C$$

(9) Evaluate $\int \frac{\sqrt{\cos(x-\theta)}}{\cos(x+\theta)}dx$

Let $I=\int \frac{\sqrt{\cos(x-\theta)}}{\cos(x+\theta)}dx$

$$=\int \sqrt{\frac{\cos(x-\theta)\cos(x-\theta)}{\cos(x+\theta)\cos(x-\theta)}}dx$$

$$=\int \frac{\cos(x-\theta)}{\sqrt{\cos(x+\theta)\cos(x-\theta)}}dx$$

$$=\int \frac{\cos(x-\theta)}{\cos^2 x-\sin^2\theta}dx$$

$$=\int \frac{\cos x\cos\theta+\sin x\sin\theta}{\sqrt{\cos^2 x-\sin^2\theta}}dx$$

$$=\int \frac{\cos x\cos\theta}{\sqrt{\cos^2 x-\sin^2\theta}}dx+\int \frac{\sin x\sin\theta}{\sqrt{\cos^2 x-\sin^2\theta}}dx$$

$$=I_1+I_2 \qquad \ldots(1)$$

Now, $I_1=\int \frac{\cos x\cos\theta}{\sqrt{(1-\sin^2 x)-(1-\cos^2\theta)}}dx$

$$=\int \frac{\cos x\cos\theta}{\sqrt{1-\sin^2 x-1+\cos^2\theta}}dx$$

$$=\int \frac{\cos x\cos\theta}{\sqrt{\cos^2\theta-\sin^2 x}}dx$$

Putting $\sin x = u$

$\cos x\, dx = du$

$$I_1=\cos\theta\int \frac{du}{\sqrt{\cos^2\theta-u^2}}$$

$$=\cos\theta\cdot\sin^{-1}\frac{u}{\cos\theta}+C_1$$

$$I_1 = \cos\theta \cdot \sin^{-1}\left(\frac{\sin x}{\cos\theta}\right) + C_1$$

and $$I_2 = \int \frac{\sin\theta \sin dx}{\sqrt{\cos^2 x - \sin^2\theta}}$$

Putting $\cos x = v$

$-\sin x\, dx = dv$

$$\therefore\ I_2 = \sin\theta \int \frac{-dv}{\sqrt{v^2 - \sin^2\theta}}$$

$$= -\sin\theta \log\left|v + \sqrt{v^2 - \sin^2\theta}\right| + C_2$$

$$= -\sin\theta \log\left|\cos x + \sqrt{\cos^2 x - \sin^2\theta}\right| + C_2$$

$$\therefore\ I = \cos\theta \cdot \sin^{-1}\frac{\sin x}{\cos\theta} - \sin\theta \log\left|\cos x + \sqrt{\cos^2 x - \sin^2\theta}\right| + C$$

Where $C_1 + C_2 = C$

$$\therefore\ \int \frac{\sqrt{\cos(x-\theta)}}{\cos(x+\theta)} = \cos\theta \cdot \sin^{-1}\frac{\sin x}{\cos\theta} - \sin\theta \log\left|\cos x + \sqrt{\cos^2 x - \sin^2\theta}\right| + C$$

(10) Evaluate $$\int \frac{1}{\left(9\sin^2 x + 4\cos^2 x\right)^2} dx$$

Let $$I = \int \frac{1}{\left(9\sin^2 x + 4\cos^2 x\right)^2} d$$

$$= \int \frac{\sec^4 x\, dx}{\left(9\tan^2 x + 4\right)^2}$$

$$= \int \frac{\left(1 + \tan^2 x\right)\sec^2 x\, dx}{\left(9\tan^2 x + 4\right)^2}$$

Putting $\tan x = t$

$\sec^2 x\, dx = du$

$$= \int \frac{\left(1 + t^2\right) dt}{\left(9t^2 + 4\right)^2}$$

$$= \frac{1}{81}\int \frac{\left(1+t^2\right)dt}{\left(t^2+\frac{4}{9}\right)^2}$$

$$= \frac{1}{81}\int \frac{\left(1+\frac{4}{9}\tan^2 z\right)\sec^2 z\, dz}{\left(\frac{4}{9}\tan^2 z+\frac{4}{9}\right)^2}$$

Putting $t = \frac{2}{3}\tan z$

$dt = \frac{2}{3}\sec^2 z\, dz$

$$= \frac{1}{81}\times\frac{2}{3}\int \frac{\left(1+\frac{4}{9}\tan^2 z\right)\sec^2 z\, dz}{\frac{16}{81}\left(\tan^2 z+1\right)^2}$$

$$= \frac{2}{3\times 16}\int \frac{\left(1+\frac{4}{9}\tan^2 z\right)\sec^2 z\, dz}{\sec^4 z}$$

$$= \frac{1}{24}\int \frac{\left(1+\frac{4}{9}\tan^2 z\right)}{\sec^2 z}$$

$$= \frac{1}{24}\times\frac{1}{9}\int\left(9\cos^2 z+4\sin^2 z\right)dz$$

$$= \frac{1}{216}\left\{\int 9\left(\frac{1+\cos 2z}{2}\right)+4\left(\frac{1-\cos 2z}{2}\right)\right\}dz$$

$$= \frac{1}{216}\times\frac{1}{2}\int\{9+9\cos 2z+4-4\cos 2z\}\,dz$$

$$= \frac{1}{432}\int(13+5\cos 2z)\,dz$$

$$= \frac{1}{432}\left[13z+\frac{5\sin 2z}{2}\right]+C$$

$$= \frac{1}{432}\left[13\tan^{-1}+\left(\frac{3}{2}t\right)+\frac{5}{2}\times 2\sin z\cos z\right]+C$$

$$= \frac{1}{432}\left[13\tan^{-1}\left(\frac{3t}{2}\right)+5\frac{3t}{\sqrt{4+9t^2}}\times\frac{2}{\sqrt{4+9t^2}}\right]+C$$

$$= \frac{1}{432}\left[13\tan^{-1}\left(\frac{3t}{2}\right)+5\left\{\frac{6t}{\sqrt{4+9t^2}}\right\}\right]+C$$

$$=\frac{1}{432}\left[13\tan^{-1}\left(\frac{3\tan x}{2}\right)+5\left\{\frac{6t}{4+9\tan^2 x}\right\}\right]+C$$

$$=\frac{1}{432}\left[13\tan^{-1}\left(\frac{3\tan x}{2}\right)+\frac{30\tan x}{4+9\tan^2 x}\right]+C$$

3.8.2 INTEGRATION OF OTHER TRANSCENDENTAL FUNCTIONS (EXPONENTIAL AND LOGARITHMIC)

(11) Evaluate $\int \frac{e^{2x}}{\sqrt{a^2+e^{4x}}}dx$

Let $I=\int \frac{e^{2x}}{\sqrt{a^2+e^{4x}}}dx$

Putting $e^{2x}=t$

$2e^{2x}dx=dt$

$$=\frac{1}{2}\int \frac{dt}{\sqrt{a^2+t^2}}$$

$$=\frac{1}{2}\log\left\{t+\sqrt{a^2+t^2}\right\}+C$$

$$=\frac{1}{2}\log\left\{e^{2x}+\sqrt{a^2+\left(e^{2x}\right)^2}\right\}+C$$

(12) $\int \frac{e^{7\log x}-e^{6\log x}}{e^{4\log x}-e^{3\log x}}dx$

Let $I=\int \frac{e^{7\log x}-e^{6\log x}}{e^{4\log x}-e^{3\log x}}dx$

$$=\int \frac{e^{\log x^7}-e^{\log x^6}}{e^{\log x^4}-e^{\log x^3}}dx \qquad \because e^{\log x}=x$$

$$=\int \frac{x^7-x^6}{x^4-x^3}dx$$

$$=\int \frac{x^3\left(x^4-x^3\right)}{\left(x^4-x^3\right)}dx$$

$$=\frac{x^4}{4}+C$$

(13) $$\int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$$

Let $$I = \int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$$

$$= \int \log(\log x)\, dx + \int \frac{1}{(\log x)^2}\, dx$$

$$= \int \log(\log x) \cdot 1 dx + \int \frac{1}{(\log x)^2}\, dx$$

Integrating first term, taking unity as second function, we get

$$I = x \log(\log(x)) - \int \frac{1}{x} \frac{1}{(\log x)} \times x\, dx + \int \frac{1}{(\log x)^2}\, dx$$

$$= x \log(\log(x)) - \int \frac{1}{\log x}\, dx + \int \frac{1}{(\log x)^2}\, dx$$

$$= x \log(\log(x)) - \left[\frac{1}{\log x} \times x - \int -\frac{1}{(\log x)^2} \times \frac{1}{x} \times x\, dx \right] + \frac{1}{(\log x)^2}\, dx$$

$$= x \log(\log(x)) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\, dx + \int \frac{1}{(\log x)^2}\, dx$$

$$= x \log(\log x) - \frac{x}{\log x} + C$$

(14) $$\int \frac{\sqrt{(x^2+1)} \left[\log(x^2+1) - 2 \log x \right]}{x^4}\, dx$$

Let $$I = \int \frac{\sqrt{(x^2+1)} \left[\log(x^2+1) - 2 \log x \right]}{x^4}\, dx$$

$$= \int \frac{\left(\sqrt{x^2+1} \right) \left[\log\left(\frac{x^2+1}{x^2} \right) \right]}{x^4}\, dx$$

$$= \int \frac{\sqrt{x^2+1}}{x^4} \log\left(1 + \frac{1}{x^2} \right) dx$$

$$= \int \sqrt{1 + \frac{1}{x^2}} \left\{ \log\left(1 + \frac{1}{x^2} \right) \times \frac{1}{x^3} \right\} dx$$

Putting $1+\frac{1}{x^2}=t$

$-\frac{2}{x^3}dx = dt$

$$\therefore\ I=\frac{1}{2}\int\sqrt{t}\cdot(\log t)\cdot(-dt)$$

$$=\frac{1}{2}\int\sqrt{t}\cdot\log t\cdot dt$$

Taking log t as first and $\sqrt{t}$ as second function, integrating it by parts, we get –

$$I=-\frac{1}{2}\left[\log t\left(\frac{t^{3/2}}{3/2}\right)-\int\frac{1}{t}\cdot\left(\frac{t^{3/2}}{3/2}\right)\cdot dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{3/2}\log t-\frac{2}{3}\int t^{1/2}\cdot dt\right]$$

$$=-\frac{1}{3}t^{3/2}\log t+\frac{1}{3}\int t^{1/2}\cdot dt$$

$$=-\frac{1}{3}t^{3/2}\log t+\frac{1}{3}+\frac{t^{3/2}}{3/2}+C$$

$$=-\frac{1}{3}t^{3/2}\log t+\frac{2}{9}t^{3/2}+C$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\log\left(1+\frac{1}{x^2}\right)-\frac{2}{9}\left(1+\frac{1}{x^2}\right)^{3/2}+C$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

EXERCISE 8.1

Evaluate the following :

Q. 1.: $\int\frac{dx}{2+\cos x}$

Q. 2.: $\int\frac{\sin}{\sqrt{1+\sin x}}dx$

Q. 3.: $\int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}dx$

Q. 4.: $\int\frac{3+4\sin x+2\cos x}{3+2\sin x+\cos x}dx$

Q. 5.: $\int\frac{\sqrt{\cos 2x}}{\cos x}$

Q. 6.: $\int\frac{1}{\cos^{5/2}\sin^{3/2}}dx$

Q. 7.: $\int\sin^{5/6}\cos^3 x\,dx$

Q. 8.: $\int\sin^{-1}\sqrt{\frac{x}{a+x}}\ dx$

Q. 9.: $\int\frac{1}{\sqrt{\cos^3 x\sin^5 x}}dx$

Q. 10.: $\int \sqrt{\tan x}\ \sec x \operatorname{cosec} x\, dx$ Q. 11.: $\int \frac{1}{6+6\cos x}dx$ Q. 12.: $\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}dx$

Q. 13.: $\int \frac{dx}{\sin x(3+2\cos x)}$ Q. 14.: $\int \frac{dx}{\cos(x+a)\cos(x+b)}$

Q. 15.: $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$

ANSWERS

(1) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right)+C$ (2) $-2\sqrt{1-\sin x}-\sqrt{2}\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right)+C$

(3) $\frac{18}{25}x+\frac{1}{25}\log(3\sin x+4\cos x)+C$ (4) $2x-3\tan^{-1}\left(1+\tan\frac{x}{2}\right)+C$

(5) $\cos^{-1}\left(\tan\frac{x}{2}\right)-\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\cos 2x}}{\sqrt{2}\sin x}\right)+C$ (6) $-2\sqrt{\cot x}+\frac{2}{3}\tan^{3/2}x+C$

(7) $\sin^{11/6}-6\sin^{23/6}\cdot\frac{x}{23}+C$ (8) $a\left(1+\frac{x}{a}\right)\tan^{-1}\sqrt{x}/a-\sqrt{x}/a+C$

(9) $-\frac{2}{3}(\tan x)^{-3/2}+2(\tan x)^{1/2}+C$ (10) $2\sqrt{\tan x}+C$

(11) $\frac{1}{6}\tan\frac{x}{2}+C$ (12) $3\log(2-\sin x)+\frac{4}{(2-\sin x)}+C$

(13) $\frac{1}{10}\log(1-\cos x)-\frac{1}{2}\log(1+\cos x)+\frac{2}{5}\log(3+2\cos x)+C$

(14) $\frac{1}{\sin(a-b)}\log\left\{\frac{\cos(a+b)}{\cos(x+a)}\right\}+C$ (15) $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2\sqrt{x}+\sqrt{(1-x)}}{\pi}+C$

3.9 DEFINITE INTEGRAL

Introduction :

We have learnt to find the area of triangle, rectangle etc. in earliar classes. These areas are the closed regions of the plane bounded by line segments. The methods which are involved in finding such areas, can not be applied in the region which is partially or wholly bounded by curves. The mathematical tool which can solve such problems, is the concept of definite integral.

The definite integral is also used in economics, finance and probability.

Definition 9.1 :

If f be a continuous function of x, defined on a closed interval $[a, b]$, then $\int_a^b f(x)dx$ is called definite integral of $f(x)$ between the limits a and b. Here a is called lower limit and b is called upper limit. The interval $[a, b]$, is called range of integrals.

$$\text{If } \int f(x)dx = F(x), \text{ when } \frac{d}{dx}f(x) = F(x), \text{ then } \int_a^b f(x) = F(b) - F(a)$$

"If a function is integrated under two limits, it is called definte integral."

We know that,

$$\int f(x)dx = F(x) + C \qquad \text{...(1) Indefinite integral of } f(x)$$

Putting $x = b$ in equation (1), we get –

$$\int f(x)dx = F(b) + C \qquad \text{...(2)}$$

Putting $x = a$ in equation (1), we get –

$$\int f(x)dx = F(a) + C \qquad \text{...(3)}$$

Subtracting (3) from (2) we get –

$$\int_a^b f(x)\,dx = \left[(F(b)+C) - (F(a)+C)\right]$$

We see that $f(b) - f(a)$ is definite integral, the constant term vanishes.

The above definte integral can also be written as,

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Where $\quad F(x) = \int f(x)dx \;$ or $\; \frac{d}{dx}F(x) = f(x)$

Remark : The value of a definite integral over any particular integral does not depend on the variable of integration, but depends on the function and the interval. If the independent variable is denoted by u or t instead of x, we simply write the integral of

$$\int_a^b f(u)\,du \quad or \quad \int_a^b f(t)\,dt \quad \text{instead of } \int_a^b f(x)\,dx$$

Hence the variable of integration is called dummy variables.

SOLVED EXAMPLES

Example 1.: Evaluate $\int_1^2 \left(x^3 + 1\right)\,dx$

Solution : We have

$$\int_1^2 \left(x^3+1\right)\,dx = \frac{x^4}{4} + x$$

$$\therefore \int_1^2 \left(x^3+1\right) dx = \left[\frac{x^4}{4}+x\right]_1^2 \qquad \text{(by definition)}$$

$$= \left[\frac{2^4}{4}+2\right]-\left[\frac{1}{4}+1\right]$$

$$= [4+2]-\left[\frac{1^4}{4}+1\right]$$

$$= 6-\frac{5}{4}$$

$$= \frac{19}{4}$$

Hence, $\int_1^2 \left(x^3+1\right) dx = \frac{19}{4}$

Example 2.: *Evaluate* $\int_0^1 \left(xe^{2x}+\sin\frac{\pi x}{2}\right) dx$

Solution : We have

$$= \int\left(xe^{2x}\sin\frac{\pi x}{2}\right)+dx$$

$$= \frac{xe^{2x}}{2}-\int 1.\frac{e^{2x}}{2}dx-\frac{2}{\pi}\cdot\cos\left(\frac{\pi x}{2}\right)$$

$$= \frac{xe^{2x}}{2}-\frac{1}{2}e^{2x}-\frac{2}{\pi r}\cos\frac{\pi x}{2}$$

$$\therefore \int_0^1 \left(xe^{2x}+\sin\frac{\pi x}{2}\right) dx = \left[xe^{2x}-\frac{e^{2x}}{2}-\frac{2}{\pi}\cos\frac{\pi x}{2}\right]_0^1$$

$$= \left[xe^{2x}\right]_0^1-\frac{1}{2}\left[e^{2x}\right]_0^1-\frac{2}{\pi}\left[\cos\frac{\pi x}{2}\right]_0^1$$

$$= \left(e^2-0\right)-\frac{1}{2}\left(e^2-e^0\right)-\frac{2}{\pi}\left[\cos\frac{\pi}{2}-\cos 0\right]$$

$$= e^2-\frac{1}{2}e^2+\frac{1}{2}-\frac{2}{\pi}[0-1]$$

$$= \frac{1}{2}e^2+\frac{2}{\pi}+\frac{1}{2}$$

Example 3.: *Evaluate* $\int_0^{\pi/4} \left(2\sec^2 x+x^3+2\right) dx$

Solution : We have

$$\int\left(2\sec^2 x + x^3 + 2\right) dx$$

$$= \int 2\sec^2 x dx + \int x^3 dx + 2\int dx$$

$$= 2\tan x + \frac{x^4}{4} + 2x$$

$$\therefore \int_0^{\pi/4}\left(2\sec^2 x + x^3 + 2\right) dx = \left[2\tan x + \frac{x^4}{4} + 2x\right]_0^{\pi/4}$$

$$= \left[2\tan x\right]_0^{\pi/4} + \frac{1}{4}\left[x^4\right]_0^{\pi/4} + 2\left[x\right]_0^{\pi/4}$$

$$= \left(2\tan\frac{\pi}{4} - \tan 0\right)_0^{\pi/4} + \left[\frac{1}{4}\left(\pi/4\right)^4 - 0\right] + 2\left[\pi/4 - 0\right]$$

$$= 2\times 1 + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\frac{\pi}{4}$$

$$= 2 + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + \frac{\pi}{2}$$

3.10 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION :

To evaluate definite integral by substitution the original variables are changed to the new variables and the limits of original integral are changed according to the new variable OR after the integration the new variables are in terms of the original variables and the integral is obtained by applying the limits of the original variables.

Example 4.: *Evaluate* $\int_0^{\pi} \frac{dx}{5 + 4\cos x}$

Solution : We have

$$\int_0^{\pi} \frac{dx}{5 + 4\cos x} = \int_0^{\pi} \frac{dx}{5 + 4\left(\frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}\right)}$$

$$\int_0^{\pi} \frac{\left(1 + \tan^2 x/2\right) dx}{5 + 5\tan^2 x/2 + 4 - 4\tan^2 x/2} \qquad \left(\therefore \cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}\right)$$

$$= \int_0^{\pi} \frac{\sec^2 x/2 \, dx}{9 + \tan^2 x/2}$$

$$\therefore \int_0^{\pi} \frac{dx}{5 + 4\cos x} = \int_0^{\infty} \frac{2dt}{9 + t^2}$$

Putting $\tan x/2 = t$

$$\therefore \frac{1}{2}\sec^2 x/2 \, dx = dt$$

$$= \frac{2}{3}\left[\tan^{-1} t/3\right]_0^{\infty}$$

where $x = 0, \ t = 0$

$x = \pi, \quad t = \infty$

$$= \frac{2}{3}\left[\tan^{-1}\infty - \tan^{-1} 0\right]_0^{\infty}$$

$$= \frac{2}{3} \times \frac{\pi}{2}$$

$$= \frac{\pi}{3}$$

Example 5.: Evaluate $\int_0^{\pi} 5(5 - 4\cos\theta)^{1/4} \sin\theta \, d\theta$

Solution : We have

$$\int_0^{\pi} 5(5 - 4\cos\theta)^{1/4} \sin\theta \, d\theta$$

Putting $\cos\theta = t$

$-\sin\theta d\theta = dt$

Where $0 = \theta, \ t = 1$

$\theta = \pi, \ t = -1$

$$\therefore \int_0^{\pi} 5(5 - 4\cos\theta)^{1/4} \sin\theta \, d\theta$$

$$= -\int_1^{-1} 5(5 - 4t)^{1/4} \, dt$$

$$= -\left[5\left(-\frac{4}{5}\frac{(5-4t)^{5/4}}{t}\right)\right]_1^{-1}$$

$$=\left[(5-4t)^{5/4}\right]_{1}^{-1}$$

$$=(5+4)^{5/4}-(5-4)^{5/4}$$

$$=9^{5/4}-1$$

$$=9\sqrt{3}-1$$

3.11 GENERAL PROPERTIES OF DEFINITE INTEGRAL

The following properties of definite integrals are more useful in the evaluation of the definite integrals more easily when the integrand is not a simple function.

Property I : $\int_a^b f(x)dx = \int_a^b f(t)dt$

Proof : Let $\int f(x)dx = F(x)$ Then $\int f(t)dt = F(t)$

and $\int_a^b f(x)dx = F(L) - F(a) = \int_a^b f(t)dt$

Property II : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof : LHS $= \int_a^b f(x)dx = F(b) - F(a)$

$$= -[F(a) - F(L)]$$

$$= -\left[\int_b^a f(x)dx\right] =$$

$$= -\left[\int_b^a f(x)dx\right] = RHS$$

Property III.: For real numbers a, b, c; where $a < c < b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof : LHS $= \int_a^b f(x)dx = F(b) - F(a)$, and

RHS $= \int_a^b f(x)dx + \int_c^b f(x)dx$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

$$= LHS$$

Property (III) can be generalised as

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_{r-1}}^{c_r} f(x)dx + \int_r^b f(x)dx$$

For $$RHS = F(c_1) - F(a) + F(c_2) - F(c_1) + \ldots F(c_r) - F(c_{r-1}) + F(b) - F(c_r)$$

Property IV : $\int_a^b f(x) = \int_a^b f\{(a+b) - x\}\,dx$

Proof : Let $a + b - x = t$, so, $dx = dt$

Where $x = a$, $t = b$, when $x = b$, $t = a$

Thus as x varies from a to b, t varies from b to a,

also $x = a + b - t$ therefore,

$$\int_a^b f(x)dx = -\int_b^a f(a+b-t)dt$$

$$= \int_a^b f(a+b-t)dt \quad \text{(by property III)}$$

$$= \int_a^b f(a+b-x)dx$$

[Since the variable of definite integral is a dummy variable]

The above integral can be divided into two parts –

$$(i)\ \int_0^a f(x)dx = \int_0^a f(a-x)dx \ \ and \ \ (ii)\ \int_b^0 f(x)dx = \int_b^0 f(b-x)dx$$

For (1) Putting $a - x = t$, so $dx = -dt$

Where $x = 0$, $t = 0$, $when$ $x = 0$, $t = a$

$$\therefore RHS\ of\ (1) = \int_a^0 f(t)dt$$

$$= \int_0^a f(t)dt \quad \text{by property (II)}$$

$$= \int_0^a f(x)dx \quad \text{(by changing the variables)}$$

Similarly we can prove (ii)

Property (v) : $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if f is an even function

i.e. $f(-x) = f(x)$

$= 0$ if f is an odd function

i.e. $f(-x) = -f(x)$

Proof : We have

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \quad \ldots(1) \text{ by property (II)}$$

Now for $\int_{-a}^{0} f(x)dx$ putting $-x = t,$ so $-dx = dt$

Where $x = -a,\ t = a$ and where $x = 0,\ t = 0$

$$\therefore \int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt$$

$$= \int_{0}^{a} f(-t)dt$$

$$= \int_{0}^{a} f(-x)dx \qquad \text{(By changing the variable } t \text{ to } x)$$

Using this in equation in (1) we have,

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{a}^{0} f(x)dx \qquad \ldots(2)$$

(i) When f is an even function, then (2) takes the form,

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx \qquad \because f(-x) = f(x)$$

$$= 2\int_{0}^{a} f(x)dx$$

(ii) When f is an odd function, the (2) becomes,

$$\int_{-a}^{0} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx \qquad \because f(-x) = -f(x)$$

$$= 0$$

Property (vi) : $\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx \qquad \text{if } f(2a-x) = f(x)$

$$= 0 \qquad \text{if } f(2a-x) = -f(x)$$

Proof : We have

$$\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{a}^{2a} f(x)dx \qquad \ldots(1) \text{ by property (II)}$$

Putting $2a - x = t,$ so that $-dx = dt$ or $dx = -dt$

Where $x = a,\ t = a$ and where $x = 2a,\ t = 0$

$$\therefore \int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt$$

$$= \int_0^a f(2a-t)dt \qquad \text{By property (iii)}$$

$$= \int_0^a f(2a-x)\,dx \qquad \text{(By changing the variable } t \text{ to } x)$$

Putting this in equation (1) we get,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \qquad ...(2)$$

Now, (i) if $\int_0^a f(2a-x) = f(x)$ then (2) becomes,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx \qquad \because f(-x) = f(x)$$

$$= 2\int_0^a f(x)dx$$

Now, (ii) if $\int_0^a f(2a-x) = -f(x)$ then (2) becomes,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx$$

$$= 0$$

IMPORTANT EXAMPLES

Example 6.: Prove that $\int_0^{\pi/2} \log(\cot x)\,dx = 0$

Solution : Let $I = \int_0^{\pi/2} \log(\cot x)\,dx \qquad ...(1)$

$$= \int_0^{\pi/2} \log\left(\cot\left(\pi/2 - x\right)\right)dx \qquad \text{by property (iv)}$$

$$= \int_0^{\pi/2} \log(\tan x)\, dx \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \log(\cot x)\, dx + \int_0^{\pi/2} \log(\tan x)\, dx$$

$$= \int_0^{\pi/2} \{\log(\cot x) + \log(\tan x)\} dx$$

$$= \int_0^{\pi/2} \log(\cot x \times \tan x)\, dx$$

$$= \int_0^{\pi/2} \log(1)\, dx$$

$$= 0$$

Example 7.: Prove that $\int_0^{\pi/2} \log(\sec x)\, dx = -\frac{\pi}{2} \log 2$

Solution : Let $I = \int_0^{\pi/2} \log(\sec x)\, dx \qquad \ldots(1)$

$$= \int_0^{\pi/2} \log\left\{\sec\left(\frac{\pi}{2} - x\right)\right\} dx$$

$$= \int_0^{\pi/2} \log(\cos x)\, dx \qquad \ldots(2) \text{ by property (iv)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \log(\sec x)\, dx + \int_0^{\pi/2} \log(\cos x)\, dx$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \{\log(\sin x) + \log(\cos x)\} dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin x \cos x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - (x)_0^{\pi/2} \log 2$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \frac{\pi}{2} \log 2 \qquad \ldots(3)$$

Now for $\int_0^{\pi/2} \log(\sin 2x)\, dx$

Putting 2x = t, $dx = \frac{1}{2} dt$, where $x = 0, t = 0, x = \pi/2, t = \pi$

$$\therefore \int_0^{\pi/2} \log(\sin 2x)\, dx = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt$$

$$= \frac{2}{2}\int_0^{\pi/2} \log(\sin t)\, dt$$ by property (vi)

and $\sin(\pi - x) = \sin x$

$$= \int_0^{\pi/2} \log(\sin t)\, dt$$

$$= \int_0^{\pi/2} \log(\sin x)\, dx$$ by changing vatiable t to x

Putting this in (3), we get,

$$2I = I - \frac{\pi}{2} \log 2$$

$$\therefore\ I = \int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2} \log 2$$

Example 8.: Prove that $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx = 0$

Solution : Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ ...(1)

$$= \int_0^{\pi/2} \frac{\sin\left(\pi/2 - x\right) - \cos\left(\pi/2 - x\right)}{1 + \sin\left(\pi/2 - x\right)\cos\left(\pi/2 - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \quad \text{...(II)}$$

Adding (I) and (II) we get,

$$I_2 = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin x - \cos x - \cos x - \sin x}{1 + \sin x \cos x}\right] dx$$

$$= \int_0^{\pi/2} 0 \cdot dx$$

$$= 0$$

Example 9.: Prove that $\int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx = \pi\left(\frac{\pi}{2} - 1\right)$

Solution : Let $I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$ ...(I)

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)} dx \quad \because \int_0^a f(x) dx = \int_0^a f(a - x) dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x \, dx}{1 + \sin x} dx \quad \text{...(II)}$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x dx}{1 + \sin x} - I$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} \left(1 - \frac{1}{1+\sin x}\right) dx$$

$$= \pi \int_0^{\pi} \left(1 - \frac{(1-\sin x)}{\cos^2 x}\right) dx$$

$$= \pi \int_0^{\pi} \left(1 - \sec^2 x + \sec x \tan x\right) dx$$

$$= \pi \left[(x - \tan x + \sec x)\right]_0^{\pi}$$

$$= \pi \left[(\pi - \tan \pi + \sec \pi) - (0 - \tan 0 + \sec 0)\right]$$

$$= \pi(\pi - 0 - 1 - 1)$$

$$= \pi(\pi - 2)$$

$$\therefore 2I = \pi(\pi - 2)$$

Hence $\quad I = \pi\left(\frac{\pi}{2} - 1\right)$

Example 10.: $\quad \int_0^2 |2x-1| dx$

Solution : $\quad$ *Let* $I = \int_0^2 |2x-1| dx$

Now, $\quad \begin{cases} |2x-1| = 2x-1, & \text{if } x \geq \frac{1}{2} \\ -(2x-1) & \text{if } x \leq \frac{1}{2} \end{cases}$

So, $\quad \int_0^2 |2x-1| dx = \int_0^{1/2} -(2x-1) dx + \int_{1/2}^{2} (2x-1) dx$

$$= \left[-\frac{2x^2}{2} + x\right]_0^{1/2} + \left[\frac{2x^2}{2} - x\right]_{1/2}^{2}$$

$$= -\frac{1}{4} + \frac{1}{2} + \left[4 - 2 - \frac{1}{4} + \frac{1}{2}\right]$$

$$= \frac{1}{4} + 2 + \frac{1}{4}$$

$$= \frac{5}{2}$$

Example 11.: Evaluate $\int_0^{\pi} |\cos x| dx$

Solution : *Let* $I = \int_0^{\pi} |\cos x| dx$

We have $|\cos x| = \begin{cases} \cos x; & if\, 0 \le x \le \frac{\pi}{2} \\ -\cos x & if\, \frac{\pi}{2} \le x \le \pi \end{cases}$

$$I = \int_0^{\pi} |\cos x| dx = \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx$$

$$= (\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}$$

$$= (\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}$$

$$= 1 + 1$$

$$= 2$$

$$\therefore I = \int_0^{\pi} |\cos x| dx = 2$$

Example 12.: Prove that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log\left(\sqrt{2} + 1\right)$

Solution : *Let* $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}$ ...(1)

$$= \int_0^{\pi/2} \frac{\sin^2\left(\pi/2 - x\right)}{\sin\left(\pi/2 - x\right) + \cos\left(\pi/2 - x\right)} dx \quad \text{By property (iv)}$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad ...(2)$$

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\cos x + \sin x}\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{1}{\sin x + \cos x}\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{1 \times 1/\sqrt{2}}{1/\sqrt{2}\sin x + 1/\sqrt{2}\cos x}\right] dx$$

$$= 1/\sqrt{2} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \pi/4\right)}$$

$$= 1/\sqrt{2} \int_0^{\pi/2} \sec\left(x - \pi/4\right) dx$$

$$= 1/\sqrt{2} \cdot \log\left[\sec\left(x - \pi/4\right) + \tan\left(x - \pi/4\right)\right]_0^{\pi/2}$$

$$= 1/\sqrt{2} \cdot \left[\log\left(\sec \pi/4 + \tan \pi/4\right) - \log\left\{\sec \pi/4 - \tan \pi/4\right\}\right]$$

$$= 1/\sqrt{2} \cdot \log\left(\sqrt{2}+1\right) - \log\left(\sqrt{2}-1\right)$$

$$= 1/\sqrt{2} \cdot \log\frac{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}$$

$$= 1/\sqrt{2} \cdot \log\frac{\left(\sqrt{2}+1\right)^2}{2-1}$$

$$= 2/\sqrt{2} \cdot \log\left(\sqrt{2}+1\right)$$

$$\therefore \; I = \frac{1}{2} \times \frac{2}{\sqrt{2}} \cdot \log\left(\sqrt{2}+1\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \log\left(\sqrt{2}+1\right)$$

Example 13.: Show that $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Solution : *Let* $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ...(1)

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\pi/2 - x\right)}}{\sqrt{\sin\left(\pi/2 - x\right)} + \sqrt{\cos\left(\pi/2 - x\right)}}\, dx \qquad \text{By property (iv)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad ...(2)$$

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right] dx$$

$$= \int_0^{\pi/2} dx$$

$$= (x)_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

Example 14.: Show that $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{12}$

Solution : *Let* $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}}$ ...(1)

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \qquad ...(2)$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\pi/6 + \pi/3 - x\right)}}{\sqrt{\sin\left(\pi/6 + \pi/3 - x\right)} + \sqrt{\cos\left(\pi/6 + \pi/3 - x\right)}}\, dx$$

By property (iv)

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad \ldots(3)$$

Adding (1) and (2) we get,

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$= \int_{\pi/6}^{\pi/3} \left[\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right] dx$$

$$= \int_{\pi/6}^{\pi/3} dx$$

$$= (x)_{\pi/6}^{\pi/3}$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

$$\therefore \; I = \frac{\pi}{12}$$

Example 15.: Show that $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab}$

Solution : *Let* $\displaystyle I = \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

$$= \int_0^{\pi/2} \frac{\sin^2 x \, dx}{a^2 + b^2 \tan^2 x}$$

Dividing Numerator and denominator by $\cos^2 x$

$$= \int_0^{\pi/2} \frac{dt}{a^2 + b^2 \tan^2 x}$$

Putting tan x = t, $\sec^2 x \, dx = dt$

Where x = 0, t = 0, when $x = \pi/2$, $t = \infty$

$$= \int_0^\infty \frac{dt}{a^2 + b^2 t^2}$$

$$= \frac{1}{b^2} \int_0^\infty \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= \frac{1}{b^2} \times \frac{b}{a} \left[\tan^{-1}\left(\frac{bt}{a}\right) \right]_0^\infty$$

$$= \frac{1}{ab} \left[\tan^{-1}\infty - \tan^{-1} 0 \right]$$

$$= \frac{1}{ab} \times \frac{\pi}{2}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab}$$

Example 16.: Prove that $\int_0^\pi \frac{x \tan x dx}{\sec x + \tan x} = \pi\left(\frac{1}{2}\pi - 1\right)$

Solution : *Let* $I = \int_0^\pi \frac{x \tan x dx}{\sec x + \tan x}$

$$= \int_0^\pi \frac{x \sin x}{1 + \sin x} dx$$

(Proceed as solved example - 9)

Example 17.: Prove that $\int_0^{\pi/4} \log(1 + \tan x)\, dx = \frac{\pi}{8} \log 2$

Solution : *Let* $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$ ...(1)

$$= \int_0^{\pi/4} \log\left\{1 + \tan\left(\pi/4 - x\right)\right\} dx \qquad \text{By property (iv)}$$

$$= \int_0^{\pi/2} \log\left\{1 + \frac{\tan \pi/4 - \tan x}{1 + \tan \pi/4 \tan x}\right\}$$

$$= \int_0^{\pi/2} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx \qquad \ldots(2)$$

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi/4} \log(1 + \tan x)\, dx + \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} [\log(2)]\, dx$$

$$= [x]_0^{\pi/4} \log 2 = \frac{\pi}{4} \log 2$$

$$\therefore \; I = \frac{\pi}{8} \log 2$$

EXERCISE 11.1

Prove the following :

Q. 1. : $\int_0^{\pi/2} \log(\cos x)\, dx = -\frac{\pi}{2} \log 2.$

Q. 2. : $\int_0^{\pi/2} \frac{(\cos x)}{(1 + \sin x)(2 + \sin x)} dx = 2\log 2 - \log 3$

Q. 3. : $\int_0^1 \tan^{-1} x\, dx = \frac{\pi}{4} - \frac{1}{2} \log 2$

Q. 4. : $\int_1^2 \frac{dx}{1 + x^2} = 2\log 2 - \frac{1}{2} \log 10$

Q. 5. : $\int_0^a \frac{dx}{x + \sqrt{a^2 + x^2}} = \frac{\pi}{4}$

Q. 6. : $\int_0^a \frac{\log(1 + x)}{(1 + x^2)} = \frac{\pi}{8} \log 2$

Q. 7. : $\int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}} \quad a > 0$

Q. 8. : $\int_0^{\pi/4} \frac{x\,dx}{1+\cos 2x + \sin 2x} = \frac{\pi}{16}\log 2$

Q. 9. : $\int_0^{\infty} \frac{\log\left(1+x^2\right)dx}{\left(1+x^2\right)} = \pi \log 2$

Q. 10. : $\int_0^{\pi/2} \frac{x \sin x \cos x}{\left(\sin^4 x + \cos^4 x\right)} = \frac{\pi^2}{16}$

Q. 11. : $\int_0^{\pi/2} \frac{\sin^{5/2} x dx}{\sin^{5/2} x dx + \cos^{5/2} x} dx = \frac{\pi}{4}$

Q. 12. : $\int_0^{\pi} \frac{x \cdot dx}{1+\cos\alpha \sin x} dx = \frac{\pi\alpha}{\sin\alpha} \qquad 0 < \alpha < \pi$

Q. 13. : $\int_0^{2\pi} \frac{\sin^2\theta \cdot d\theta}{a - b\cos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{a^2-b^2}\right] \; (a > b > 0)$

Q. 14. : $\int_0^{1} \frac{\sin^{-1} x}{x} dx = \frac{\pi}{2} \log 2$

Q. 15. : $\int_0^{\pi} \frac{x \sin x}{2+\cos 2x} dx = \frac{\pi}{\sqrt{2}} \tan^{-1}\sqrt{2}$

Q. 16. : $\int_0^{\pi} \frac{x \sin x}{1+\alpha^2\cos^2 x} dx = \frac{\pi}{\alpha} \tan^{-1}\alpha, \quad (\alpha < 1)$

Q. 17. : $\int_0^{\pi/4} \frac{\sin x + \cos x}{\cos^2 x + \sin^4 x} dx = \frac{\pi}{4} + \frac{1}{\sqrt{3}} \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)$

Q. 18. : $\int_0^{\infty} \log\frac{\left(x+\frac{1}{x}\right)}{1+x^2} dx = \pi \log 2$

Q. 19. : $\int_0^{\pi/2} \log\frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx = \frac{\pi^2}{16}$

Q. 20. : $\int_0^{\infty} \log \frac{xdx}{(1+x)(1+x^2)} dx = \frac{\pi}{4}$

3.12 DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Definition 14.1 : If f be a single real valued continuous function of independent variable x, in the interval $[a, b]$, where $b > a$ and if the interval $[a, b]$, is subdivided into n equal parts, $a+b, a+2h, a+3h, \ldots, a+(n-1)\,h$, then,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h) +\ldots+f\{a+(n-1)\}h\right]$$

$$= \lim_{n\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots +\ldots+f\{a+(n-1)h\}\right] \quad \ldots(1)$$

Where $h = \frac{b-a}{n}$, *or* $b = a + nh$, is called definite integral as the limit of a sum.

Equation (1) can also be written as–

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$$

$$= \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$$

Further if $\frac{d}{dx}F(x) = f(x)$ *i.e.* $\int f(x)\,dx = F(x)$ then,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) = F(b)-F(a)$$

Note : Integration by summation is also called integration from the first principle or integration *ab initio.*

SOLVED EXAMPLES

Example 1 : Evaluate $\int_a^b x\,dx$ as the limit of a sum.

Solution : Here $f(x) = x$, $\frac{b-a}{n} = h$

We know that,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\left[f(a)+f(a+h)+f(a+2h)\right.$$

$$+\ldots+f\{a+(n-1)\}h]$$

$$\therefore \int_a^b x\,dx = \lim_{n\to\infty} h[a+a+h+a+2h+\ldots a+(n-1)h]$$

$$= \lim_{n\to\infty} h[a+a+\ldots n\,times + h\{1+2+3\ldots(n-1)\}]$$

$$= \lim_{n\to\infty} h\left[na + h\cdot\frac{(n-1)n}{2}\right]$$

$$= \lim_{n\to\infty}\left[nah + h^2\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\not{n}a\frac{(b-a)}{\not{n}} + \frac{(b-a)^2}{\not{n}^2}\frac{\not{n}\cdot\not{n}\left(1-\frac{1}{n}\right)}{2}\right]$$

$$= ab - a^2 + \frac{1}{2}\left(b^2+a^2-2ab\right)$$

$$= \left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{1}{2}\left(b^2-a^2\right)$$

Example 2 : Evaluate $\int_1^2 x^2dx$ from the definition of definite integral as the limit of a sum.

Solution : We know that from the definition –

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a)+f(a+h)+\ldots+f\{a+(n-1)\}h]$$

Here $a = 1, b = 2,\ f(x) = x^2,\ h = \frac{2-1}{n} = \frac{1}{n}$

$$\therefore \int_1^2 x^2dx = \lim_{n\to\infty} h\left[1^2+(1+h)^2+(1+2h)^2+\ldots\{1+(n-1)h\}^2\right]$$

$$= \lim_{n\to\infty} h[1+1+\ldots n\,times + (1+2+3+\ldots(n-1)2h$$

$$+h^2\left(1^2+2^2+3^2+\ldots(n-1)^2\right)\Big]$$

$$= \lim_{n\to\infty} h\left[n + 2h\frac{(n-1)n}{2} + h^2\frac{(n-1)n(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n + 2\cdot\frac{1}{n}\cdot\frac{n(n-1)}{2} + \frac{1}{n^2}\cdot\frac{(n-1)n(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[n\cdot\frac{1}{n} + \frac{2}{n^2}\cdot\frac{n^2\left(1-\frac{1}{n}\right)}{2} + \frac{1}{n^3}\cdot\frac{n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}\right]$$

$$= 1 + 1 + \frac{2}{6}$$

$$= 2 + \frac{1}{3}$$

$$= \frac{7}{3}$$

Example 3 : Evaluate $\int_0^2 e^x dx$ as the limit of a sum.

Solution : By definition we have –

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\left[f(a) + f(a+h) + \ldots + f\{a+(n-1)\}h\right]$$

Here $a = 0, b = 2, \quad f(x) = e^x, \quad h = \frac{2-0}{n} = \frac{2}{n}$

$$\therefore \int_0^2 e^x dx = \lim_{n\to\infty} h\left[e^0 + e^{0+h} + e^{0+2h} + \ldots + e^{(n-1)h}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[e^0 + e^{2/n} + e^{4/n} + \ldots + e^{2n-2/n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^{2n/n} - 1}{e^{2/n} - 1}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2 - 1}{e^{2/n} - 1}\right]$$

$$= \left[\frac{2\left(e^2 - 1\right)}{\lim_{n\to\infty} \frac{e^{2/n} - 1}{2/n}\cdot 2}\right] \qquad \therefore = \lim_{n\to\infty} \frac{e^{2/n} - 1}{2/n} = 1$$

$$= \left(e^2 - 1\right)$$

Example 4 : Evaluate $\int_1^2 (x^2+x)$ as the limit of a sum.

Solution : By definition we have –

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\left[f(a)+f(a+h)+\ldots+f\{a+(n-1)\}h\right]$$

Here $a = 1, b = 2,\ h=\frac{2-1}{n}=\frac{1}{n}\ \&\ f(x)=x^2+x$

$$\therefore \int_1^2 (x^2+x)\,dx = \lim_{n\to\infty} h\Big[(1^2+1)+(1+h)^2+(1+h)+(1+2h)^2$$
$$+1+2h+\ldots+(1+(n-1)h)^2+\{1+(n-1)h\}\Big]$$

$$= \lim_{n\to\infty} h\Big[1^2+1^2+1^2+\ldots n\ times+1+1+1+\ldots n\ times$$
$$+h^2\left\{1^2+2^2+3^2+\ldots(n-1)^2\right\}$$
$$+2h\big[1+2+3+\ldots(n-1)$$
$$+h\big(1+2+3+\ldots(n-1)\big)\big\}\Big]$$

$$\therefore \int_1^2 (x^2+x)\,dx = \lim_{n\to\infty} h\left[(n+n)+h^2\frac{(n-1)n(2n-1)}{6}+2h\frac{(n-1)n}{2}\right.$$
$$\left.+h\cdot\frac{(n-1)n}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[2n+\frac{1}{n^2}\frac{n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{3}{n}\cdot\frac{n^2\left(1-\frac{1}{n}\right)}{2}\right]$$

$$= \lim_{n\to\infty} \left[2n\times\frac{1}{n}+\frac{1}{n^3}\frac{n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{3}{n^2}\cdot\frac{n^2\left(1-\frac{1}{n}\right)}{2}\right]$$

$$= 2+\frac{1}{3}+\frac{2}{3}$$

$$= \frac{23}{6}$$

Example 5 : Evaluate $\int_a^b \cos x dx$ as the limit of a sum.

Solution : Here $f(x)=\cos x$. By definition we have –

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f\{a+(n-1)\}h\right]$$

$$\therefore \int_a^b \cos dx = \lim_{n\to\infty} h\left[\cos a + \cos(a+h) + \cos(a+2h) + \ldots + \cos(a+(n-1)h)\right]$$

$$= \lim_{n\to\infty} h \cos\cdot\left(\frac{a+a+(n-1)h}{2}\right)\frac{\sin\frac{nh}{2}}{\sin\frac{nh}{2}}$$

$$= \lim_{h\to 0} 2\cdot\frac{h/2}{\sin h/2}\left[\cos\left(a+\frac{1}{2}\left(\frac{b-a}{h}-1\right)h\right)\right]\sin\left(\frac{b-a}{2}\right)$$

$$= \lim_{h\to 0} 2\cdot\frac{h/2}{\sin h/2}\left[\cos\left(a+\frac{b-a}{2}-2h\right)\right]\sin\left(\frac{b-a}{2}\right)$$

$$= 2\times 1\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) = \sin b - \sin a$$

EXERCISE 12 .1

Evaluate the following integrals as the limit of a sum :

Q. 1. : $\int_1^2 x\,dx.$ **Q. 2. :** $\int_0^4 \left(x^2 - x\right)dx$ **Q. 3. :** $\int_0^4 \left(x + e^{2x}\right)dx$

Q. 4. : $\int_a^b e^x dx$ **Q. 5. :** $\int_a^b \frac{1}{x^2}dx$ **Q. 6. :** $\int_a^b \sin x\,dx$

Q. 7. : $\int_0^{\pi/4} \sec^2 x\,dx$ **Q. 8. :** $\int_0^{\pi/2} \sin\theta d\theta$ **Q. 9. :** $\int_1^2 \left(3x^2 + 2x\right)dx$

Q. 10. : $\int_a^b \sin^2 x dx$ **Q. 11. :** $\int_0^1 (3x-2)dx$

ANSWERS

(1) $\frac{3}{2}$ **(2)** $\frac{27}{2}$ **(3)** $\frac{15+e^x}{2}$ **(4)** $e^b - e^a$ **(5)** $\frac{1}{a} - \frac{1}{b}$

(6) $\cos a - \cos b$ **(7)** 1 **(8)** 1 **(9)** 10

(10) $\frac{1}{2}(b-a) + \frac{1}{2}(\sin a\cos a - \sin b\cos b)$

3.13 APPLICATION OF DEFINITE INTEGRAL TO FIND THE SUM OF INFINITE SERIES.

In previous article we have expressed the definite integral as the limit of a sum which is nothing but sums of series as definite integral.

We have

$$\int_a^b f(x)\,dx = \lim_{h\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$$

Where $b-a=nh$

Putting a = 0, b = 1, we get

$$nh = 1-0 = 1, \therefore h = \frac{1}{n}$$

$$\int_0^1 f(x)\,dx = \lim_{h\to\infty} \frac{1}{n}\sum f\left(\frac{r}{n}\right)$$

15.1 Rule to find the sum of the series :

First we find the r[th] term of the series and express it as $\frac{1}{n} f\left(\frac{r}{n}\right)$. Then the series can be written as $\lim_{n\to\infty} \sum \frac{1}{n} f\left(\frac{r}{n}\right)$. The corresponding definite integral can be obrained by replacing r/h by x and $\frac{1}{n}$ by dx and $\lim_{h\to\infty} \sum_{r=0}^{n-1}$ by $\int_6^1$.

SOLVED EXAMPLES

Example 1.: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+1}+\ldots+\frac{1}{2n}\right]$

Solution : The general terms (r[th] term) $= \frac{1}{n+r}$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r} = \lim_{n\to\infty} \frac{1}{n\left(1+r/n\right)}$$

The given series can be written as

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{\left(1+r/h\right)} \quad \left(r^{th} \ \ term\right)$$

Putting $\frac{r}{n} = x, \ \ and \ \ \frac{1}{n} = dx,$

when r = h, $x = n/n \to 1 \ \ as \ \ n \to \infty$

$$\therefore I = \int_0^1 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^1$$

$$= \log 2 - \log 1$$

$$= \log 2$$

Example 2.: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right]$

Solution : The given series can be written as

$$\lim_{h\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{1+\frac{5n}{n}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{\left(1+r/n\right)}$$

By putting $\frac{r}{n} = x,\ \ and\ \ \frac{1}{n} = dx,$

when $r = 1,\quad x = 1/n \to 0\ \ as\ \ n\to\infty$

and $r = 5n,\ x \to \frac{5n}{n} = 5 \quad n\to\infty$

$$\therefore\quad I = \int_0^5 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^5$$

$$= \log 6 - \log 1$$

$$= \log 6$$

Example 3.: Evaluate $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\ldots+\frac{1}{n}\right]$

Solution : The given series can be written as

$$\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\ldots+\frac{n+n}{n^2+n^2}\right]$$

(Last term = $\frac{n+n}{n^2+n^2}$)

$$= \lim_{n\to\infty}\frac{n}{n^2}\left[\frac{1+\frac{1}{n}}{1+1^2/n^2}+\frac{1+\frac{1}{n}}{1+1^2/n^2}+\ldots+\frac{1+1}{1+1}\right]$$

By putting $\frac{r}{n} = x$, *and* $\frac{1}{n} = dx$,

when r = 1, $x = \frac{1}{n} \to 0$ *as* $n \to \infty$

when r = n, $x \to \frac{n}{n} = 1$

$$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \frac{\left(1+\frac{r}{n}\right)}{\left(1+\frac{r^2}{n^2}\right)} = \int_0^1 \frac{1+x}{1+x^2} dx$$

$$= \int_0^1 \frac{1}{1+x^2} dx + \int_0^1 \frac{x}{1+x^2}$$

$$= \left[\tan^{-1} x\right]_0^1 + \frac{1}{2}\left[\log\left(1+x^2\right)\right]_0^1$$

$$= \left[\tan^{-1} - \tan^{-1} 0\right] + \frac{1}{2}\left[\log(1+1) - \log 1\right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\log 2$$

Example 4.: Evaluate $\lim_{n\to\infty} \frac{1}{n}\left[(n+1)+(n+2)+\ldots+(n+n)\right]^{1/n}$

Solution : The given series can be written as

Let $$A = \lim_{n\to\infty} \frac{1}{n} \cdot n\left[\left(1+\frac{1}{n}\right)\times\left(1+\frac{2}{n}\right)\times\ldots\times\left(1+\frac{n}{n}\right)\right]^{1/n}$$

Taking log on both sides, we get –

$$\log A = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)\times\log\left(1+\frac{2}{n}\right)\times\ldots\times\log\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\log\left(1+\frac{r}{n}\right)$$

By putting $\frac{r}{n} = x$, *and* $\frac{1}{n} = dx$,

when r = 1, $x = \frac{1}{n} \to 0 \quad (n \to \infty)$

when r = n, $x \to \frac{n}{n} = 1$

Taking unity as first function and log (1+x) as second fuction and integrating by parts –

$$\therefore \quad \log A = \int_0^1 \log(1+x)\,dx$$

$$= \left[x\log(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \log 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right) dx$$

$$= \log 2 - \int_0^1 dx + \int_0^1 \frac{1}{1+x}\,dx$$

$$= \log 2 - [x]_0^1 + \left[\log(1+x)\right]_0^1$$

$$= \log 2 - 1 + \log 2 = 2\log 2 - 1$$

$$= -1 + \log 4$$

or $$= -1 + \log(2\times 2)$$

$$\log A = -1 + \log 4$$

or $$\log A - \log 4 = -1$$

$$\log \frac{A}{4} = -1$$

or $$\frac{A}{4} = e^{-1}$$

$$\therefore A = \frac{4}{e}$$

Example 5.: Evaluate $\lim_{n\to\infty} \left[\frac{n!}{n^n}\right]^{1/n}$

Solution : We have

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)..3\cdot 21}{n^n}$$

$$= \left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right]$$

$$\therefore \lim_{n\to\infty} \left[\frac{n!}{n^n}\right]^{1/n} = \lim_{n\to\infty} \left[\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\cdots\frac{n}{n}\right]^{1/n}$$

Let $$A = \lim_{n\to\infty} \left[\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\cdots\frac{n}{n}\right]^{1/n}$$

Taking log on both sides, we get –

$$\log A = \frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \ldots \log\frac{n}{n}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \log\left(\frac{r}{n}\right)$$

By putting $\frac{r}{n} = x$, *and* $\frac{1}{n} = dx$, and

when r = 1, $x = 1/n \to 0 \quad (n \to \infty)$

r = n, $x = \frac{n}{n} = 1$

$$\therefore \quad \log A = \int_0^1 \log x$$

$$= [x \log x - x]_0^1$$

$$= 1\log 1 - 1 - 0$$

$$= -1$$

$$A = e^{-1} = \frac{1}{e}$$

EXERCISE 13.1

Evaluate the following integrals as the limit of a sum :

Q. 1. : $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)-\ldots\left(1+\frac{n^2}{n^2}\right)^{1/n}$

Q. 2. : $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}$

Q. 3. : $\lim_{n\to\infty}\frac{1}{2n}+\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\ldots+\frac{1}{\sqrt{3n^2+2n-1}}$

Q. 4. : $\lim_{n\to\infty}\frac{1}{n^{p+1}}\left[1^p+2^p+\ldots n^p\right]=\frac{1}{p+1}$

Q. 5. : $\lim_{n\to\infty}\frac{1}{n}\left[\tan\frac{\pi}{4n}+\tan\frac{2\pi}{4n}+\ldots\tan\frac{n\pi}{4n}\right]$

Q. 6. : $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots\frac{1}{8n}\right]$

Q. 7. : $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\frac{1}{n+2n}\right]$

Q. 8. : $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+2^2}+\frac{n}{n+4^2}+\ldots\frac{1}{n^2+(2n-2)^2}\right]$

Q. 9. : $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\ldots\frac{1}{n}\sec^2 1\right]=\frac{1}{2}\tan 1$

Q. 10. : $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\frac{3^4}{n^4}\right)^{1/3}\ldots\left(1+\frac{n^4}{n^4}\right)^{1/4}\right]$

ANSWERS

(1) $2e^{\pi-4/2}$ **(2)** $-1+\sqrt{5}$ **(3)** $\frac{\pi}{6}$ **(4)** $\frac{1}{p+1}$ **(5)** $\frac{2}{\pi}\log 2$

(6) $\frac{3}{8}$ **(7)** $\log 3$ **(8)** $\frac{1}{2}\tan^{-1}2$ **(9)** $\frac{1}{2}\tan 1$ **(10)** $e^{\frac{\pi^2}{48}}$

4 Consumers and Producers Surplus

Chapter Includes:

1. Consumers' Surplus
2. Producers' Surplus

4.1 CONSUMERS' SURPLUS

A demand curve for a commodity shows the amount of the commodity that will be bought by people at any given price p. Suppose that the prevailing market price is p_0. At this price an amount x_0 of the commodity determined by the demand curve will be sold. However there are buyers who would be willing to pay a price higher than p_0. All such buyers will gain from the fact that the prevailing market price is only p_0. This gain is called Consumers' Surplus. It is represented by the area below the demand curve $p = f(x)$ and above the line $p = p_0$.

Thus Consumers' Surplus,

CS = [Total area under the demand function bounded by $x = 0$, $x = x_0$ and x-axis – Area of the rectangle OAPB]

$$\therefore \quad CS = \int_0^{x_0} f(x)\,dx - p_0x_0$$

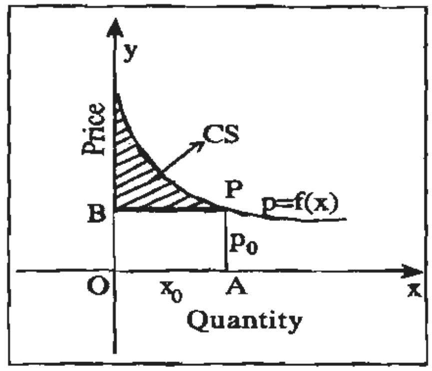


Fig. 4.1

Example 1

Find the consumers' surplus for the demand function $p = 25 - x - x^2$ when $p_0 = 19$.

Solution :

Given that,

The demand function is $p = 25 - x - x^2$

$p_0 = 19$

$\therefore \quad 19 = 25 - x - x^2$

$\Rightarrow \quad x^2 + x - 6 = 0$

$\Rightarrow \quad (x + 3)(x - 2) = 0$

$\Rightarrow \quad x = 2 \text{ (or) } x = -3$

$\therefore \quad x_0 = 2$ [demand cannot be negative]

$\therefore \quad p_0 x_0 = 19 \times 2 = 38$

$$CS = \int_0^{x_0} f(x)\,dx - p_0x_0$$

$$= \int_0^2 (25 - x - x^2)dx - 38$$

$$= \left[25x - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 - 38$$

$$= [25(2) - 2 - \frac{8}{3}] - 38 = \frac{22}{3} \text{ units}$$

Example 2

The demand of a commodity is $p = 28 - x^2$ Find the consumers' surplus when demand $x_0 = 5$

Solution :

Given that,

The demand function, $p = 28 - x^2$

when $x_0 = 5$

$p_0 = 28 - 25$

$= 3$

$\therefore p_0x_0 = 15$

$$CS = \int_0^{x_0} f(x)\,dx - p_0x_0$$

$$= \int_0^5 (28 - x^2)dx - 15$$

$$= \left[28x - \frac{x^3}{3}\right]_0^5 - 15$$

$$= [28 \times 5 - \frac{125}{3}] - 15 = \frac{250}{3} \text{units}$$

Example 3

The demand function for a commodity is $p = \frac{12}{x+3}$. Find the consumers' surplus when the prevailing market price is 2.

Solution :

Given that, Demand function, $p = \frac{12}{x+3}$

$p_0 = 2 \Rightarrow 2 = \frac{12}{x+3}$

or $2x + 6 = 12$ or $x = 3$ $\therefore$ $x_0 = 3 \Rightarrow p_0x_0 = 6$

$$CS = \int_0^{x_0} f(x)\,dx - p_0x_0 = \int_0^3 \frac{12}{x+3}dx - 6$$

$$= 12\,[\log(x+3)]_0^3 - 6.$$

$$= 12[\log 6 - \log 3] - 6 = 12 \log \frac{6}{3} - 6 = 12 \log 2 - 6$$

4.2 PRODUCERS' SURPLUS

A supply curve for a commodity shows the amount of the commodity that will be brought into the market at any given price p. Suppose the prevailing market price is p_0. At this price an amount x_0 of the commodity, determined by the supply curve, will be offered to buyers. However, there are producers who are willing to supply the commodity at a price lower than p_0. All such producers will gain from the fact that the prevailing market price is only p_0. This gain is called 'Producers' Surplus'. It is represented by the area above the supply curve $p = g(x)$ and below the line $p = p_0$.

Thus Producers' Surplus,

PS = [Area of the whole rectangle OAPB – Area under the supply curve bounded by $x = 0$, $x = x_0$ and x - axis]

$$\therefore \ PS = p_0 x_0 - \int_0^{x_0} g(x)\,dx$$

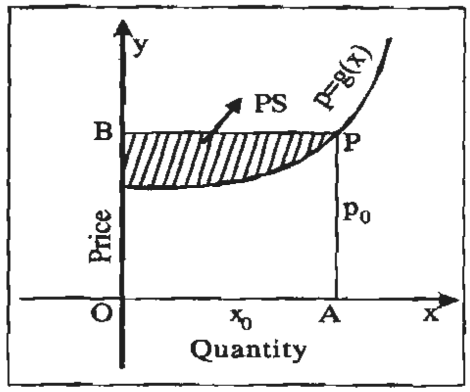


Fig 4.2

Example 4

The supply function for a commodity is $p = x^2 + 4x + 5$ where x denotes supply. Find the producers' surplus when the price is 10.

Solution :

Given that,

Supply function, $p = x^2 + 4x + 5$

For $p_0 = 10$,

$10 = x^2 + 4x + 5 \quad \Rightarrow x^2 + 4x - 5 = 0$

$\Rightarrow \quad (x + 5)(x - 1) = 0 \Rightarrow x = -5 \quad$ or $\quad x = 1$

Since supply cannot be negative, $x = -5$ is not possible.

$\therefore \quad x = 1$

$\therefore \quad p_0 = 10$ and $x_0 = 1 \quad \Rightarrow \quad p_0 x_0 = 10$

Producers' Surplus,

$$PS = p_0 x_0 - \int_0^{x_0} g(x)\,dx$$

$$= 10 - \int_0^1 (x^2 + 4x + 5)\,dx$$

$$= 10 - \left[\frac{x^3}{3} + \frac{4x^2}{2} + 5x\right]_0^1$$

$$= 10 - [\frac{1}{3} + 2 + 5] = \frac{8}{3} \text{ units.}$$

Example 5

Find the producers' surplus for the supply function $p = x^2 + x + 3$ when $x_0 = 4$.

Solution :

Given that,

supply function $p = x^2 + x + 3$

when $x_0 = 4$, $p_0 = 4^2 + 4 + 3 = 23$

$\therefore \ p_0 x_0 = 92.$

Producers' Surplus

$$PS = p_0 x_0 - \int_0^{x_0} g(x)\,dx \quad = 92 - \int_0^4 (x^2 + x + 3)\,dx$$

$$= 92 - \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x\right]_0^4$$

$$= 92 - \left[\frac{64}{3} + \frac{16}{2} + 12\right] = \frac{152}{3} \text{ units.}$$

Example 6

Find the producers' surplus for the supply function $p = 3 + x^2$ when the price is 12.

Solution :

Given that,

supply function, $p = 3 + x^2$. When $p_0 = 12$,

$12 = 3 + x^2$ or $x^2 = 9$ or $x = \pm 3$

Since supply cannot be negative,

$x = 3$. i.e. $x_0 = 3$,

$\therefore\ p_0x_0 = 36$.

Producers' Surplus,

$$PS = p_0x_0 - \int_0^{x_0} g(x)\,dx$$

$$= 36 - \int_0^3 (3 + x^2)\,dx = 36 - \left[3x + \frac{x^3}{3}\right]_0^3$$

$$= 36 - \left[9 + \frac{27}{3} - 0\right] = 18 \text{ units.}$$

Example 7

The demand and supply functions under pure competition are $p_d = 16 - x^2$ and $p_s = 2x^2 + 4$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

Solution :

For market equilibrium,

Quantity demanded = Quantity supplied

$\Rightarrow\ 16 - x^2 = 2x^2 + 4 \ \Rightarrow\ 3x^2 = 12$

$\Rightarrow\ x^2 = 4 \ \Rightarrow\ x = \pm 2$ But $x = -2$ is inadmissible.

$\therefore\ x = 2$ (i.e.) $x_0 = 2$

$\therefore\ p_0 = 16 - (2)^2 = 12$

$\therefore\ p_0x_0 = 12 \times 2 = 24$.

Consumers' Surplus,

$$CS = \int_0^{x_0} f(x)\,dx - p_0x_0$$

$$= \int_0^2 (16 - x^2)\,dx - 24$$

$$= \left[16x - \frac{x^3}{3}\right]_0^2 - 24 = 32 - \frac{8}{3} - 24 = \frac{16}{3} \text{ units.}$$

Producers' Surplus

$$PS = p_0x_0 - \int_0^{x_0} g(x)\,dx$$

$$= 24 - \int_0^2 (2x^2 + 4)\,dx \qquad = 24 - \left[\frac{2x^3}{3} + 4x\right]_0^2$$

$$= 24 - \frac{2 \times 8}{3} - 8 \quad = \quad \frac{32}{3} \text{ units.}$$

EXERCISE

11) The area of the region bounded by $y = x + 1$ the x - axis and the lines $x = 0$ and $x = 1$ is

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{3}{2}$ (d) 1

12) The area bounded by the demand curve $xy = 1$, the x - axis, $x = 1$ and $x = 2$ is

(a) log 2 (b) $\log \frac{1}{2}$ (c) 2 log 2 (d) $\frac{1}{2}$ log 2

13) If the marginal cost function $MC = 3e^{3x}$, then the cost function is

(a) $\frac{e^{3x}}{3}$ (b) $e^{3x}+k$ (c) $9e^{3x}$ (d) $3e^{3x}$

14) If the marginal cost function $MC = 2 - 4x$, then the cost function is

(a) $2x-2x^2+k$ (b) $2-4x^2$ (c) $\frac{2}{x} - 4$ (d) $2x - 4x^2$

15) The marginal revenue of a firm is $MR = 15 - 8x$. Then the revenue function is

(a) $15x-4x^2+k$ (b) $\frac{15}{x} - 8$ (c) -8 (d) $15x - 8$

16) The marginal revenue $R'(x) = \frac{1}{x+1}$ then the revenue function is

(a) $\log |x+1| + k$ (b) $-\frac{1}{(x+1)}$ (c) $\frac{1}{(x+1)^2}$ (d) $\log \frac{1}{x+1}$

17) The consumers' surplus for the demand function $p = f(x)$ for the quantity x_0 and price p_0 is

(a) $\int_0^{x_0} f(x)\,dx - p_0 x_0$ (b) $\int_0^{x_0} f(x)\,dx$

(c) $p_0 x_0 - \int_0^{x_0} f(x)\,dx$ (d) $\int_0^{p_0} f(x)\,dx$

18) The producers' surplus for the supply function $p = g(x)$ for the quantity x_0 and price p_0 is

(a) $\int_0^{x_0} g(x)\,dx - p_0 x_0$ (b) $p_0 x_0 - \int_0^{x_0} g(x)\,dx$

(c) $\int_0^{x_0} g(x)\,dx$ (d) $\int_0^{p_0} g(x)\,dx$

5 Matrices and Determinants

Chapter Includes:

1. Matrices to describe NETWORK
2. Order of a Matrix
3. Types of MatriCES
4. Algebra of Matrices
5. Transpose of a Matrix
6. addition and subtraction of matrices
7. Multiplication of Matrices
8. Symmetric and Skew - Symmetric Matrices Orthogonal Matrix
9. Nilpotent Matrix
10. Periodic Matrix
11. Idempotent Matrix
12. Involutery Matrix
13. Determinant of a Square Matrix
14. Singular and Non-Singular Matrices
15. Minors and Cofactors
16. Expansion of a determinantS
17. Elementary Properties of Determinants
18. Application of Determinants
19. Adjoint of a Square Matrix
20. Inverse of a Matrix
21. Elementary Operations on matrices
22. Echelon form of a Matrix
23. Solution of System of Linear Equations by Matrix Method
24. Solution of System of Linear Equations by Elementary Transformation (Operations)

INTRODUCTION :

Whenever we perform a journey by train/bus, we go to railway station/bus station and see the time table of trains/buses for our destination. The time of arrival and departure of trains/buses along with destinations are arranged in a rectangular arrays.

The student seating in class/examination hall, the cadets in parade ground, the price list of different articles in a shop, the days and dates in a calendar are arra ged in a rectangular arrays (rows and column).

Date / Days					
SUN		05	12	19	26
MON		06	13	20	27
TUE		07	14	21	28
WED	01	08	15	22	29
THU	02	09	16	23	30
FRI	03	10	17	24	
SAT	04	11	18	25	

The table is the shortest method of finding a lot of information. The information can be written without heading as shown below :

Let us consider the following table

Seating arrangement of students in an examination hall.

1	6	11	16	21
2	7	12	17	22
3	8	13	18	23
4	9	14	19	24
5	10	15	20	25

(iii) Price list of articles in three shops (in paise).

Shop / Articles	A	B	C
Tea	40	42	44
Sugar	60	62	64
Milk	42	44	46
Sliced bread	10	12	15

The above information can be written as :

Let us write the above information in the square brackets for good looking :

$$\begin{matrix} 1 & 6 & 11 & 16 & 21 \\ 2 & 7 & 12 & 17 & 22 \\ 3 & 8 & 13 & 18 & 23 \\ 4 & 9 & 14 & 19 & 24 \\ 5 & 10 & 15 & 20 & 25 \end{matrix} \qquad \begin{matrix} 40 & 42 & 44 \\ 60 & 62 & 64 \\ 42 & 44 & 46 \\ 10 & 12 & 15 \end{matrix}$$

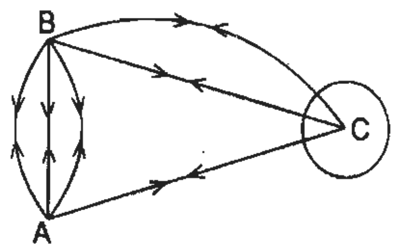
$$\begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 2 & 7 & 12 & 17 & 22 \\ 3 & 8 & 13 & 18 & 23 \\ 4 & 9 & 14 & 19 & 24 \\ 5 & 10 & 15 & 20 & 25 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 40 & 42 & 49 \\ 60 & 62 & 64 \\ 42 & 44 & 46 \\ 10 & 12 & 15 \end{bmatrix}$$

The rectangular arrangement of numbers inside the square brackets (more precisely brackets), is called Matrix.

5.1 MATRICES TO DESCRIBE NETWORK

Let us consider these cities A, B and C which are connected as shown in the figure.

It is seen clearly, that there are three routes in going from city A to city B. There is no any route from city A to A, and city B to B, but there are routes from city C to city C.



Let us denote no routes, one route, two routes, and three routes by the numbers 0, 1, 2, and 3 respectively

$$\text{from} \; \begin{array}{c} \\ A \\ B \\ C \end{array} \overset{\text{To}}{\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \end{array}} \qquad \text{from} \; \begin{array}{c} A \\ B \\ C \end{array} \overset{\text{To}}{\begin{bmatrix} A & B & C \\ 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}}$$

The above information can be represented by a matrix, which tells that matrices are the storage of information.

Clearly it is known that the geometrical model has been converted in to arithmetical table.

There are so many such examples which can be obtained from our practical life problems.

Definition 1.1 : A rectangular array of numbers having m and n-columns which represents $m \times n$ elements, is called a matrix of order $m \times n$.

The matrices are generally denoted by capital letters and their elements by small letters of English alphabets.

The $m \times n$ elements of a matrix can be written as :

$$A = \begin{bmatrix} a_{11}\ a_{12}\ a_{13}\ \cdots\cdots\ a_{ij}\cdots a_{1n} \\ a_{21}\ a_{22}\ a_{23}\ \cdots\cdots\ a_{ij}\cdots a_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{i1}\ a_{i2}\ a_{i3}\ \cdots\cdots a_{ij}\cdots a_{in} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}\ a_{m2}\ a_{m3}\ \cdots\cdots a_{mj}\cdots a_{mn} \end{bmatrix}_{m\times n}$$

here a_{ij} represents three elements of i^{th} row and j^{th} column. The suffix $'i'$ represents row and suffix $'j'$ represents column. Thus $(i, j)^{th}$ the element of first row are $((a_{11}, a_{12}, \ldots, a_{1n}]$ and that of first column are :

$$\begin{matrix} a_{11} \\ a_{12} \\ \cdots \\ \cdots \\ a_{1n} \end{matrix}$$

The Matrix A, can also be written as $A = [a_{ij}]_{m\times n}$

where $i = 1, 2, \ldots, m$ and

$j = 1, 2, \ldots, n$

The horizontal lines (→) and vertical lines (↓) in the matrix are called row (row vectors) and column (or column vectors).

5.2 ORDER OF A MATRIX

The order of a matrix having *m* rows and *n* columns is $m \times n$.

For example

The element '11' occurs in third row and second column.

Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 7 & 8 & 10 \\ 5 & 11 & 13 \end{bmatrix}_{3\times 3}$ is a 3×3 matrix, (as it has three rows and three columns. Here the element '11' occurs in the third row and second column.

5.3 TYPES OF MATRICES

(i) **Horizontal Matrix :** A matrix in which the number of rows is less than the number of columns, is called a Horizontal Matrix.

For Example : $\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix}$ is a horizontal matrix because it has three rows and two columns.

(ii) **Virtical Matrix :** A matrix in which the number of rows is more than the number of columns, is called a Virtical Matrix.

For Example :

$$\begin{bmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}$$ is a virtical matrix because it has four rows and three columns.

(iii) **Square Matrix :** A matrix in which number of rows is equal to the number of columns, is called a square matrix.

The matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix if and only if $m = n$, we write-

And $B = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2}$ or $D = \begin{bmatrix} 6 & 7 & 8 \\ 2 & 4 & 5 \\ 9 & 3 & 11 \end{bmatrix}_{3\times 3}$

$$A = \begin{bmatrix} a_{11}, a_{12}, \ldots, a_{ij}, a_{1n} \\ a_{11}, a_{12}, \ldots, \ldots, a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}, a_{n2}, \ldots\ldots a_{nn}, \end{bmatrix}_{n \times n} \quad \textit{is a square matrix of order } n (\textit{or } n \times n)$$

are examples of square matrices.

Diagonal elements of a Square Matrix : The elements $[a_{11}, a_{22}, a_{33}]$ in B, the element $[1, 5]$ in C and $[6, 4, 11]$ in D are the diagonal elements if these squares matrices. the diagonal elements of A are $[a_{11}, a_{12}, \ldots, a_{mn}]$ as $[a_{11}]$.

(vi) **Zero or Null Matrix :** A matrix in which all the elements are zero, is called a zero matrix, or a null matrix.

Example : $[0]_{1\times 1}$, $[0, 0]_{1\times 2}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

are the examples of null matrices.

(vii) **Diagonal Matrix :** A square matrix in which all the elements are zero except the diagonal elements

Example :

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}_{3\times 3}$$

is an example of diagonal matrix.

Particular Cases : The diagonal elements of a diagonal matrix may also be zero. For example

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}_{2\times 2}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

are also diagonal matrices, while their diagonal elements are zero.

Scalar Matrix : A diagonal matrix in which all the diagonal elements are equal, is called a scalar matrix.

Example :

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are scalar matrices as the diagonal elements is A are 4, in B are 3, and in C are 0.

Identity Matrix : A diagonal matrix in which the diagonal elements are equal to 1, is called an identity matrix or unit matrix. An identity matrix is denoted by I.

Example : $I_1 = [1]_{1\times 1}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

are identity matrices.

Note : An identity matrix is a scalar matrix as well as a diagonal matrix.

Triangular Matrices : Triangular matrices are of two types :

(a) **Lower Triangular Matrices :** A square matrix in which the elements above the main diagonal are all zero, is called a lower triangular matrix.

Example :

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 6 & 7 \end{bmatrix}_{3\times 3}$$

are lower triangular matrices.

(b) Upper Triangular Matrices : A square matrix in which the elements below the main diagonal are all zero, is called a uper triangular matrix.

Example :

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 9 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

are upper triangular matrices.

5.4 ALGEBRA OF MATRICES

5.4.1 (i) Equal Matrices : Two matrices $A = [a_{ij}], B = [b_{ij}]$ are said to be equal , if they have the same order and having the corresponding elements equal.

$$A = [a_{ij}]_{m\times n}, \quad B = [b_{ij}]_{m\times n}$$

the A = B, if $a_{ij} = b_{ij}$ $(i, j)^{th}$ element of A is equal to $(i, j)^{th}$ element of B.

Note : Same order means number of rows and the number of columns are equal in both the matrices.

5.5 TRANSPOSE OF A MATRIX

A matrix which is obtained by interchanging rows and columns of a given matrix is called transpose of the matrix.

The transpose of a matrix A is denoted by A' or A^T.

Example : Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 7 & 8 & 9 \end{bmatrix}_{2\times 3}$, then by interchanging row and columns of A,

We get $A' = \begin{bmatrix} 1 & 7 \\ 2 & 8 \\ 5 & 9 \end{bmatrix}_{3\times 2}$ Now transpose of $A' = \begin{bmatrix} 1 & 2 & 5 \\ 7 & 8 & 9 \end{bmatrix} = A$

Thus, the transpose of a transpose of a matrix, is the matrix itself i.e : $(A')' = A$

Thus we see that in the matrix A, the number of rows 2 changed to three rows in its transpose, and number of columns 3 are changed to two columns.

5.6 ADDITION AND SUBTRACTION OF MATRICES

(a) Addition of Matrices : If A and B are two matrices having the same order, then they can be added and the resulting matrix can be obtained by adding the corresponding elements of the matrices A and B. The sum is denoted by A + B.

for if $\quad A = \begin{bmatrix} 3 & 7 & 4 \\ 5 & 2 & -1 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 9 & 0 & 5 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3}$

then $$A+B=\begin{bmatrix}3+9 & 7+0 & 4+5\\ 5+2 & 2+3 & -1+4\end{bmatrix}$$

$$=\begin{bmatrix}12 & 7 & 9\\ 7 & 5 & 3\end{bmatrix}_{2\times 3}$$

In particular, if $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{ij}\right]_{m\times n}$

then $A+B=\left[a_{ij}+b_{ij}\right]_{m\times n}$

(b) Subtraction of Matrices : In subtraction of Matrices of the same order, the difference of corresponding elements are obtained and the order of the resulting matrix is same as the order of the two matrices.

For if $A=\begin{bmatrix}2 & 4\\ 3 & 6\\ 5 & 7\end{bmatrix}_{2\times 3}$ $B=\begin{bmatrix}1 & 3\\ 2 & 5\\ 4 & 6\end{bmatrix}_{2\times 3}$

then $$A-B=\begin{bmatrix}2-1 & 4-3\\ 3-2 & 6-5\\ 5-4 & 7-6\end{bmatrix}$$

$$=\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}_{2\times 3}$$

In particular, if $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{ij}\right]_{m\times n}$

then $A-B=\left[a_{ij}-b_{ij}\right]_{m\times n}$

5.6.1 PROPERTIES OF MATRIX ADDITION :

Property (i) : Matrix addition satisfies the commutative property,

As $A+B=B+A$

If $A=\left[a_{ij}\right]$; $B=\left[b_{ij}\right]$

then $(i,j)^{th}$ element of $A+B=\left[a_{ij}+b_{ij}\right]$ and of $B+A=\left[b_{ij}+a_{ij}\right]$

which are same as a_{ij} and b_{ij} are numbers and satisfy the commutative property of addition.

Proof (i) :

Let $A = [a_{ij}]$, $B = [b_{ij}]$ then

$$A + B = ([a_{ij}] + [b_{ij}]) = [a_{ij} + b_{ij}]$$

$$= [b_{ij} + a_{ij}] \text{ (Addition of numbers is commutative)}$$

$$= ([b_{ij}] + [a_{ij}])$$

$$= B + A$$

Property (ii) : Matrix addition satisfies the associative property,

i. e. $A + (B + C) = (A + B) + C$

> If $A = [a_{ij}]$; $B = [b_{ij}]$ and $C = [c_{ij}]$
> then $(i, j)^{th}$ element of $A + (B + C)$ will be $a_{ij} + (b_{ij} + c_{ij})$
> and of $(A + B) + C$ is $[(a_{ij} + b_{ij}) + c_{ij})]$

Proof (ii) :

Let $C = [c_{ij}]$, then

$$(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$$

$$= [a_{ij} + b_{ij}] + [c_{ij}] \quad \text{(By definition of } A + B\text{)}$$

$$= [(a_{ij} + b_{ij}) + c_{ij}] \quad \text{(Addition of numbers is associative)}$$

$$= [(a_{ij}) + (b_{ij} + c_{ij})]$$

$$= [a_{ij}] + [b_{ij} + c_{ij}]$$

$$= [a_{ij}] + ([b_{ij} + c_{ij}])$$

$$= A + (B + C)$$

Property (iii) : Existence of Identity for Matrix Addition

If A be a matrix of order $m \times n$, and O be null matrix of he same order $m \times n$, then

$$A + O = O + A$$

For $A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 9 \end{bmatrix}_{2\times 3}$ $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

Then $A + O = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & 3+0 & 4+0 \\ 5+0 & 7+0 & 9+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 9 \end{bmatrix}$$

And $$O + A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 0+3 & 0+4 \\ 0+5 & 0+7 & 0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 9 \end{bmatrix}_{2\times 3}$$

$$A + O = O + A$$

Proof (iv) : Existence of Additive Inverse

For every matrix A, there exists a matrix B of the same order that of A, such that

$A + B = O = B + A$

Where O is a null matrix whose order is equal to the order of A (or B).

B is called additive inverse O or negative of matrix A.

Proof (v) : Cancellation Law for Addition of Matrices

If A, B, C are three matrices of the same order then

$A + B = A + C \Rightarrow B = C$ (1)

And $B + A = C + A \Rightarrow B = C$ (2)

Equations (1) and (2) show left cancellation law and right cancellation law respectively.

5.7 MULTIPLICATION OF MATRICES

5.7.1 Multiplication of Matrix by a Scalar : Let A be any matrix of order $m \times n$ and k be a scalar (Complex or real). Then the matrix kA, which is obtained by multiplying each element of A by the scalar k *is called scalar multiple of A.*

Example :

Let $A = \left[a_{ij}\right]_{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{ij} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Then $$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & \cdots & ka_{ij} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \cdots & ka_{2j} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ ka_{i1} & ka_{i2} & ka_{i3} & \cdots & ka_{ij} & \cdots & ka_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & ka_{m3} & \cdots & ka_{mj} & \cdots & ka_{mn} \end{bmatrix}$$

$\therefore A = [a_{ij}]_{m\times n}$ then $kA = [ka_{ij}]_{m\times n}$

Particularly, If k = 3, and $A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 7 & 8 \end{bmatrix}_{2\times 3}$

Then $$3A = \begin{bmatrix} 3\times 1 & 3\times(-1) & 3\times 5 \\ 3\times 6 & 3\times 7 & 3\times 8 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 3 & -3 & 15 \\ 18 & 21 & 24 \end{bmatrix}_{2\times 3}$$

5.7.2 Properties of Multiplication of a Matrix by a Scalar :

(i) Scalar Multiplication is distributive over matrix addition. For, if A and B are any two matrices of the same order, then

$k(A + B) = kA + kB$

(ii) For any two scalar r and s and the matrix A of any order $m \times n$, then

$$(r+s)A = rA + sA \quad \text{and}$$
$$r(sA) = (rs)A$$

(iii) For any matrix A of order $m \times n$ and scalar k

$$(-k)A = -(kA) = k(-A)$$

(iv) For any matrix A of order $m \times n$.

(a) $1A = A$ (b) $(-1)A = -A$

(v) For any two matrices A and B

$$-(A+B) = -A - B$$

5.7.3 Multiplication of a Matrix by another Matrix :

Two matrices A and B can be multiplied if the number of columns in A is same as the number of rows in B. And as such the matrices are said to be conformable for multiplication.

If A be matrix of order $m \times n$ and B be matrix of order $n \times p$ then (Product) AB can be bt i d b t BA t b bt i d

The product of Matrices A and B of orders $m \times n$ and $n \times p$ respectively denoted by (is or order $m \times p$ for

$$A = [a_{ij}]_{m\times n}, \quad B = [b_{ij}]_{n\times p}, \quad then \quad C = [c_{ij}]_{m\times p}$$

is given by -

$$c_{ij} = \sum_{j=1}^{n} a_{ij} b_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots \; a_{in}b_{nk}$$

$$c_{ij} = a_{i1}b_{i1} = a_{i2}b_{2k} + \dots \; a_{in}b_{nk}$$

$$= \sum_{j=1}^{n} a_{ij} b_{jk}; \quad i = 1,2 \ldots m \quad k = 1,2 \ldots \; p.$$

$= a_{ij}b_{jk}$, where i the dummy suffix.

Diagrammatically

$$\begin{bmatrix} c_{11} \ldots c_{1k} \ldots c_{1p} \\ \ldots\ldots\ldots\ldots \\ c_{i1} \ldots c_{ik} \ldots c_{ip} \\ \ldots\ldots\ldots\ldots \\ c_{mi} \ldots c_{mk} \ldots c_{mp} \end{bmatrix} = \begin{bmatrix} a_{11} a_{12} \ldots a_{ij} \ldots a_{1n} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{i1} a_{i2} \ldots a_{ij} \ldots a_{in} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{m1} a_{m2} \ldots a_{mj} \ldots a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} \ldots b_{1k} \ldots b_{1p} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ b_{j1} \ldots b_{jk} \ldots b_{jp} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ b_{n1} \ldots b_{nk} \ldots b_{np} \end{bmatrix}$$

Thus, to obtain c_{ik}, we multiply each element in the i^{th} zero of A by the corresponding elements in j^{th} column of B and find the sum of all the terms. matrix product AB,A is called pre-multiplier and B the post multiplier.

Particularly, if

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 0 \end{bmatrix}_{2\times3} \text{ and } B = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 4 \\ 3 & -2 & 6 \end{bmatrix}_{3\times3}$$

Now $$AB = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 0 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 3 & 4 \\ 5 & 9 & 4 \\ 3 & -2 & 6 \end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix} 3\times1 + 4\times5 + 6\times3 & 1\times3 + 4\times9 + 6\times(-2) & 3\times4 + 4\times4 + 6\times6 \\ 1\times1 + 2\times5 + 0\times3 & 1\times3 + 2\times9 + 0\times(-2) & 1\times4 + 2\times4 + 0\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 27 & 64 \\ 11 & 21 & 12 \end{bmatrix}_{2\times3}$$

5.7.4 Properties of Matrix Multiplication :

(i) Associative Property

For the matrices A B C of order $m \times n$ $n \times p$ and $p \times r$ respectively then

$$(AB)C = (AB)C$$

(ii) Distributive Property :

For the matrices A, B, C of order $m \times n$, $n \times p$ and $n \times p$

$$A(B+C) = AB + AC$$

i. e. Multiplication distributive addition.

(iii) Matrix Multiplication is not always commutative :

i. e. If A and B are matrices such that AB and BA is defined, then it is not necessary that AB = BA

for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} -7 & 2 \\ 1 & 4 \end{bmatrix}_{2\times 2}$

Since the both matrices have the same order, therefore AB and BA both are defined. Further -

$$AB = \begin{bmatrix} -7+2 & 2+8 \\ -21+4 & 6+16 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ -17 & 22 \end{bmatrix}$$

$$BA = \begin{bmatrix} -7+6 & -14+8 \\ 1+12 & 2+16 \end{bmatrix} = \begin{bmatrix} -1 & -6 \\ 13 & 18 \end{bmatrix}$$

Clearly $AB \neq BA$

Note : (i) If AB = BA, then A and B are said to be commute.

(ii) If AB = –BA, the matrices A and B are said to be anti - commute.

(iv) The product of two matrices A and B can be a zero matrix, it does not meant that either A or B is a null matrix or both the matrices are null.

i. e. AB = O does not mean that either A= O or B = O or both be a zero matrix.

For $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0\times 3+1\times 0 & 0\times 4+1\times 0 \\ 0\times 3+2\times 0 & 0\times 4+2\times 0 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Thus AB = O, i.e. the product of two non-zero matrix is a zero matrix.

5.8 SYMMETRIC AND SKEW - SYMMETRIC MATRICES

5.8.1 Symmetric Matrix :

A square matrix is said to be symmetric, if it is equal to its transpose.

If $A = A'$ then the square matrix of A is said to be symmetric.

Example :

Let $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

$\because A = A'$ and consequently A is a symmetric matrix.

If $A = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & c_2 & a_2 \\ b_2 & a_2 & a_3 \end{bmatrix}$, then $A' = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & c_2 & a_2 \\ b_2 & a_2 & a_3 \end{bmatrix}$

Hence A is symmetric.

$A = \begin{bmatrix} a & h & g \\ h & b & g \\ g & j & c \end{bmatrix}$ then $A' = \begin{bmatrix} a & h & g \\ h & b & g \\ g & j & c \end{bmatrix}$

And $A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 0 \end{bmatrix}$, then $A' = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

are examples of symmetrical matrices.

5.8.2 Skew - Symmetric Matrix :

A square matrix is said to be skew-symmetric, if its transpose is equal to (–1) times the matrix.

If A is square matrix, and if

$A' = -A$, then A is called skew symmetric.

Example : $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -a & -c & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -4 & -6 \\ 4 & 0 & 8 \\ -6 & 8 & 0 \end{bmatrix}$

are example of skew-symmetric Matrices.

Note : It is clear that the diagonal elements of a skew symmetric matrix are all zero.

(Let $A' = A$, and $A = [a_{ij}]$ then $A' = [a_{ji}]$, as A is skew symmetric.

or $2\,a_{ij} = 0$

For diagonal elements $i = j$

$\therefore$ i.e. $a_{ij} = 0$

$a_{11} = a_{22} = a_{23} \;\dots\; a_{mn} = 0$

5.9 ORTHOGONAL MATRIX

A square matrix is said to be orthogonal if ϕ the product of matrix and its transpose is equal to an identity matrix of the same order.

If A be a square matrix such that $AA' = I$ then A is said be orthogonal.

Example : Let $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$

$$A' = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}$$

Now, $$AA' = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\alpha + \cos^2\alpha & -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha & \cos^2\alpha + \sin^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

5.10 NILPOTENT MATRIX

A square matrix A is said to be nilpotent matrix of index n, if $A^n = O$, where O is null matrix of the same order as A.

In particular if $A^2 = O$, then matrix A is called nilpotent matrix of index 2,

Example : $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, is a nilpotent of order 2.

5.11 PERIODIC MATRIX

A square matrix A is said to be periodic of period k, if $A^{k+11} = A$, when k is the least positive integer.

Example : $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$, is periodic of period 2.

5.12 IDEMPOTENT MATRIX

A square matrix is said to be idempotent if is square is the matrix itself.

Let A be a square matrix such that $A^2 = A$, then it is called idempotent matrix.

Example : The matrix $A = \begin{bmatrix} 2 & -2 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$, is an idempotent matrix

5.13 INVOLUTERY MATRIX

A square matrix is said to be involutery if its square is an identity matrix of the same order.

If A is a square matrix and if $A^2 = I$ then A is called involutery

Example : The matrix $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$, is an involutery matrix.

SOLVED EXAMPLE

Example 1 : Let $A = \begin{bmatrix} 0 & -\tan x/2 \\ \tan x/2 & 0 \end{bmatrix}$ and I, the identity matrix of order 2, then

prove that $I + A = (I - A)\begin{bmatrix} \cos x & -\sin x \\ \sec x & \cos x \end{bmatrix}$

Solution : $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\tan\frac{x}{2} \\ \tan\frac{x}{2} & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -\tan\frac{x}{2} \\ \tan\frac{x}{2} & 1 \end{bmatrix}$$

$$(I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{x}{2} \\ \tan\frac{x}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot & \tan\frac{x}{2} \\ -\tan\frac{x}{2} & 1 \end{bmatrix}$$

Now, $(I - A)\begin{bmatrix} \cos x & -\sin x \\ \sec x & \cos x \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{x}{2} \\ \tan\frac{x}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

$$= \begin{bmatrix} \cos x + \tan x/2 \sin x & -\sin x + \tan x/2 \cos x \\ -\tan x/2 \cos x + \sin x & +\tan x/2 \sin x + \cos x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\frac{x}{2} - \sin^2\frac{x}{2} + \frac{\sin x/2}{\cos x/2} \times 2\sin\frac{x}{2}\cos\frac{x}{2}, & -2\sin\frac{x}{2} \times \cos\frac{x}{2} + \frac{\sin x/2}{\cos x/2}(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}) \\ -\frac{\tan x/2}{\cos x/2}(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}) + 2\sin\frac{x}{2}\cos\frac{x}{2}, & \frac{\sin x/2}{\cos x/2} \times 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\frac{x}{2}+\sin^2\frac{x}{2} & -2\sin\frac{x}{2}\cos\frac{x}{2}+\sin\frac{x}{2}\times\cos\frac{x}{2}-\frac{\sin^3 x/2}{\cos x/2} \\ -\sin\frac{x}{2}\cos\frac{x}{2}+\frac{\sin^3 x/2}{\cos^2 x/2}+2\sin\frac{x}{2}\cos\frac{x}{2} & 2\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{\sin x/2}{\cos x/2}\ (\sin^2\frac{x}{2}\ \cos^2\frac{x}{2}) \\ \frac{\sin x/2}{\cos x/2}\ (\sin^2\frac{x}{2}\ \cos^2\frac{x}{2}) & \sin^2\frac{x}{2}+\cos^2\frac{x}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{x}{2} \\ \tan\frac{x}{2} & 1 \end{bmatrix}$$

EXERCISE 1.1

1. **Construct a 3×4 matrix whose elements are :**

   (i) $a_{ij} = 2i - j$ (ii) $a_{ij} = i + j$ (iii) $a_{ij} = \frac{i}{j}$ (iv) $a_{ij} = \frac{i-j}{i+j}$

2. If $A = \begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+20 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$, find x, y, z

3. If $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$. then prove that $(AB)' = B'A'$ where A' and B' are respectively transpose of matrices A and B.

4. Show that the elements on the main diagonal of a skew-symmetric matrix are all zeros.

5. If $f(x) = x^2 - 5x + 6$ find $f(A)$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

6. If A be a square matrix, show that $\frac{1}{2}(A + A')$ is a symmetric and $\frac{1}{2}(A - A')$ is a skew symmetric matrix.

7. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ find A^2 and show by mathematical induction that

   $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ for every positive integer n.

8. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find A^2 and show by mathematical induction that

   $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ for every positive integer n.

9. If $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, $B=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$ find AB and BA and show that $A^2B+B^2A=A$.

5.14 DETERMINANT OF A SQUARE MATRIX

Every square matrix $A=\left[a_{ij}\right]$ is associated with a number called determinant of A, and denoted by del A or $|A|$ or $|a_{ij}|$ and sometimes also by the symbol Δ.

Thus only the square matrices have their determinant.

5.14.1 Determinant of 1×1, 2 × 2 and 3 × 3 matrices :

(i) Let $A=[a]_{1\times1}$ matrix

Then del $A=|A|=|a|=a$

(ii) Let $A=\left[a_{ij}\right]_{2\times2}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$

Then del $A=|A|=\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}=a_{11}\times a_{22}-a_{12}\times a_{21}$

(iii) Let $A=\left[a_{ij}\right]_{3\times3}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}_{3\times3}$ then

$$\text{del } A=|A|=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}$$

$$=a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}$$

$$=a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}a_{31}\right)$$
$$+a_{13}\left(a_{21}a_{32}-a_{22}a_{31}\right)$$

5.14.2 The determinant of a diagonal matrix : The determinant of a diagonal matrix is equal to the product of diagonal elements.

For $A=\begin{bmatrix}2&0\\0&4\end{bmatrix}$ then det $A=\begin{vmatrix}2&0\\0&4\end{vmatrix}=2\times4-0\times0=8$

and $B=\begin{bmatrix}3&0&0\\0&4&0\\0&0&5\end{bmatrix}$, then det $B=\begin{vmatrix}3&0&0\\0&4&0\\0&0&5\end{vmatrix}$

$$\det \quad A = 3\begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} 0 & 7 \\ 0 & 0 \end{vmatrix}$$

$$= 3 \times (4 \times 5 - 0) - 0(0 \times 5 - 0) + 0 \times (0 - 0)$$

$$= 3 \times 4 \times 5$$

Thus in general if...

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \dots & \dots \\ 0 & \dots & \dots & a_{nn} \end{bmatrix} \text{than del } A = \begin{vmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \dots & \dots \\ 0 & \dots & \dots & a_{mn} \end{vmatrix}$$

Then det $A = a_{11} \times a_{22} \times a_{33} \times \dots a_{mn}$, which is the product of the diagonal elements of the Matrix A.

5.15 SINGULAR AND NON-SINGULAR MATRICES

5.15.1 Singular Matrix : If the value of the determinant of a square matrix is zero, it is called singular matrix.

If A be a square matrix and $|A| = 0$, then A is called singular matrix.

Example : $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}_{2\times3}$ $|A| = \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 4 \times 3 - 2 \times 6 = 12 - 12 = 0$

$$B = \begin{bmatrix} 4 & 6 & 12 \\ 4 & 8 & 16 \\ 6 & 10 & 20 \end{bmatrix} \quad \text{for del } B = \begin{vmatrix} 4 & 6 & 12 \\ 4 & 8 & 16 \\ 6 & 10 & 20 \end{vmatrix}$$

Let $B = \begin{vmatrix} 8 & 16 \\ 10 & 20 \end{vmatrix} - 6\begin{vmatrix} 4 & 16 \\ 6 & 20 \end{vmatrix} + 12\begin{vmatrix} 4 & 8 \\ 6 & 10 \end{vmatrix}$

$$= 2(8 \times 20 - 16 \times 10) - 6|4 \times 20 - 6 \times 16| + 12(4 \times 10 - 8 \times 6)$$

$$= 2(160 - 160) - 6(80 - 96) + 12(40 - 48)$$

$$= 2 \times 0 - 6 \times (-16) + 12 \times (-8)$$

$$= +96 - 96$$

$$= 0$$

are singular matrices.

Note : the determinant of singular matrix have a single value equal to zero irrespective of its order.

5.15.2 Non - Singular Matrix : If the value of the determinant of a square matrix, is not equal to zero it is called a non-singular matrix.

If A is a square matrix and if del A or $|A|$, is not equal to zero it is called non-singular matrix.

Example : $A = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$ is a non-singular matrix.

as $\det A = |A| = \begin{vmatrix} 6 & 4 \\ 2 & 8 \end{vmatrix} = 6 \times 8 - 4 \times 2$

$= 48 - 8 = 40 \neq 0$

Note : the determinant of non-singular matrix can have any value other than zero.

5.16 MINORS AND COFACTORS

5.16.1 Minor of an element of a determinant :

Consider the square matrix $A = \left[a_{ij}\right]_{m \times n}$ then its determinant $|A| = |a_{ij}|$

$$\text{Now } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \qquad (1)$$

The $(i, j)^{th}$ element is a_{ij}. If we supress the elements of first row and first column, in the above determinant then we get a determinant det B, (Δ_1) whose order is less thanone that of the determinant A. If $|A|$ is of order n, then the order of the determinant B (Δ_1) will be $(n-1)$.

$$\text{Then} \quad \det B = \Delta_1 = \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & & & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

This determinant Δ_1 is defined as the minor of the element a_{11}.

Definition 16.1 : Thus minor of an element is defined as the determinant obtained by deleting the elements of the corresponding row and column in which the element lies. The minor of element a_{ij} can be obtained by deleting the elements of the j^{th} row and j^{th} column. The minor of the element, as of a determinant is denoted by M_{ij}.

5.16.2 COFACTOR OF AN ELEMENT OF A DETERMINANT

The cofactor of an element is defined as the product of $(-1)^{i+j}$ and its minor. If M_{ij} be the minor of a_{ij}, then its cofactor C_{ij} is given by :.

$C_{ij} = (-1)^{i+j} M_{ij}$ i, and j represent the corresponding row and column inwhich element lies.

Note : if $i+j =$ even number the cofactor of the element is same as its minor. If $i+j =$ odd, then (c_{ij}) is negative for corresponding minor.

Example : Find all minors and cofactor of the determinant.

$$|A| = \Delta = \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix}$$

Solution : Minor of a $= M_{ij} = \begin{vmatrix} m & n \\ q & r \end{vmatrix}$

Cofactor of a $= c_{11} = (-1)^{1+1} \begin{vmatrix} m & n \\ q & r \end{vmatrix} = \begin{vmatrix} m & n \\ q & r \end{vmatrix}$

Here M_{11} and C_{11} means minor and cofactor of the element of first row and first column respectively.

Minor of $b = M_{12} = \begin{vmatrix} l & m \\ p & r \end{vmatrix}$, and $C_{12} = (-1)^{1+2} \begin{vmatrix} l & m \\ p & r \end{vmatrix} = -\begin{vmatrix} l & m \\ p & r \end{vmatrix}$

Minor of $c = M_{13} = \begin{vmatrix} l & m \\ p & q \end{vmatrix}$, and $C_{13} = (-1)^{1+3} \begin{vmatrix} l & m \\ p & q \end{vmatrix} = \begin{vmatrix} l & m \\ p & q \end{vmatrix}$

Minor of $l = M_{21} = \begin{vmatrix} b & c \\ q & r \end{vmatrix}$, and $C_{21} = (-1)^{2+1} \begin{vmatrix} b & c \\ q & r \end{vmatrix} = -\begin{vmatrix} b & c \\ q & r \end{vmatrix}$

Minor of $m = M_{22} = \begin{vmatrix} a & c \\ p & r \end{vmatrix}$, and $C_{22} = (-1)^{2+2} \begin{vmatrix} a & c \\ p & r \end{vmatrix} = \begin{vmatrix} a & c \\ p & r \end{vmatrix}$

Minor of $n = M_{23} = \begin{vmatrix} a & b \\ p & q \end{vmatrix}$, and $C_{23} = (-1)^{2+3} \begin{vmatrix} a & b \\ p & q \end{vmatrix} = -\begin{vmatrix} a & b \\ p & q \end{vmatrix}$

Minor of $p = M_{31} = \begin{vmatrix} b & c \\ m & n \end{vmatrix}$, and $C_{31} = (-1)^{3+1} \begin{vmatrix} b & c \\ m & n \end{vmatrix} = \begin{vmatrix} b & c \\ m & n \end{vmatrix}$

Minor of $q = M_{32} = \begin{vmatrix} a & c \\ l & m \end{vmatrix}$, and $C_{32} = (-1)^{3+2} \begin{vmatrix} a & c \\ l & m \end{vmatrix} = -\begin{vmatrix} a & c \\ l & m \end{vmatrix}$

Minor of $r = M_{33} = \begin{vmatrix} a & b \\ l & m \end{vmatrix}$, and $C_{33} = (-1)^{3+3} \begin{vmatrix} a & b \\ l & m \end{vmatrix} = \begin{vmatrix} a & b \\ l & m \end{vmatrix}$

Now, it is easy to define the value of the determinant $|A| = |a_{ij}|$ of order n, by

$$|A| = \Delta = \sum_{0=1}^{n} (-1)^{i+j} a_{ij} \ M_{ij}$$

$$= \sum_{0=1}^{n} a_{ij} \ C_{ij} \qquad \ldots(1)$$

When M_{ij} and C_{ij} respectively denote the minors and cofactors of the elements a_{ij}

In the expression (1) the summation $j = 1, \ldots n$. shows that the determinant has been expanded along i^{th} row.

If the determinant is expanded along j^{th} column, its value will be given by ;

$$|A| = \Delta = \phi \sum_{0=1}^{n} (-1)^{i+j} \ a_{ij} M_{ij}$$

$$= \sum_{i=1}^{n} i + j \qquad a_{ij} C_{ij}$$

The value of the determinant is not affected by expanding it along any row or column, this can be verified by the following example.

5.17 EXPANSION OF A DETERMINANTS

Let us consider a determinant of order 3 of a square matrix -

$$A = [a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \ A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Now, to expand the above determinant we assign the sign to each element. The first element a_{11} is assigned the + sign and a_{12} assigned − sign. a_{13} is assigned + sign and so on. This assignment is made for the element of the column too.

$$\text{Thus we have del } A = |A| = \Delta = \begin{vmatrix} \overset{+}{a}_{11} & \overset{-}{a}_{12} & \overset{+}{a}_{13} \\ \overset{-}{a}_{21} & \overset{+}{a}_{22} & \overset{-}{a}_{23} \\ \overset{+}{a}_{31} & \overset{-}{a}_{32} & \overset{+}{a}_{33} \end{vmatrix}$$

In the expression of the above determinant along any row (or column, the element of the row (or column) are written with the assigned sign and then multiplied by their corresponding minors. The expansion along first row, of the above determinant is given below :

$$\Delta = a_{11}\begin{vmatrix} a_{21} & a_{22} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the expression is done by deleting the row and columns in which the element belongs.

Note : A determinant can be expressed along any row or column but the sign of the elements must be according the following rule.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Example : Let $\Delta = \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix}$

Now expanding the above determinant

(i) Along first row -

$$\Delta = a\begin{vmatrix} m & n \\ q & r \end{vmatrix} - b\begin{vmatrix} l & n \\ p & r \end{vmatrix} + c\begin{vmatrix} l & m \\ p & q \end{vmatrix}$$

$$= a(mr - nq) - b(lr - np) + c(lq - mp)$$

$$= amr - anq - blr - bnp + clq - cmp$$

(ii) Along first column

$$\Delta = a\begin{vmatrix} m & n \\ q & r \end{vmatrix} - l\begin{vmatrix} b & c \\ p & r \end{vmatrix} + p\begin{vmatrix} b & c \\ m & n \end{vmatrix}$$

$$= a(mr - nq) - l(br - cq) + p(bn - cm)$$

$$= amr - anq - lbr - lcq) + pbn - pcm$$

(iii) Along third column

$$\Delta = c\begin{vmatrix} l & m \\ p & q \end{vmatrix} - n\begin{vmatrix} a & b \\ p & q \end{vmatrix} + r\begin{vmatrix} a & b \\ l & m \end{vmatrix}$$

$$= c(lq - mp) - n(aq - bp) + r(am - bl)$$

$$= clq - cmp - naq - nbp + ram - rbl$$

Clearly the expression for Δ in (i), (ii) and (iii) are same.

5.18 ELEMENTARY PROPERTIES OF DETERMINANTS

The expansion of determinants can be done by very easily by using the following properties. These properties can be applied for determinants of any order. However, we, shall describe the properties for the determinants of order 2 and 3.

Property 1: If every element of a row (or column) is zero, then the value of determinant is also zero.

Example :

$$\begin{vmatrix} 4 & 10 & 8 \\ 0 & 0 & 0 \\ 4 & 8 & -4 \end{vmatrix} = -0\begin{vmatrix} 10 & 8 \\ 8 & -4 \end{vmatrix} + 0\begin{vmatrix} 4 & 8 \\ 4 & -4 \end{vmatrix} - 0\begin{vmatrix} 4 & 10 \\ 4 & 8 \end{vmatrix}$$

$$= 0$$

Expanding along second row.

Property 2: If the rows and columns of a determinant are exchanged the value of the determinant remains same.

Example :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad and \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

$$\therefore \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Similarly, $\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = (er - qf) - b(dr - pf) + c(dq - ep)$

and $\begin{vmatrix} a & d & p \\ b & e & q \\ c & f & r \end{vmatrix} = a(er - fq) - d(br - cq) + p(bf - ce)$

$$= a(er - qf) - b(dr - pf) + c(dq - ep)$$

from above $\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & d & p \\ b & e & q \\ c & f & r \end{vmatrix}$

Property 3 : If any two adjacent row (or columns) of a determinant are interchanged the sign of the determinant is changed.

Example :

$$\begin{vmatrix} c & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

$$= -(ad - bc)$$

$$= -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Similarly for, $\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = a(er - qf) - b(dr - pf) + c(dq - ep)$

and $\begin{vmatrix} d & e & f \\ a & b & c \\ p & q & r \end{vmatrix} = d(br - qc) - e(ar - cp) + f(aq - bp)$

$$= -[a(er - qf) - b(dr - pf) + c(dq - ep)$$

$$= -\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}$$

$$\begin{vmatrix} d & e & f \\ a & b & c \\ p & q & r \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}$$

Note : If the rows (or columns) are exchanged odd times the sign of the determinant is changed but if they are exchanged even times the sign of the determinant does not change.

Property 4 : If any two rows (or two columns) of a determinants are identical, the value of determinant becomes zero.

For $\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ p & q & r \end{vmatrix}$, then first and second rows are identical.

Interchanging first and second row and using property 3.

$$\Delta = -\begin{vmatrix} a & b & c \\ a & b & c \\ p & q & r \end{vmatrix} = -\Delta$$

$$\therefore \Delta + \Delta = 0 \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$$

or $2\Delta = 0$ or $\Delta = 0$

Note : The value of the determinant used above will also be zero, if we expand it.

$$\Delta = a(br - qc) - b(ar - cp) + c(aq - bp)$$

$$\Delta = abr - aqc - bar + bcp) + caq - cbp = 0$$

Property 5 : If any row (or column) of a determinant is multiplied by any non-zero number then the value of the determinant gets multiplied by the number.

For $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}$, Let us multiply first row by k, then

$$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ p & q & r \end{vmatrix} = ka(er - qf) - kb(dr - pf) + kc(dq - ep)$$

$$= k[(er - qf) - b(dr - pf) + c(dq - ep)]$$

$$= k\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = k\,\Delta$$

Similarly $$= \begin{vmatrix} ka & kb & kc \\ ld & le & lf \\ p & q & r \end{vmatrix} = kl\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = k\,l\,\Delta$$

Note : Property 5 is of great importance. It is clear that if any row (or column) has common factor, then it can be taken out of the determinant.

Conversely, if we multiply any row (or column) of a determinant, by a constant then we divide the entire determinant by the same constant to keep the same value of the determinant.

$$\therefore \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = \frac{1}{k}\begin{vmatrix} ka & kb & kc \\ d & e & f \\ p & q & r \end{vmatrix}$$

Property 6 : If each element of a row (or column) of a determinant is the sum of two quantities then the determinant can be written as the sum of two determinant of the
same order.

For $$\begin{vmatrix} a+l & b & c \\ d+m & e & f \\ p+n & q & r \end{vmatrix} = (a+l)(er - qf) - (b+m)(br - cq) + (p+n)(bf - ec)$$

(On expanding along first column)

$$= a(er - qf) + d(br - cq) + p(bf - ec) + l(er - qf) - m(br - cq) + n(bf - ec)$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} + \begin{vmatrix} l & b & c \\ m & e & f \\ n & q & r \end{vmatrix}$$

Similarly we can show that

$$\begin{vmatrix} a+k_1 & b+l_1 & c \\ d+k_2 & e+l_2 & f \\ p+k_3 & q+l_3 & r \end{vmatrix} = \begin{vmatrix} a & b+l_1 & c \\ d & e+l_2 & f \\ p & q+l_3 & r \end{vmatrix} + \begin{vmatrix} k_1 & b+l_1 & c \\ k_2 & e+l_2 & f \\ k_3 & q+l_3 & r \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & l_1 & c \\ d & l_2 & f \\ p & l_3 & r \end{vmatrix} + \begin{vmatrix} k_1 & b & c \\ k_2 & e & f \\ k_3 & q & r \end{vmatrix} + \begin{vmatrix} k_1 & l_1 & c \\ k_2 & l_2 & f \\ k_3 & l_3 & r \end{vmatrix}$$

Property 7 : If each element of any row (or column) is added with the multiple of corresponding element of another row (or column) the value of the determinant is unchanged.

For let $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a+kp & b+kq & c+kr \\ d & e & f \\ p & q & r \end{vmatrix}$

here Δ_1 is the determinant obtained by Δ where k times of each element of the third row is added to the corresponding elements of the first row.

Now, $\Delta_1 = \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} + k\begin{vmatrix} p & q & r \\ d & e & f \\ p & q & r \end{vmatrix}$ (by property 5 and 6)

$= \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} + k0$ (by property of 4) as two rows are identical.

$\therefore \Delta = \Delta_1$

Property 8 : The sum of the product of the element of any row (or column) with the cofactor of the corresponding elements of another row (or column) is zero.

For $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}$

Let the cofactors of elements of first row be A_1, A_2 *and* A_3 are multiplied by the elements then of the second row and added, we get

$$dA_1 + eA_2 + fA_3 = d\begin{vmatrix} e & f \\ q & r \end{vmatrix} + e\left\{-\begin{vmatrix} d & f \\ p & r \end{vmatrix}\right\} + f\begin{vmatrix} d & e \\ p & q \end{vmatrix}$$

$$= d(er - qf) - e(dr - pf) + f(dq - ep)$$

$$= der - dqf - edr - epf + fdq - fep$$

$$= 0$$

Property 9 : If the determinant contains a variable x, and the element are polynomials in x such that putting x = a, the value of determinant becomes zero, then (x-a) is a factor of the determinant.

Since the element of the determinant are polynomials in x, therefore the expansion of thedeterminant will also be a polynomial in x. As the value of determinant becomes zero by putting $x = a$, then $(x - a)$ is a factor of the polynomial after expansion. Thus $(x - a)$ is a factor of the determinant.

for

$$\text{let} \quad \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

In second row if we write $y = x$, then the first and second rows become identical consequently the value of the determinant becomes zero. Therefore, $(x - y)$ is a factor of the determinant.

Similarly putting $y = z$ and $z = x$, the value of determinant becomes zero in each case. So $(y - z)$ *and* $(z - y)$ are also the factors of the determinant.

$$\therefore \Delta = \lambda(x - y)(y - z)(z - x) \qquad \ldots(1)$$

where λ *is a* constants the value of which is to be determined. It is clear that each term of the determinant after expansion is of degree three and each term of the product in right side is also of degree three in x, y, z.

We observe that yz^2 is a term in the expansion, its coefficient is 1, and the coefficient of yz^2 in the RHS of (1) is λ.

$\therefore$ comparing the coefficients of corresponding terms on both sides of equation (1), we get $\lambda = 1$

$$\therefore \Delta = (x - y)(y - z)(z - x)$$

5.18.1 Working Rule for Finding values of Determinants :

To evaluate a determinant we apply the properties of the determinants and try to bring maximum number of zeros in any row (or column) and then the determinant is xpanded along the elements of the same row (or column).

Note : First, second, third ... rows are denoted by R_1, R_2, and R_3 where as first, second, third columns are denoted by C_1, C_2, C_3.... .

SOLVED EXAMPLES

Example 1 : Evaluate the following :

(i) $\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -1 \\ -a & b \end{vmatrix}$ (iii) $\begin{vmatrix} 21 & 11 & 10 \\ 34 & 12 & 22 \\ 61 & 25 & 36 \end{vmatrix}$

Solution : (i) $\Delta = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - (3)(-1) = 8 + 3 = 11$

(ii) $\Delta = \begin{vmatrix} 1 & -1 \\ -a & b \end{vmatrix} = 1 \times b - (-1)(-a) = b - a$

(iii) $\begin{vmatrix} 21 & 11 & 10 \\ 34 & 12 & 22 \\ 61 & 25 & 36 \end{vmatrix} = \begin{vmatrix} 11+10 & 11 & 10 \\ 12+22 & 12 & 22 \\ 25+36 & 25 & 36 \end{vmatrix}$

$= \begin{vmatrix} 11 & 11 & 10 \\ 12 & 12 & 22 \\ 25 & 25 & 36 \end{vmatrix} + \begin{vmatrix} 10 & 11 & 10 \\ 22 & 12 & 22 \\ 36 & 25 & 36 \end{vmatrix}$

$= 0 + 0$ (two columns in each determinants are identical)

$= 0$

Alternatively :

$\Delta = \begin{vmatrix} 21 & 11 & 10 \\ 34 & 12 & 22 \\ 61 & 25 & 36 \end{vmatrix} = \begin{vmatrix} 21-11 & 11 & 10 \\ 34-12 & 12 & 22 \\ 61-25 & 25 & 36 \end{vmatrix} \quad C_1 - C_2 \to C_3$

$= \begin{vmatrix} 10 & 11 & 10 \\ 22 & 12 & 22 \\ 36 & 25 & 36 \end{vmatrix} = 0$ $(C_1 = C_3)$ (first and third column are identical)

Example 2 :

Prove that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

Solution : Expanding the determinant along the first row (R_1)

$\text{LHS} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$= a(bc - f^2) - h(hc - gf) + g(hf - bg)$

$= abc - af^2 - h^2c - ghf + ghf - bg^2$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

$= RHS$

Example 3 : If a, b, c are all different and;

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0 \quad \text{show that} \quad 1+abc=0$$

Solution : $$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 - R_2 \rightarrow R_3,\ R_2 - R_3 \rightarrow R_2$ in both the determinant and taking a, b and c common from first row, second row and third row respectively in second determinant.

$$= \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b & 0 \\ 1 & b+c & 0 \\ c & c^2 & 1 \end{vmatrix} + abc(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ c & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) + abc(a-b)(b-c)(c-a)$$

Expanding first determinant along third column and second determinant along first column.

Now, $(a-b)(b-c)(c-a)(1+abc)=0$

But a, b, c are all different that is $a \neq b \neq c$ (given) $\therefore 1+abc=0$

Example 4 :

Prove that $$\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = (x-y)(y-z)(z-x)$$

Solution : $$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$ Applying $R_1 - R_2 \rightarrow R_3,\ R_2 - R_3 \rightarrow R_2$

$$= \begin{vmatrix} 0 & x-y & z(y-x) \\ 0 & y-z & x(z-y) \\ 1 & z & xy \end{vmatrix}$$

$$= (x-y)(y-z) \begin{vmatrix} 0 & 1 & -z \\ 0 & 1 & -x \\ 1 & z & xy \end{vmatrix} \quad \text{(Expanding along } C_1\text{)}$$

$$\Delta = (x-y)(y-z)(1(-x+z)$$

$$\Delta = (x-y)(y-z)(z-x)$$

Example 5 : Evaluate

$$\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix}$$

Solution :

Let $\Delta = \begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix}$ Applying $C_1 + C_2 + C_3 \to C_1$

$$\Delta = \begin{vmatrix} 0 & y-z & z-x \\ 0 & z-x & x-y \\ 0 & x-y & y-z \end{vmatrix} = 0 \quad \text{by property of determinant}$$

Example 6 : Evaluate

$$\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$$

Solution :

$\Delta = \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$ Applying $R_1 - R_2 \to R_3,\ R_2 - R_3 \to R_2$

$$\Delta = \begin{vmatrix} 0 & x-y & y-z \\ 0 & y-z & z-y \\ 1 & z & x+y \end{vmatrix} = (x-y)(y-z) \begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & z & x+y \end{vmatrix}$$

Expanding along C_1

$$\Delta = (x-y)(y-z)\ 1 \times (-1+1)$$

$$(\quad)(\quad)\ 0$$

$\Delta = 0$

Example 7 : Prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$

Solution :

Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ (taking x, y, z common from first, second and third columns respectively)

$$= x\,y\,z\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= x\,y\,z\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$ Applying $R_2 - R_1 \to R_2$ and $R_3 - R_1 \to R_3$

$$= x\,y\,z\begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= x\,y\,z\{(y-x)(z^2-x^2)-(z-x)(y^2-x^2)\}$$

$$= x\,y\,z\{(y-x)(z-x)\{z+x-(y+x)\}\}$$

$$= x\,y\,z(x-y)(y-z)(z-x)$$

Example 8 : Without expanding Prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

Solution :

Let $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$

$$\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}, \quad \text{(Interchanging rows and columns)}$$

$$\Delta_1 = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}, \quad \text{(Interchanging } \mathbf{C_1} \text{ and } \boldsymbol{C_2}\text{)}$$

$$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}, \quad \text{(Interchanging } R_1 \text{ and } R_2\text{)}$$

$$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \Delta_2$$

Further $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ (Interchanging R_1 and R_2)

$$= -\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} \quad \text{(Again Interchanging } R_2 \text{ and } R_3\text{)}$$

$$= (-1)^2 \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \Delta_3$$

$$\Delta_1 = \Delta_2 = \Delta_3$$

Example 9 : Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y-z)(z-x)(x-y)(x+y+z)$$

Solution :

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ Applying $C_1 - C_2 \to C_2,\ C_1 - C_3 \to C_3$

$$\Delta = 1\begin{vmatrix} 1 & 0 & 0 \\ x & x-y & x-z \\ x^3 & x^3-y^3 & x^3-z^3 \end{vmatrix}$$ Expanding along C_1

$$\Delta = 1\begin{vmatrix} (x-y) & x-z \\ x^3-y^3 & x^3-z^3 \end{vmatrix}$$

$$= \begin{vmatrix} (x-y) & x-z \\ (x-y)(x^2+xy+y^2) & (x-z)(x^2+xz+z^2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ (x-y)(x^2+xy+y^2) & (x-z)(x^2+xz+z^2) \end{vmatrix}$$

(Taking (x-y) and (x-z) common from C_1 and C_2 respectively).

$$=(x-y)(x-z)\{(x^2+zx+z^2-x^2-xy-y^2\}$$

$$=(x-y)(x-z)(x(z-y)+(z-y)(z-y))$$

$$=(x-y)(x-z)(z-x)(x+y+z)$$

$$=(x-y)(y-z)(z-x)(x+y+z)$$

Example 10 : Prove that

$$\begin{vmatrix} x+y & y & z \\ y+z & z & x \\ z+x & x & y \end{vmatrix} = 3x\,y\,z - x^3 - y^3 - x^3$$

Solution :

Let $\Delta = \begin{vmatrix} x+y & y & z \\ y+z & z & x \\ z+x & x & y \end{vmatrix}$ Applying $C_1 + C_3 \to C_1$

$$= \begin{vmatrix} x+y+z & y & z \\ y+z+x & z & x \\ z+x+y & x & y \end{vmatrix},$$ Taking $(x+y+z)$ common from C_1

$$\Delta = (x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & z & x \\ 1 & x & y \end{vmatrix},$$ Applying $R_2 - R_1 \to R_2,\ R_3 - R_1 \to R_3$

$$= (x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & z-y & x-z \\ 0 & x-y & y-z \end{vmatrix}, \text{ Expanding along } C_1$$

$$= (x+y+z)\begin{vmatrix} z-y & x-z \\ x-y & y-z \end{vmatrix}$$

$$= (x+y+z)[\{(z-y)(y-z)-(x-y)(x-z)\}]$$

$$= (x+y+z)\{zy - z^2 - y^2 + yz - x^2 + xz + yx - yz\}$$

$$= (x+y+z)\{xy + yz + zx - x^2 - y^2 - z^2\}$$

$$= -(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$= -(x^3+y^3+z^3-3xyz)$$

$$= 3xyz - x^3 - y^3 - z^3 \qquad \text{Proved.}$$

Example 11 : Find the value(s) of x if;

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0$$

Solution : Given equation

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0$$

$$\begin{vmatrix} 1+x & 0 & -1-x \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0 \qquad \text{Applying } R_1 - R_3 \to R_1$$

$$\begin{vmatrix} 1+x & 0 & 0 \\ 1 & 7+x & 7 \\ 2 & 5 & 5+x \end{vmatrix} = 0 \qquad \text{Applying } C_1 + C_3 \to C_3$$

$$= (1+x)[(7+x)(5+x)-35] = 0 \qquad \text{Expanding along } R_1$$

$$= (1+x)[35 + 12x + x^2 - 35] = 0$$

$$= x(x+1)(x+12) = 0$$

$$\therefore\ x = 0, \quad x = -1, \quad x = -12$$

Example 12 : Prove that

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Solution :

Let $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$ Applying $C_1 + C_2 + C_3 \to C_1$

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x\cdot 2y & y \\ 2(x+y+z) & x & z+x+2x\cdot 2y \end{vmatrix}$$

Taking $2(x+y+z)$ common from C_1

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x\cdot 2y & y \\ 1 & x & z+x+2x\cdot 2y \end{vmatrix}$$

Applying $R_2 - R_1 \to R_2,\ R_3 - R_1 \to R_3$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$ Expanding along C_1

$$= 2(x+y+z)\begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\,(x+y+z)^2$$

$$= 2(x+y+z)^3$$

EXERCISE 1.2

Q. 1. $\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

Q. 2. $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$

Q. 3. $\begin{vmatrix} x & y+z & x^2 \\ y & z+x & y^2 \\ z & x+y & z^2 \end{vmatrix} = (x+y+z)(x-y)(y-z)(z-x)$

Q. 4. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = ab$

Q. 5. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Q. 6. $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$ Where w is one of the imaginary cube roots of unity.

Q. 7. $\begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+4y+2z \\ 3x & 6x+3y & 10x+6y+3z \end{vmatrix} = x^3$

Q. 8. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix} = 0$

Q. 9. $\begin{vmatrix} -x^2 & xy & zx \\ xy & -y^2 & yz \\ xz & yz & -z^2 \end{vmatrix} = 4x^2y^2z^2$

Q. 10. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xy+yz+zx+xyz$

Q. 11. Prove that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Represents the equation of a line which passes through the points (x_1, y_1) and (x_2, y_2).

Q. 12. Prove that $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Represents the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Q. 13. Find the area of a triangle whose vertices are (2, + 4), (+2, 6) and (5, 4).

5.19 APPLICATION OF DETERMINANTS

5.19.1 Solution of System of Linear Equations by Determinants : (CRAMER'S RULE)

Consider the system -

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

On solving the above equation we get,

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

$$\therefore \quad \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = b_2c_1 - b_1c_2, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - c_1a_2 \text{ and } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The values of x and y can be given in terms of the determinants. Thus we have,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} c_1 & c_1 \\ c_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

Here we see that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the determinant of the coefficients of the variables as

$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ is the determinant obtained by replacing the coefficients of x in two equations by corresponding constants, and $\begin{vmatrix} a_1 & c_1 \\ c_2 & c_2 \end{vmatrix}$ is the determinant obtained by replacing the coefficients of y in two equation by constants in the equations.

Now consider the following system of linear equations in three variables *x, y, z.*

$$a_1x + b_1y + c_1z = d_1 \qquad \ldots(1)$$

$$a_2x + b_2y + c_2z = d_2 \qquad \ldots(2)$$

$$a_3x + b_3y + c_3z = d_3 \qquad (3)$$

Now,

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

Putting the value of d_1, d_2 *and* d_3 from equation (1), (2) and (3)

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix}$$

$$= x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + y\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + z\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$$= x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + y.0 + z.0$$ *(Determinants in 2nd and 3rd terms vanish as they have identical columns)*

$$\because \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or $x = \dfrac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$, Similarly $y = \dfrac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$ and $z = \dfrac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ *(Determinant of the coefficients of the variables in the three equations.)*

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \textit{and} \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, *and* $z = \dfrac{D_z}{D}$ If $(D \neq 0)$

In determinants D_x, D_y, D_z the column of coefficient of variables x, y and z are replaced by the constants respectively.

The solution of system of linear equation discussed above can be used to solve a system of n equations in n unknowns as follows -

Consider the following system of n linear equations in n unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \dots a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 \dots a_{2n}x_n &= c_2 \\ \dots\dots\dots\dots &\dots\dots \\ \dots\dots\dots\dots &\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 \dots a_{nn}x_n &= c_n \end{aligned} \qquad \text{(A)}$$

$$\text{Let} \quad D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{vmatrix} \neq 0$$

Let D_{x_1} be to determinant obtained by D after replacing first column by $\begin{vmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{vmatrix}$

Then $x_1 = \frac{D_{x_1}}{D}$, Similarly $x_2 = \frac{D_{x_2}}{D}$, $x_3 = \frac{D_{x_3}}{D}$..., $x_n = \frac{D_{x_n}}{D}$

The solution of n linear equations in n- unknowns given above is known as Cramer's Rule after the Swiss Mathematician Gabriel Cramer (1704-1752).

SOLVED EXAMPLES

Example 13 : Apply Cramer's Rule to solve the following system of linear equation.

$$\begin{aligned} 2x + y &= 6 \\ 3x - 2y &= 9 \end{aligned}$$

Solution : Here $D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$, $D_x = \begin{vmatrix} 6 & 1 \\ 9 & 2 \end{vmatrix}$ $D_y = \begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix}$

Now, $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$

$$x = \frac{\begin{vmatrix} 6 & 1 \\ 9 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}$$

$$x=\frac{(-12-9)}{-4-3},\qquad y=\frac{(18-18)}{-4-3}$$

$$x=\frac{-21}{-7},\qquad y=\frac{0}{-7}$$

$$x=3\qquad y=0$$

Example 14 : Apply Cramer's rule to solve the following system of equations.

$$x+y+z=4$$
$$2x-y+2z=5$$
$$x-2y-z=-3$$

Solution :

Here $D=\begin{vmatrix}1 & 1 & 1\\ 2 & -1 & 2\\ 1 & -1 & -2\end{vmatrix}\begin{array}{l}=1(1+4)-1(-2+2)+1(-4+1)\\ =5+4-3=6\neq 0\end{array}$

$$D_x=\begin{vmatrix}4 & 1 & 1\\ 5 & -1 & 2\\ -3 & -2 & -1\end{vmatrix}\begin{array}{l}=4(1+4)-1(-5+6)+1(-10-3)\\ =20-1-13=6\end{array}$$

$$D_y=\begin{vmatrix}1 & 4 & 1\\ 2 & 5 & 2\\ 1 & -3 & -1\end{vmatrix}\begin{array}{l}=1(-5+6)-4(-2-2)+1(-6-5)\\ =1+16-11=6\end{array}$$

$$D_z=\begin{vmatrix}1 & 1 & 4\\ 2 & -1 & 5\\ 1 & -2 & -3\end{vmatrix}\begin{array}{l}=1(3+10)-1(-6-5)+4(-4+1)\\ =13+11-12=12\end{array}$$

$$\therefore\ x=\frac{D_x}{D}=\frac{6}{6}=1\qquad y=\frac{D_y}{D}=\frac{6}{6}=1\qquad z=\frac{D_z}{D}=\frac{12}{6}=2$$

$$x=1,\quad y=1,\quad z=2$$

Note : **(i)** In the system of n equations in n unknowns given by (A); if $c_1=c_2=\ldots c_n=0$, then each $D_{x_1}, D_{x_2}, D_{x_3}, \ldots, D_{xn}=0$ and if $D\neq 0$, the system has only the trivial solution $x_1=x_2=\ldots x_n=0$

(ii) Cramer's rule is not applicable if $D=0$.

(iii) If D = 0 and either $D_x\neq 0$, (or any other $Dx_i\neq 0$) the system has no solution.

(iv) If D = 0, and $D_{x_1}=D_{x_2}=\ldots D_{x_n}=0$, the system has infinite number of solutions.

For consider the s stem of eq ations

$$x + y = 7 \qquad ...(1)$$

$$2x+2y = 14. \qquad ...(2)$$

Here,

$$D = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0, \quad 2-2=0, \; D_x = \begin{vmatrix} 7 & 1 \\ 14 & 2 \end{vmatrix} = 14-14=0 \text{ and } D_y = \begin{vmatrix} 1 & 7 \\ 2 & 14 \end{vmatrix} = 14-14=0$$

Thus we see that $D = 0, \quad D_x = D_y = 0$

Now putting $x = k$ in equation (1), k is arbitrary and can take any value.

$$y = 7 - k$$

thus the system has solutions and it has infinite number of solution.

> If we draw the graph of these two equations we get coincident lines, and as such there are infinite number of points where these lines meet, the system has infinite number of solutions.

Example 15 : Solve the following system of equations :

$$x - y + 3z = 6$$

$$x - 3y + 3z = -4$$

$$5x - 3y + 3z = 10$$

Here,

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 5 & 3 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & -2 \\ 5 & 8 & -4 \end{vmatrix} \qquad \text{Applying } C_1 + C_2 \to C_3, \quad C_3 + C_1 \to C_3$$

$= 0$ (as second and third columns are scalar multiple of each other)

$$D_x = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix} = 3$$

$$= \begin{vmatrix} 6 & -1 & 1 \\ -4 & 3 & -1 \\ 10 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 & 0 \\ 6 & 6 & 0 \\ 10 & 3 & 3 \end{vmatrix} \quad \text{Applying } R_1 + R_2 \to R_1, \quad R_2 + R_3 \to R_2$$

$$= 3\begin{vmatrix} 2 & 2 \\ 6 & 6 \end{vmatrix} = 0$$

Similarly it can be shown that $D_y = D_z = 0$

Now consider the first two equations,

$$x - y + 3z = 6$$

$$x - 3y + 3z = -4 \quad \text{these equations can be written as}$$

$$x - y = 6 - 3z$$

$$x + 3y = 3z - 4 \quad \text{Treating z as constant we have,}$$

Here,

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}, \quad D_x = \begin{vmatrix} 6-3z & -1 \\ 3z-4 & 3 \end{vmatrix}, \quad D_y = \begin{vmatrix} 1 & 6-3z \\ 1 & 3z-4 \end{vmatrix}$$

Or,

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 6-3z & -1 \\ 3z-4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}}$$

$$x = \frac{18 - 9z - 4}{3+1} = \frac{14 - 6z}{4} = \frac{7 - 3z}{2}$$

And,

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 1 & 6-3z \\ 1 & 3z-4 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{3z - 4 - 6 + 3z}{3+1}$$

$$= \frac{6z - 10}{4} = \frac{3z - 5}{2}$$

Since we have taken z as constant we can give it arbitrary values and accordingly we can get the corresponding value of x and y. Thus we have infinite number of solutions which satisfy the first two equations. These are also satisfying the third equation.

EXERCISE 1.3

Using determinants (Cramer's Rule) solve the following system of equations.

Q. 1. (*i*) $x + 14y = -4$, $8x + 12y = -6$

(*ii*) $x + 3y + z = 8$, $4x + y = 7$, $x - 3y - 3z = -2$

(*iii*) $x + y = -5$, $x + z = -6$, $x + y - 2z = 3$

(*iv*) $2x - 3y - z = 0$, $x + 3y - 2z = 0$, $x \quad 3y \quad 0$

(*v*) $2x + 5y - z = 9$, $3x - 3y + 2z = 7$, $2x \quad 4y + 3z \quad 1$

(*vi*) $3x + y + 2z = 3$, $2x + 3y - z = -3$, $x + 2y + z \quad 4$

5.20 ADJOINT OF A SQUARE MATRIX

Consider

$$A = [a_{ij}]_{n\times n}$$

Then
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let the cofactor of the element a_{ij} be A_{ij} then the cofactor matrix can be written as -

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Taking the transpose of $[A_{ij}] \equiv [A_{ji}]$

Thus
$$[A_{ji}] = \begin{bmatrix} A_{11} & A_{11} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ A_{13} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

The transpose of cofactor matrix of the elements of the matrix A, is called adjoint of A and is written as *adj A*

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

If A_{ij} denote the cofactor of the element a_{ij}, then the cofactor matrix of A is given by :

Cofactor matrix of
$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

5.21 INVERSE OF A MATRIX

For every non-singular Matrix A, there exists a matrix B of the same order as that of A. such that $AB = BA = I$ where I is the identity matrix of the same order as of A.

The inverse of matrix A is denoted by A^{-1}.

If A is a square matrix $[a_{ij}]$ and $|A| \neq 0$ i.e. the matrix A is non-singular then

$$A^{-1} = adj\, A \times \frac{1}{|A|}$$

SOLVED EXAMPLE

Example 16 : If $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -2 \end{bmatrix}$ *find* A', *adj* A and A^{-1}

Solution : $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -2 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -6 \\ -1 & 5 & -2 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & 8 \\ 0 & -6 & -2 \end{vmatrix}$ Applying $C_1 + C_3 \to C_3$

$= 1(-8+48) = 40 \neq 0$ Expanding along first row.

$\therefore$ A is invertible.

Now,

$A_{11} = -8+30 = 22,\quad A_{21} = -(+6) = -6,\quad A_{31} = 4$

$A_{12} = -(-6-0) = 6,\quad A_{22} = -2 = -2,\quad A_{32} = -(5+3) = -8$

$A_{13} = -18-0 = -18,\quad A_{23} = -(-6) = 6,\quad A_{33} = 4$

$\therefore Adj\ A = \begin{bmatrix} 22 & -6 & 4 \\ 6 & -2 & -8 \\ -18 & 6 & 4 \end{bmatrix}$ $\therefore A^{-1} = adj\, A \times \frac{1}{[A]} = \frac{1}{40}\begin{bmatrix} 22 & -6 & 4 \\ 6 & -2 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

$$= \begin{bmatrix} \frac{11}{20} & -\frac{3}{20} & \frac{1}{10} \\ \frac{3}{20} & -\frac{1}{20} & -\frac{1}{5} \\ -\frac{9}{20} & \frac{3}{20} & \frac{1}{10} \end{bmatrix}$$

EXERCISE 1.4

Q. 1. Show that $A = \begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$, is invertible. Find $A', adj\ A$ *and* A^{-1}.

Q. 2. Find the inverse of $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

Q. 3. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix}$.

Q. 4. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Q. 5. Find $A', adj\ A$, and A^{-1}, if $A = \begin{bmatrix} 2 & 1 & 3 \\ -4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

5.22 ELEMENTARY OPERATIONS ON MATRICES

Consider the matrices $A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 8 & 10 \\ 3 & 4 & 5 \end{bmatrix}$ here matrix B is obtained by interchanging first and second rows in the matrix A.

Further $C = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 4 & 6 \\ 14 & 16 & 18 \end{bmatrix}$

Clearly, Matrix D is obtained by multiplying each element of Matrix C by 2.

Also, $E = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 6 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 3 & 5 \\ 5 & 10 & 16 \end{bmatrix}$

Matrix *F* is obtained by multiplying second row of Matrix *E* by 2 and adding it to the second row.

The above operations on rows of a matrix are called elementary row operations.

Thus an elementary operation is either elementary row operations or elementary column operations and is of three types.

(i) Interchanging any two rows (or columns).

(ii) Multiplication of the elements of any row (row column) by any non-zero number.

(iii) Addition of a non-zero scalar multiple of any row (or column) to another row (or column).

5.22.1 Elementary Matrices :

A matrix obtained by a single elementary operation from an identity matrix, is called elementary matrix.

Let $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ On interchanging first and second rows, we get a matrix of the form

$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Multiplying first row of I_3 by 3, we get another matrix

$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ The matrices A and B are elementary matrices.

5.22.2 Equivalent Matrices : Two matrices are said to be equivalent if one can be obtained from the other by elementary row transformations.

If matrix B is obtained from matrix A by elementary row transformation, then we write A ~ B.

5.22.3 Inverse of a Matrix by Elementary Transformations :

If a matrix A is reduced to identity Matrix I by elementary transformation, then

$PA = I$, where $P = P_n P_{n-1} \ldots, P_2 P_1$, Matrices

$\therefore\ P = A^{-1}$

To find the inverse of a matrix A, we write

$A = IA$

Now we perform elementary operations on A in left side and same elementary operations on I in right hand side so that A is reduced to I in left side and I on right side reduce to P, getting $I = PA$ then P is inverse of A.

5.23 ECHELON FORM OF A MATRIX

A matrix is said to be in Echelon form if its first element is one and the elements below the diagonal are all zeros.

Examplet :

(i) (ii) $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

are the examples of Matrices in Echelon form.

Note : $\begin{bmatrix} \ddots 1 & 2 & 3 \\ 0 & \ddots 1 & 7 \\ 0 & 0 & \ddots 3 \end{bmatrix}$ A matrix can be reduced to Echelon form by elementary row operation on it.

The following steps are used to reduce a matrix in Echelon form.

(i) First we reduce the element of first row and first column ie. (1, 1)th element as unity i.e. 1 by suitable elementary row operations.

(ii) We reduce all the elements below the element of first row and first column i.e. 1 to zero.

(iii) We reduce the element of second row and second column i.e. (2, 2)th element as unity by suitable elementary row operation.

(iv) We reduce all the element in the second column to zero, below the second row.

In the similar manner we proceed and the matrix is reduced in Echelon form.

SOLVED EXAMPLES

Example 17:

Reduce $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 6 & 12 \\ 2 & 4 & 8 \end{bmatrix}$ to Echelon form.

Solution : We have

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 6 & 12 \\ 2 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 \\ 3 & 6 & 12 \\ 2 & 4 & 8 \end{bmatrix} \qquad \text{Applying } R_2 - R_1 \to R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 9 \\ 0 & -1 & -5 \\ 2 & 4 & 8 \end{bmatrix} \qquad \text{Applying } R_2 - (R_1 + R_3) \to R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 9 \\ 0 & -1 & -5 \\ 1 & 2 & 4 \end{bmatrix} \qquad \text{Applying } \frac{1}{2} R_3 \to R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 5 \\ 1 & 2 & 4 \end{bmatrix} \qquad \text{Applying } -R_2 \to R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_2 - (R_1 + R_3) \to R_2$$

Whi h i i E h l f

Example 18 : Reduce $A=\begin{bmatrix}1 & 2 & 3\\2 & 1 & 3\\3 & 1 & 2\end{bmatrix}$ in Echelon form.

Solution : We have

$$A=\begin{bmatrix}1 & 2 & 3\\2 & 1 & 3\\3 & 1 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3\\0 & -2 & -4\\3 & 1 & 2\end{bmatrix}$$

Applying $R_3-(R_1+R_2)\rightarrow R_2$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 2\end{bmatrix} \qquad \text{Applying } -\frac{1}{2}R_2$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\-2 & 1 & 1\end{bmatrix} \qquad \text{Applying } R_1-R_3\rightarrow R_3$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\-1, & \frac{1}{2}, & \frac{1}{2}\end{bmatrix} \qquad \text{Applying } \frac{1}{2}R_3\rightarrow R_3$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & \frac{5}{2}, & \frac{7}{2}\end{bmatrix} \qquad \text{Applying } R_1+R_3\rightarrow R_3$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 5 & 7\end{bmatrix} \qquad \text{Applying } 2R_3\rightarrow R_3$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 3\end{bmatrix} \qquad \text{Applying } 5R_2-R_3\rightarrow R_3$$

Which is Echelon form of A.

Example 19 :

Reduce $A=\begin{bmatrix}1 & 2 & 3 & 4\\2 & 1 & 3 & 2\\3 & 1 & 2 & 4\end{bmatrix}$ to Echelon form.

Solution : We have

$$\begin{bmatrix}1 & 2 & 3 & 4\\2 & 1 & 3 & 2\\3 & 1 & 2 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -3 & -6\\3 & 1 & 2 & 4\end{bmatrix} \quad \text{Applying } R_2 - 2R_1 \to R_2$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -3 & -6\\0 & -5 & -7 & -8\end{bmatrix} \quad \text{Applying } R_3 - 3R_1 \to 3R_1$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 1 & 2\\0 & -5 & -7 & -8\end{bmatrix} \quad \text{Applying } -\frac{1}{3}R_2 \to R_2$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 1 & 2\\0 & 0 & -2 & 2\end{bmatrix} \quad \text{Applying } R_3 + 5R_2 \to R_3$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 1 & 2\\0 & 0 & -1 & 1\end{bmatrix} \quad \text{Applying } \frac{1}{2}R_3 \to R_3$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 1 & 2\\0 & 0 & 1 & -1\end{bmatrix} \quad \text{Applying } -R_3 \to R_3$$

Example 20 : Find the inverse of $\begin{bmatrix}6 & -6 & 8\\4 & -6 & 8\\0 & -2 & 2\end{bmatrix}$ by elementary transformation.

Solution :

Let $A = \begin{bmatrix}6 & -6 & 8\\4 & -6 & 8\\0 & -2 & 2\end{bmatrix}$

We have,

$$\begin{bmatrix}6 & -6 & 8\\4 & -6 & 8\\0 & -2 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \qquad (\because A = IA = AI)$$

$$\begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{Applying } \frac{1}{6}R_1 \to R_1$$

$$\begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -2 & \frac{8}{3} \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{Applying } R_2 - 4R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} A \qquad \text{Applying } \frac{-R_2}{2} \to R_2 \ \textit{and} \ \frac{R_3}{2} \to R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} A \qquad \text{Applying } R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & -\frac{1}{2} & 0 \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} A \qquad \text{Applying } R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & -\frac{1}{2} & -2 \\ -1 & \frac{3}{2} & -4 \end{bmatrix} A \qquad \text{Applying } R_2 - 4R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{2} & -2 \\ 3 & -\frac{9}{2} & \frac{4}{3} \end{bmatrix} A \qquad \text{Applying } -3R_3 \to R_3$$

$$\therefore\ A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{3} & -\frac{1}{2} & -2 \\ 3 & -\frac{9}{2} & \frac{4}{3} \end{bmatrix}$$

Example 21 : Find the inverse of

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$ by elementary transformation.

Solution :

Let $$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad (\because\ A = IA = AI)$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} A \qquad \text{Applying } \begin{matrix} R_1 - R_2 \to R_2 \\ R_1 - R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \qquad \text{Applying } \begin{matrix} -R_2 \to R_2 \\ -R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \qquad \text{Applying } R_1 - 3R_2 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \qquad \text{Applying } R_1 - 3R_3 \to R_1$$

$$\therefore\ A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

5.24 SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD.

Consider the following system of linear equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The above system of linear equation can be written in Matrix form as :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad \ldots(1)$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, *and* $|A| \neq 0$

Now multiplying (1) by A^{-1} we get

$$A^{-1}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$I\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad (\because A^{-1}A = I)$$

And $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad \because I\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $X = A^{-1}B$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and

Now comparing the corresponding element on matrices on either side , we get the value of x, y and z.

Example 22 : Solve the following system of linear equations by matrix method.

$$2x + y + z = 1$$
$$x - 2y - 3z = 1$$
$$3x + 2y + 4z = 5$$

Solution : The above system of linear equations can be written as,

Coefficient Matrix	$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix}$
Variables Matrix	$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
Constant Matrix	$B = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \qquad ...(1)$$

Hence $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix}$ $\quad \because |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{vmatrix}$

Now, $|A| = 2\begin{vmatrix} -2 & -3 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix}$

$$= 2(-8+6) - 1(4+9) + 1(2+6)$$

$$= -4 - 13 + 8 = -9 \neq 0$$

$$\because |A| \neq 0$$

$\therefore$ A is invertible matrix.

Now cofactors c^ the element of $|A|$ are given by,

$A_{11} = \begin{vmatrix} -2 & -3 \\ 2 & 4 \end{vmatrix} = -8+6 = -2,$ $\qquad A_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -(4-2) = -2$

$A_{12} = -\begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix} = -(4+9) = -13,$ $\qquad A_{22} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8-3 = 5$

$A_{13} = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 2+6 = 8,$ $\qquad A_{23} = -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -(4-3) = -1$

$A_{31} = \begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} = -3+2 = -1,$ $\qquad A_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -(-6-1) = 7$

$A_{33} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4-1 = -5$

$$\therefore \text{Cofactor Matrix of } |A| = \begin{bmatrix} -2 & -13 & 8 \\ -2 & 5 & -1 \\ -1 & 7 & -5 \end{bmatrix}$$

and $\text{Adj. } A = \begin{bmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{bmatrix}$ $\quad \therefore A^{-1} \dfrac{Adj\, A}{|A|} = -\dfrac{1}{9} \cdot \begin{bmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{bmatrix}$.

Multiplying equation (1) by A^{-1}

$$-\frac{1}{9}\begin{bmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} -2 & -2 & - \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} +\frac{2}{9} & +\frac{2}{9} & \frac{1}{9} \\ \frac{13}{9} & -\frac{5}{9} & -\frac{7}{9} \\ -\frac{8}{9} & \frac{1}{9} & \frac{5}{9} \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +\frac{2}{9} & +\frac{2}{9} & \frac{1}{9} \\ \frac{13}{9} & -\frac{5}{9} & -\frac{7}{9} \\ -\frac{8}{9} & \frac{1}{9} & \frac{5}{9} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{9}+\frac{2}{9}+\frac{3}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} & \frac{2}{9}-\frac{6}{9}+\frac{4}{9} \\ \frac{26}{9}-\frac{5}{9}-\frac{21}{9}, & \frac{13}{9}+\frac{10}{9}-\frac{14}{9}, & \frac{13}{9}+\frac{15}{9}-\frac{28}{9} \\ -\frac{16}{9}+\frac{1}{9}+\frac{15}{9} & -\frac{8}{9}-\frac{2}{9}+\frac{10}{9} & -\frac{8}{9}-\frac{3}{9}+\frac{20}{9} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{9}+\frac{2}{9}+\frac{5}{9} \\ \frac{13}{9}-\frac{5}{9}-\frac{35}{9} \\ -\frac{8}{9}+\frac{1}{9}+\frac{25}{9} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Comparing the corresponding elements of above matrices, we get

$x = 1, \quad y = -3, \quad z = 2$

Note : If $|A| \neq 0$ the solution of system can be written as $X = A^{-1}B$

Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Thus in the above example we have, $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{2}{9} & \frac{5}{9} \\ \frac{13}{9} & \frac{-5}{9} & \frac{-7}{9} \\ \frac{-8}{9} & \frac{1}{9} & \frac{25}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & +\frac{2}{9} & +\frac{5}{9} \\ \frac{13}{9} & -\frac{5}{9} & -\frac{35}{9} \\ \frac{-8}{9} & +\frac{1}{9} & +\frac{25}{9} \end{bmatrix} \text{ or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Consequently x = 1, y = -3, z = 2

Example 23 : Solve the following system of linear equations :

$$2x - 3y + 3z = 1$$

$$2x - 3y + 3z = 2$$

$$3x + 2y + 2z = 3$$

Solution : The above system of equations can be written as :

$$\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad ...(1)$$

Let $\therefore A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{vmatrix}$ $= 2(4+6) + 3(4-9) + 13(-4-6)$ $= 20 - 15 - 30$ $= -25 \neq 0$

$\therefore A$ is invertible.

Cofactors of elements of $|A|$ are

$A_{11} = (4+6) = 10$, $A_{12} = -(4-9) = 5$, $A_{13} = (-4-6) = -10$,

$A_{21} = -(-6+6) = 0$, $A_{22} = (4-9) = -5$, $A_{23} = -(-4+9) = -5$

$A_{31} = (-9-6) = -15$, $A_{32} = -(6-6) = 0$, $A_{33} = (4+6) = 10$

Adj. $A = \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$ $\therefore A^{-1} \frac{1}{|A|}$ Adj $A = -\frac{1}{25} \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$

$$\text{Or } A^{-1} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

Now, multiplying (1) by A^{-1}, we get

$$\begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & +0 & +\frac{9}{5} \\ -\frac{1}{5} & +\frac{2}{5} & +0 \\ \frac{2}{5} & +\frac{2}{5} & -\frac{6}{5} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix} \quad (\because A^{-1}A = I)$$

Comparing the corresponding elements on both sides.

$x = 7/5, \quad y = 1/5, \quad z = -2/5$

5.25 SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY ELEMENTARY TRANSFORMATION (OPERATIONS)

Consider the following system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_{11}$$
$$a_{21}x + a_{22}y + a_{23}z = b_{12}$$
$$a_{31}x - a_{32}y + a_{33}z = b_{13}$$

Solution : The system of equations can be written as :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix}$$

If we denote $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix}$

Then we have $AX = B$

Clearly A is the coefficient matrix (coefficient of the variable) X is the matrix representing the variables and B is column matrix representing the constants on right hand side of above equations.

To solve the system of linear equations we reduce the coefficient matrix A into Echelon form by elementary row operations and then write the corresponding equations and then we solve.

The system of linear equation is said to be consistent if it has solution and inconsistent if it has no solution.

Example 24 : Solve the following system of equations by elementary row transformation.

$$2x + y + z = 1$$
$$x - 2y - 3z = 1$$
$$3x + 2y + 4z = 5$$

Solution : The matrix form of the given system of equations is-

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$
$$\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

Applying $R_3 - (R_1 + R_2) \rightarrow R_2$
$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 6 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Applying $\frac{1}{3}R_2 \rightarrow R_2$
$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

Applying $R_3 - 3R_1 \rightarrow R_3$
$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 8 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Applying $\frac{1}{8}R_3 + R_2$
$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & \frac{13}{8} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{4} \end{bmatrix}$$

Applying $R_2 - R_3 \to R_3$
$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & \frac{3}{8} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3/4 \end{bmatrix}$$

Writing the corresponding linear equation for the above,

$$x - 2y - 3z = 1$$
$$y + 2z = 1$$
$$\frac{3}{8}z = \frac{3}{4}$$

From the above equation $z = 2,\ y = -3,\ x = 1$

i. e. $x = 1, \quad y = -3, \quad z = 2$

Example 25 : Solve the following system of equations by elementary transformations.

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

Solution : Writing the given system of linear equations in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Applying $R_1 - R_2 \to R_2$ & $R_2 - R_3 \to R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Applying $-R_2 \to R_2 \frac{1}{2} R_3 \to R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Applying $R_2 + R_3 \to R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Now writing the corresponding linear equation for the above,

$$x + y + z = 3$$
$$y + 2z = 1$$
$$-z = 0$$

$\therefore\ z = 0, \quad y = 1, \quad x = 2$

EXERCISE 1.5

1. Reduce the following matrices in to echelon form.

(i) $\begin{bmatrix} 3 & -10 & 5 \\ -1 & 12 & -2 \\ 1 & -5 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 & 4 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 9 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & -8 \end{bmatrix}$

2. Find the inverse of the following matrices by using elementary transformation-

(i) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{2}{9}, & \frac{2}{9}, & \frac{2}{9} \\ -\frac{13}{9}, & -\frac{5}{9}, & \frac{7}{9} \\ -\frac{8}{9}, & \frac{1}{9}, & \frac{5}{9} \end{bmatrix}$

3. Solve the following systems of equations by matrix method (inverse of a matrix.)

(i) $\begin{aligned} x+y+z&=6 \\ x-y+z&=2 \\ 2x+y-z&=1 \end{aligned}$ (ii) $\begin{aligned} x+y+z&=3 \\ x+2y+3z&=4 \\ 2x+4y+9z&=6 \end{aligned}$ (iii) $\begin{aligned} x-2y+3z&=11 \\ 3x+y-z&=2 \\ 5x+3y+2z&=3 \end{aligned}$

(iv) $\begin{aligned} 2x-y+3z&=9 \\ x+y+z&=6 \\ x-y+z&=2 \end{aligned}$ (v) $\begin{aligned} 5x-y+z&=4 \\ 3x+2y-3z&=2 \\ 5x+3y-2z&=5 \end{aligned}$

4. Solving the following system of equations by elementary row transformation (by coeffecient matrix in echelon form).

(i) $\begin{aligned} x+y+z&=3 \\ x+2y+3z&=4 \\ x+4y+9z&=6 \end{aligned}$ (ii) $\begin{aligned} x-2y+3z&=11 \\ 3x+y-z&=2 \\ 5x+3y+2z&=3 \end{aligned}$ (iii) $\begin{aligned} 2x-y+3z&=9 \\ x+y+z&=6 \\ x-y+z&=2 \end{aligned}$

(iv) $\begin{aligned} x+y+z&=4 \\ 2x-2y+2z&=5 \\ x-2y-z&=-3 \end{aligned}$ (v) $\begin{aligned} 2x-y+z&=3 \\ x+3y-2z&=11 \\ 3x-2y+4z&=1 \end{aligned}$ (vi) $\begin{aligned} 4x+y+4z&=7 \\ 2x+3y+2z&=6 \\ 6x+9y+2z&=14 \end{aligned}$

Ans.: (i) $x=1,\ y=2,\ z=3$ (ii) $x=2,\ y=-3,\ z=1$ (iii) $x=2,\ y=-3,\ z=1$

(iv) $x=1,\ y=2,\ z=3$ (v) $x=1,\ y=-2,\ z=-3$

6 Linear Programming Formulation of LPP

Chapter Includes:

6.1 Introduction

Linear programming is the general technique of optimum allocation of limited resources such as labour, material, machine, capital etc., to several competing activities such as products, services, jobs, projects, etc., on the basis of given criterion of optimality.

The term limited here is used to describe the availability of scarce resources during planning period. The criterion of optimality generally means either performance, return on investment, utility, time, distance etc., The word linear stands for the proportional relationship of two or more variables in a model. Programming means 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. It is an extremely useful technique in the decision making process of the management.

6.2 Structure of Linear Programming Problem (LPP)

The LP model includes the following three basic elements.

(i) Decision variables that we seek to determine.

(ii) Objective (goal) that we aim to optimize (maximize or minimize)

(iii) Constraints that we need to satisfy.

One of the major applications of linear algebra involving systems of linear equations is in finding the maximum or minimum of some quantity, such as profit or cost. In mathematics the process of finding an extreme value (maximum or minimum) of a quantity (normally called a function) is known as **optimization**. **Linear programming (LP)** is a branch of Mathematics which deals with modeling a decision problem and subsequently solving it by mathematical techniques. The problem is presented in a form of a linear function which is to be optimized (i.e maximized or minimized) subject to a set of linear constraints. The function to be optimized is known as the **objective function**.

Linear programming finds many uses in the business and industry, where a decision maker may want to utilize limited available resources in the best possible manner. The limited resources may include material, money, manpower, space and time. Linear Programming provides various methods of solving such problems. In this unit, we present the basic concepts of linear programming problems, their formulation and methods of solution.

6.3 Formulation of Linear Programming Problems

Mathematically, the general linear programming problem (LPP) may be stated as:

$$\begin{aligned}
\text{Maximize or Minimize} \quad & Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \qquad (1) \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}$$

where

(i) the function Z is the objective function.

(ii) $x_1, x_2, \ldots, x_n$ are the decision variables.

(iii) the expression $(\leq, =, \geq)$ means that each constraint may take any one of the three signs.

(iv) c_j $(j = 1, \ldots, n)$ represents the per unit cost or profit to the j^{th} variable.

(v) b_i $(i = 1, \ldots, m)$ is the requirement or availability of the i^{th} constraint.

(vi) $x_1, x_2, \ldots, x_n \geq 0$ is the set of non-negative restriction on the LPP. In real life problems negative decision variables have no valid meaning.

In this module we shall only discuss cases in which the constraints are strictly inequalities (either have a $\leq$ *or* $\geq$).

In formulating the LPP as a mathematical model we shall follow the following four steps.

1. Identify the **decision variables** and assign symbols to them (eg x, y, z,... or x_1, x_2, x_2, ...). These decision variables are those quantities whose values we wish to determine.
2. Identify the set if constraints and express them in terms of inequalities involving the decision variables.
3. Identify the objective function and express it is terms of the decision variables.
4. Add the *non-negativity condition.*

We will use the following product mix problem to illustrate the formulation of an LPP.

Example : Prototype Example *A paint manufacturer produces two types of paint, one type of standard quality (S) and the other of top quality (T). To make these paints, he needs two ingredients, the pigment and the resin. Standard quality paint requires 2 units of pigment and 3 units of resin for each unit made, and is sold at a profit of R1 per unit. Top quality paint requires 4 units of pigment and 2 units of resin for each unit made, and is sold at a profit of R1.50 per unit. He has stocks of 12 units of pigment, and 10 units of resin. Formulate the above problem as a linear programming problem to maximize his profit?*

We make the following table from the given data.

	Product		Available
Ingredients	S-Type	T-Type	Stock
Pigment	2	4	12
Resin	3	2	10
Profit (R/Unit)	1.0	1.5	

We follow the four steps outlined above for solving LP problems.

1. In our prototype Example , the number of units of S-type and T-type paint are the decision variables.

2. The first constraint is the number of units of pigment available, while the second constraint is the number of units of resin available. It is required that the total pigment and resin used does not exceed 12 and 10, respectively.

Pigment:	for S is 2	Resin:	S = 3
	for T is 4		T = 2

Therefore the required mathematical expressions for the constraints are

$$2S + 4T \leq 12$$
$$3S + 2T \leq 10$$

3. If we let P be the profit, then the objective in our example is to maximize profits

$$P = S + 1.5T,$$

i.e. the number of units of S times R1 *plus* the number of units of T times R1.5 .

4. In addition to the given constraints, there are nonnegativity constraints which ensure that the solution is meaningful. This is a requirement that whatever the decision, the decision variables should not be negative.

$$S \geq 0, T \geq 0$$

We can now write the complete mathematical model of the problem described in Example as

$$\begin{aligned} \text{Maximise:} \quad & P = S + 1.5T \\ \text{Subject to:} \quad & 2S + 4T \leq 12 \\ & 3S + 2T \leq 10 \\ & S \geq 0, T \geq 0 \end{aligned} \qquad (2)$$

The above problem is an example of a maximization LPP. Maximization LPPs are usually identified by the $\leq$ in all the constraints. Minimization problems can be identified by a $\geq$ in all the constraints.

In the next example we formulate a minimization LPP.

Example :

(Diet problem) A house wife wishes to mix two types of food F_1 and F_2 in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food F_1 costs E60/Kg and Food F_2 costs E80/kg. Food F_1 contains 3 units/kg of vitamin A and 5 units/kg of vitamin B while Food F_2 contains 4 units/kg of vitamin A and 2 units/kg of vitamin B. Formulate this problem as a linear programming problem to minimize the cost of the mixtures.

We make the following table from the given data.

Vitamin content	Food (in Kg)		Requirement (in units)
	F_1	F_2	
Vitamin A (units/kg)	3	4	8
Vitamin B (units/kg)	5	2	11
Cost (E/Kg)	60	80	

In formulating the LPP we use the following steps:

1. The number of kilograms of the foods F_1 and F_2 contained in the mixture are the decision variables. Let the mixture contain x_1 Kg of Food F_1 and x_2 Kg of food F_2.

2. In this example, the constraints are the minimum requirements of the vitamins. The minimum requirement of vitamin A is 8 units. Therefore

$$3x_1 + 4x_2 \geq 8$$

   Similarly, the minimum requirement of vitamin B is 11 units. Therefore,

$$5x_1 + 2x_2 \geq 11$$

3. The cost of purchasing 1 Kg of food F_1 is E60.

   The cost of purchasing 1 Kg of food F_2 is E80.

   The total cost of purchasing x_1 Kg of food F_1 and x_2 Kg of food F_2 is

$$C = 60x_1 + 80x_2$$

   which is the objective function.

4. The non-negativity conditions are

$$x_1 \geq 0, \quad x_2 \geq 0$$

Therefore the mathematical formulation of the LPP is

$$\begin{aligned} \text{Minimize:} \quad & C = 60x_1 + 80x_2 \\ \text{Subject to:} \quad & 3x_1 + 4x_2 \geq 8 \\ & 5x_1 + 2x_2 \geq 11 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

6.4 The Graphical Method of Solution

The **graphical method** of solving a linear programming problem is used when there are only two decision variables. If the problem has three or more variables, the graphical method is not suitable. In that case we use the **simplex method** which is discussed in the next section.

We begin by giving some important definitions and concepts that are used in the methods of solving linear programming problems.

1. **Solution** A set of values of decision variables satisfying all the constraints of a linear programming problem is called a *solution* to that problem.

2. **Feasible solution** Any solution which also satisfies the non-negativity restrictions of the problem is called a *feasible solution*.

3. **Optimal feasible solution** Any feasible solution which maximizes or minimizes the objective function is called an *optimal feasible solution*.

4. **Feasible region** The common region determined by all the constraints and non-negativity restriction of a LPP is called a *feasible region.*

5. **Corner point** A *corner point* of a feasible region is a point in the feasible region that is the intersection of two boundary lines.

The following theorem is the fundamental theorem of linear programming .

Theorem : *If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the corner points of the feasible region.*

To solve a linear programming problem with two decision variables using the graphical method we use the procedure outlined below;

	Graphical method of solving a LPP
Step 1.	Formulate the linear programming problem.
Step 2.	Graph the feasible region and find the corner points. The coordinates of the corner points can be obtained by either inspection or by solving the two equations of the lines intersecting at that point.
Step 3.	Make a table listing the value of the objective function at each corner point.
Step 4.	Determine the optimal solution from the table in step 3. If the problem is of maximization (minimization) type, the solution corresponding to the largest (smallest) value of the objective function is the optimal solution of the LPP.

We will now use this procedure to solve some LPP where the model has already been determined. We use example (0.1.1) for illustration purposes The graph of the LPP is shown in Figure 1.

Step 2

The boundary of the feasible region consists of the lines obtained from changing the inequalities to equalities; i.e. The lines

$$2S + 4T = 12 \qquad \text{and} \qquad 3S + 2T = 10$$

Step 3

The corner points (or extreme points) and their corresponding objective functional values are.

Extreme Points	Profit ($P = S + 1.5T$)
$(0,0)$	0
$(\frac{10}{3},0)$	$\frac{10}{3}$
$(2,2)$	5
$(0,3)$	4.5

Step 4

We therefore deduce that the optimal solution is $S = 2, T = 2$ corresponding to a profit $P = 5$. Thus profits are maximized when 2 units of standard quality and 2 units of top quality type paint are produced.

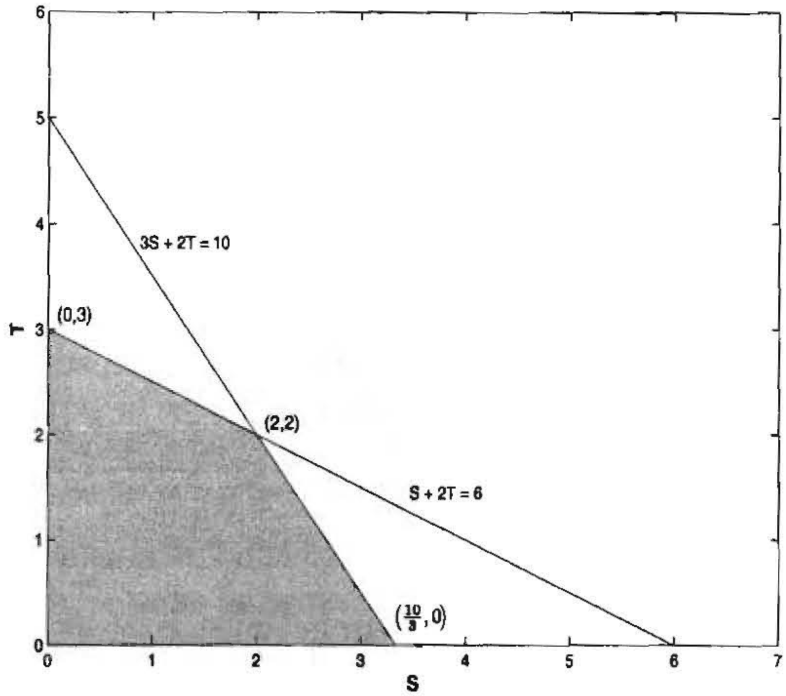


Figure 1: Graphical solution of the model of prototype example

Example:

A furniture company produces inexpensive tables and chairs. The production process for each is similar in that both require a certain number of hours of carpentry work and a certain number of labour hours in the painting department.

Each table takes 4 hours of carpentry and 2 hours in the painting department. Each chair requires 3 hours of carpentry and 1 hour in the painting department. During the current production period, 240 hours of carpentry time are available and 100 hours in painting is available. Each table sold yields a profit of E7; each chair produced is sold for a E5 profit.

Find the best combination of tables and chairs to manufacture in order to reach the maximum profit.

Solution:

We begin by summarizing the information needed to solve the problem in the form of a table. This helps us understand the problem being faced.

	Hours required to make 1 Unit		
Department	Tables	Chairs	Available Hours
Carpentry	4	3	240
Painting	2	1	100
Profit	7	5	

The objective is to maximize profit.

The constraints are

1. *The hours of carpentry time used cannot exceed 240 hours per week.*
2. *The hours of painting time used cannot exceed 100 hours per week.*
3. *The number of tables and chairs must be non-negative.*

The decision variables that represent the actual decision to be made are defined as

$$x_1 = \text{number of tables to be produced}$$
$$x_2 = \text{number of chairs to be produced}$$

Now we can state the linear programming (LP) problem in terms of x_1 *and* x_2 *and Profit* (P).

maximize	$P = 7x_1 + 5x_2$	*(Objective function)*
subject to	$4x_1 + 3x_2 \leq 240$	*(hours of carpentry constraint)*
	$2x_1 + x_2 \leq 100$	*(hours of painting constraint)*
	$x_1 \geq 0, \quad x_2 \geq 0$	*(Non-negativity constraint)*

To find the optimal solution to this LP using the graphical method we first identify the region of feasible solutions and the corner points of the of the feasible region. The graph for this example is plotted in figure (2)

In this example the corner points are (0,0), (50,0), (30,40) and (0,80). Testing these corner points on $P = 7x_1 + 5x_2$ *gives*

Corner Point	*Profit*
(0, 0)	0
(50, 0)	350
(30, 40)	410
(0, 80)	400

Because the point (30,40) produces the highest profit we conclude that producing 30 tables and 40 chairs will yield a maximum profit of E410.

Example :

A small brewery produces Ale and Beer. Suppose that production is limited by scarce resources of corn, hops and barley malt. To make Ale 5kg of Corn, 4kg of hops and 35kg of malt are required. To make Beer 15kg of corn, 4 kg of hops and 20kg of malt are required. Suppose that only 480 kg of corn, 160kg of hops and 1190 kg of malt are available. If the brewery makes a profit of E13 for each kg of Ale and E23 for each kg of Beer, how much Ale and Beer should the brewer produce in order to maximize profit?

Solution:

The given information is summarized in the table below.

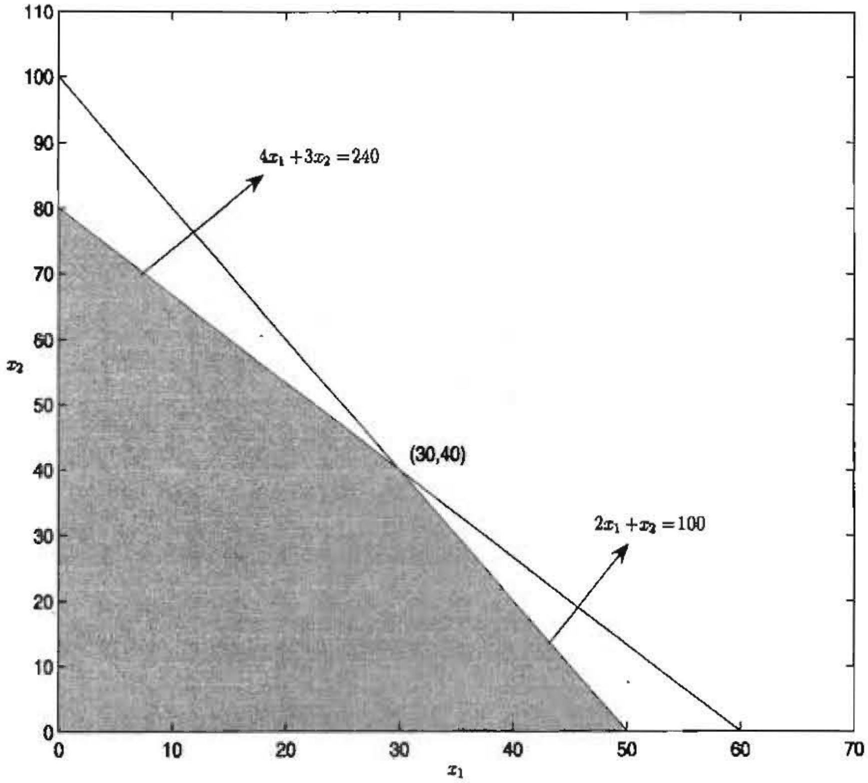


Figure 2: Graphical solution of the carpentry/painting model

Ingredients	*Beverages*		*Available quantity*
	Ale	*Beer*	
Corn(Kg)	*5*	*15*	*480*
Hops (Kg)	*4*	*4*	*160*
Malt (Kg)	*35*	*20*	*1190*
Profit	*13*	*23*	

The decision variables are

1. x_1 *the amount of Ale to be produced.*

2. x_2 *the amount of Beer to be produced.*

The profit function is given by $P = 13x_1 + 23x_2$. *Thus the LP problem can be formulated as follows:*

$$\begin{aligned} \textit{Maximize} \quad & P = 13x_1 + 23x_2 \\ \textit{Subject to} \quad & 5x_1 + 15x_2 \leq 480 \\ & 4x_1 + 4x_2 \leq 160 \\ & 35x_1 + 20x_2 \leq 1190 \\ & x_1 \geq 0\ , \quad x_2 \geq 0 \end{aligned}$$

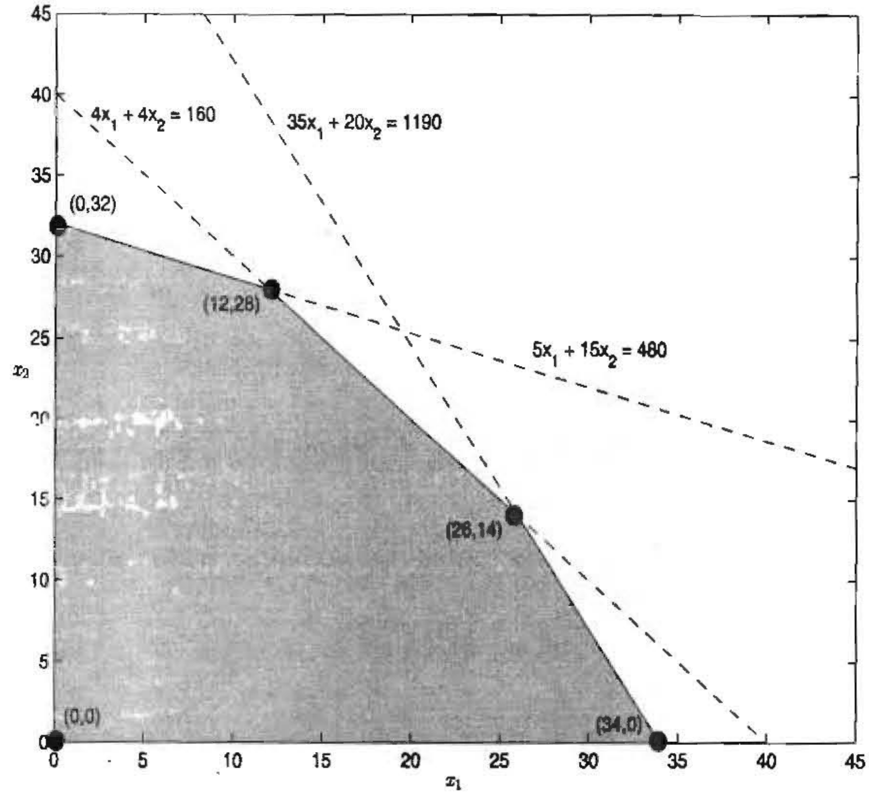


Figure 3: Graphical solution of the brewery model

The graph for this example is plotted in figure (3)

The corner points in this example are (0,0), (0,32), (12,28), (26,14) and (34,0). Testing these corner points on $P = 13x_1 + 23x_2$ *gives*

Corner Point	*Profit*
(0, 0)	0
(0, 32)	736
(12, 28)	800
(26, 14)	660
(34, 0)	442

Because the point (12,28) produces the highest profit we conclude that producing 12 Kg of Ale and 28 Kg of Beer will yield a maximum profit of E800.

Example: *(Medicine) A patient in a hospital is required to have at least 84 units of* drug A *and 120 units of* drug B *each day. Each gram of substance M contains 10 units of* drug A *and 8 units of* drug B, *and each gram of substance N contains 2 units of* drug A *and 4 units of* drug B. *Now suppose that both M and N contain an undesirable* drug C, *3 units per gram in M and 1 unit per gram in N. How many grams of substances M and N should be mixed to meet the minimum daily requirements at the same time minimize the intake of* drug C*? How many units of the undesirable* drug C *will be in this mixture?*

Solution: *We start by summarizing the given data in the following table;*

	AMOUNT OF DRUG PER GRAM		MINIMUM DAILY
	Substance M	Substance N	REQUIREMENT
Drug A	10 Units	2 units	84 units
Drug B	8 units	4 units	120 units
Drug C	3 units	1 unit	

To form the mathematical model, we start by identifying the decision variables.

Let: x_1 = *Number of grams of substance M used.*
x_2 = *Number of grams of substance N used.*

The objective is to minimize the intake of drug C. In terms of the decision variables, the objective function is

$$C = 3x_1 + x_2$$

which gives the amount of the undesirable drug C in x_1 grams of M and x_2 grams of N.

The following conditions must be satisfied to meet daily requirements:

$$\begin{pmatrix} \text{Number of units of} \\ \text{drug A} \\ \text{in } x_1 \text{ grams of substance M} \end{pmatrix} + \begin{pmatrix} \text{Number of units of} \\ \text{drug A} \\ \text{in } x_2 \text{ grams of substance N} \end{pmatrix} \geq 84$$

$$\begin{pmatrix} \text{Number of units of} \\ \text{drug B} \\ \text{in } x_1 \text{ grams of substance M} \end{pmatrix} + \begin{pmatrix} \text{Number of units of} \\ \text{drug B} \\ \text{in } x_2 \text{ grams of substance N} \end{pmatrix} \geq 120$$

$$(\text{Number of grams of substance M used}) \geq 0$$

$$(\text{Number of grams of substance N used}) \geq 0$$

Writing the above constraint inequalities in terms of the decision variables x_1 and x_2 and including the objective function we obtain the following linear programming model.

Minimize $C = 3x_1 + x_2$

Subject to $10x_1 + 2x_2 \geq 84$

$8x_1 + 4x_2 \geq 120$

$x_1 \geq 0\,, \quad x_2 \geq 0$

Figure 4 shows the graph of the feasible region obtained by plotting the system of inequalities. The evaluation of the objective function at each corner point is show in the table below.

CORNER POINT (x_1, x_2)	$C = 3x_1 + x_2$
(0,42)	42
(4,22)	34
(15,0)	45

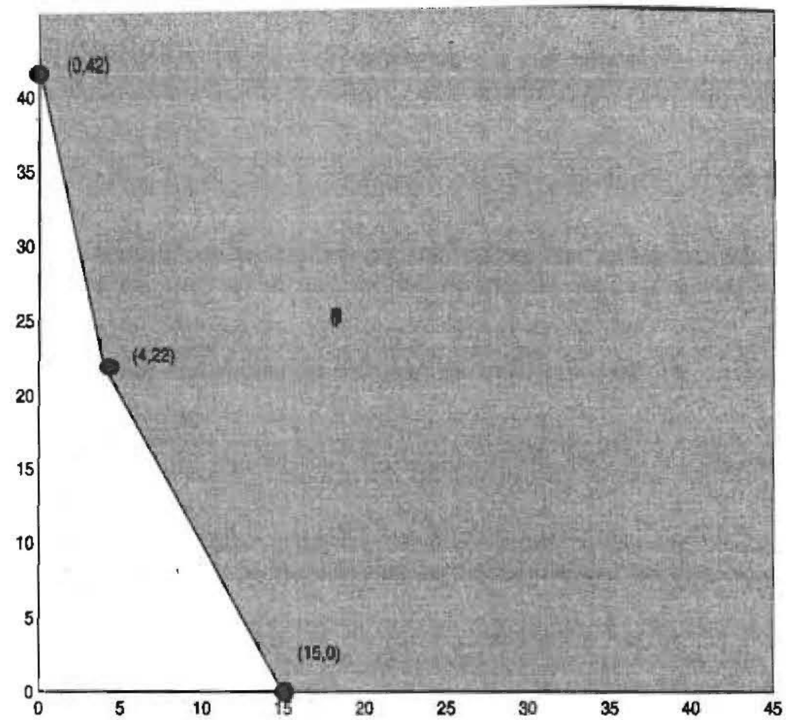


Figure 4: Graphical solution of the medicine minimization example

The graphical method is the easiest way to solve a small LP problem. However this method is useful only when there are two decision variables. When there are more than two decision variables, it is not possible to plot the solution on a two-dimensional graph and we must turn to more complex methods.

The graphical nature of the above method makes its use limited to problems involving only two decision variables. For such problems it is possible to represent the constraints graphically. A graphical solution for a problem with a higher number of decision variables than two cannot be practically obtained because of the complexity of the graphs in higher dimensional spaces. An additional limitation of this method is that if the graph is not good, the answer may be very inaccurate.

A very useful method of solving linear programming problems of any size is the so called Simplex method. The use of computers has made this method a viable tool for solving linear programming problems involving a very large number of decision variables.

6.5 Simple Linear Programming Problems

When both the objective and all the constraints in Expression are linear functions, then the optimization problem is called a *linear programming problem*. This has the general form:

$$\begin{cases} \max\ z(x_1,\ldots,x_n) = c_1x_1 + \cdots + c_nx_n \\ s.t.\ \ a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \qquad \vdots \\ \qquad a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\ \qquad h_{11}x_1 + \cdots + h_{n1}x_n = r_1 \\ \qquad \vdots \\ \qquad h_{l1}x_1 + \cdots + h_{ln}x_n = r_l \end{cases}$$

Definition (Linear Function). A function $z : \mathbb{R}^n \to \mathbb{R}$ is *linear* if there are constants $c_1, \ldots, c_n \in \mathbb{R}$ so that:

$$z(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$$

Lemma (Linear Function). *If* $z : \mathbb{R}^n \to \mathbb{R}$ *is linear then for all* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ *and for all scalar constants* $\alpha \in \mathbb{R}$ *we have:*

$$z(\mathbf{x}_1 + \mathbf{x}_2) = z(\mathbf{x}_1) + z(\mathbf{x}_2)$$
$$z(\alpha\mathbf{x}_1) = \alpha z(\mathbf{x}_1)$$

Exercise Prove Lemma 2.2.

For the time being, we will eschew the general form and focus exclusively on linear programming problems with two variables. Using this limited case, we will develop a graphical method for identifying optimal solutions, which we will generalize later to problems with arbitrary numbers of variables.

Example. Consider the problem of a toy company that produces toy planes and toy boats. The toy company can sell its planes for \$10 and its boats for \$8 dollars. It costs \$3 in raw materials to make a plane and \$2 in raw materials to make a boat. A plane requires 3 hours to make and 1 hour to finish while a boat requires 1 hour to make and 2 hours to finish. The toy company knows it will not sell anymore than 35 planes per week. Further, given the number of workers, the company cannot spend anymore than 160 hours per week finishing toys and 120 hours per week making toys. The company wishes to maximize the profit it makes by choosing how much of each toy to produce.

We can represent the profit maximization problem of the company as a linear programming problem. Let x_1 be the number of planes the company will produce and let x_2 be the number of boats the company will produce. The profit for each plane is \$10 − \$3 = \$7 per plane and the profit for each boat is \$8 − \$2 = \$6 per boat. Thus the total profit the company will make is:

$$z(x_1, x_2) = 7x_1 + 6x_2$$

The company can spend no more than 120 hours per week making toys and since a plane takes 3 hours to make and a boat takes 1 hour to make we have:

$$3x_1 + x_2 \leq 120$$

Likewise, the company can spend no more than 160 hours per week finishing toys and since it takes 1 hour to finish a plane and 2 hour to finish a boat we have:

$$x_1 + 2x_2 \leq 160$$

Finally, we know that $x_1 \leq 35$, since the company will make no more than 35 planes per week. Thus the complete linear programming problem is given as:

$$\begin{cases} \max \ z(x_1, x_2) = 7x_1 + 6x_2 \\ s.t. \ 3x_1 + x_2 \leq 120 \\ \quad x_1 + 2x_2 \leq 160 \\ \quad x_1 \leq 35 \\ \quad x_1 \geq 0 \\ \quad x_2 \geq 0 \end{cases}$$

Exercise . A chemical manufacturer produces three chemicals: A, B and C. These chemical are produced by two processes: 1 and 2. Running process 1 for 1 hour costs \$4 and yields 3 units of chemical A, 1 unit of chemical B and 1 unit of chemical C. Running process 2 for 1 hour costs \$1 and produces 1 units of chemical A, and 1 unit of chemical B (but none of Chemical C). To meet customer demand, at least 10 units of chemical A, 5 units of chemical B and 3 units of chemical C must be produced daily. Assume that the chemical manufacturer wants to minimize the cost of production. Develop a linear programming problem describing the constraints and objectives of the chemical manufacturer. [Hint: Let x_1 be the amount of time Process 1 is executed and let x_2 be amount of time Process 2 is executed. Use the coefficients above to express the cost of running Process 1 for x_1 time and Process 2 for x_2 time. Do the same to compute the amount of chemicals A, B, and C that are produced.]

Modeling Assumptions in Linear Programming

Inspecting Example (or the more general Problem) we can see there are several assumptions that must be satisfied when using a linear programming model. We enumerate these below:

Proportionality Assumption: A problem can be phrased as a linear program only if the contribution to the objective function *and* the left-hand-side of each constraint by each decision variable $(x_1, \ldots, x_n)$ is proportional to the value of the decision variable.

Additivity Assumption: A problem can be phrased as a linear programming problem only if the contribution to the objective function *and* the left-hand-side of each constraint by any decision variable x_i $(i = 1, \ldots, n)$ is completely independent of any other decision variable x_j $(j \neq i)$ and additive.

Divisibility Assumption: A problem can be phrased as a linear programming problem only if the quantities represented by each decision variable are infinitely divisible (i.e., fractional answers make sense).

Certainty Assumption: A problem can be phrased as a linear programming problem only if the coefficients in the objective function and constraints are known with certainty.

The first two assumptions simply assert (in English) that both the objective function and functions on the left-hand-side of the (in)equalities in the constraints are linear functions of the variables $x_1, \ldots, x_n$.

The third assumption asserts that a valid optimal answer could contain fractional values for decision variables. It's important to understand how this assumption comes into play-even in the toy making example. Many quantities can be divided into non-integer values (ounces, pounds etc.) but many other quantities cannot be divided. For instance, can we really expect that it's reasonable to make 1/2 a plane in the toy making example? When values must be constrained to true integer values, the linear programming problem is called an *integer programming problem.*

6.6 Graphically Solving Linear Programs Problems with Two Variables (Bounded Case)

Linear Programs (LP's) with two variables can be solved graphically by plotting the feasible region along with the level curves of the objective function. We will show that we can find a point in the feasible region that maximizes the objective function using the level curves of the objective function. We illustrate the method first using the problem from Example.

Example (Continuation of Example). Let's continue the example of the Toy Maker begin in Example . To solve the linear programming problem graphically, begin by drawing the feasible region. This is shown in the blue shaded region of Figure .

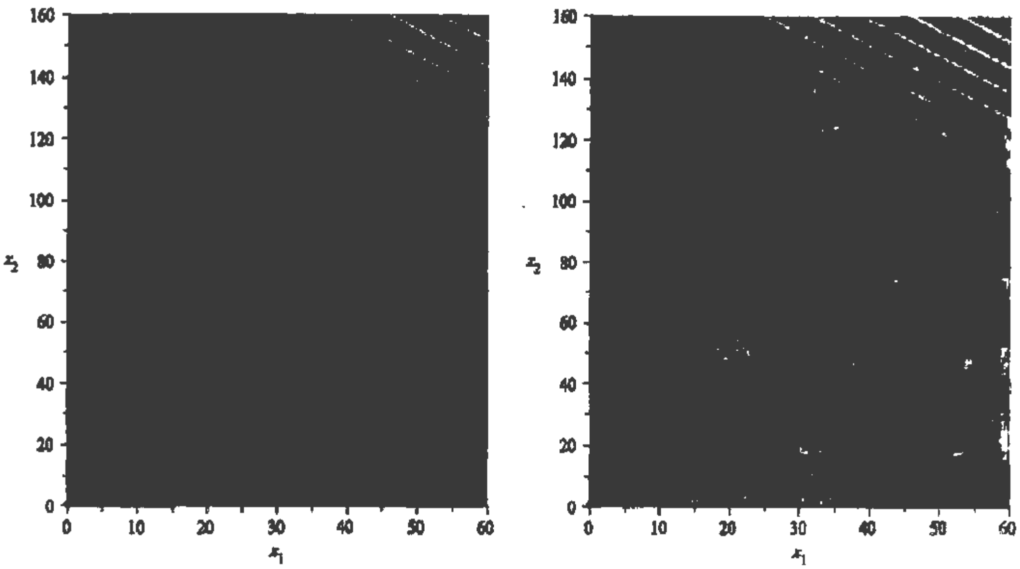


Figure Feasible Region and Level Curves of the Objective Function: The shaded region in the plot is the feasible region and represents the intersection of the five inequalities constraining the values of x_1 and x_2. On the right, we see the optimal solution is the "last" point in the feasible region that intersects a level set as we move in the direction of increasing profit.

After plotting the feasible region, the next step is to plot the level curves of the objective function. In our problem, the level sets will have the form:

$$7x_1 + 6x_2 = c \implies x_2 = \frac{-7}{6}x_1 + \frac{c}{6}$$

This is a set of parallel lines with slope $-7/6$ and intercept $c/6$ where c can be varied as needed. The level curves for various values of c are parallel lines. In Figure they are shown in colors ranging from red to yellow depending upon the value of c. Larger values of c are more yellow.

To solve the linear programming problem, follow the level sets along the gradient (shown as the black arrow) until the last level set (line) intersects the feasible region. If you are doing this by hand, you can draw a single line of the form $7x_1 + 6x_2 = c$ and then simply draw parallel lines in the direction of the gradient $(7, 6)$. At some point, these lines will fail to intersect the feasible region. The last line to intersect the feasible region will do so at a point that maximizes the profit. In this case, the point that maximizes $z(x_1, x_2) = 7x_1 + 6x_2$, subject to the constraints given, is $(x_1^*, x_2^*) = (16, 72)$.

Note the point of optimality $(x_1^*, x_2^*) = (16, 72)$ is at a corner of the feasible region. This corner is formed by the intersection of the two lines: $3x_1 + x_2 = 120$ and $x_1 + 2x_2 = 160$. In this case, the constraints

$$3x_1 + x_2 \leq 120$$
$$x_1 + 2x_2 \leq 160$$

are both *binding*, while the other constraints are non-binding. In general, we will see that when an optimal solution to a linear programming problem exists, it will always be at the intersection of several binding constraints; that is, it will occur at a corner of a higher-dimensional polyhedron.

Formalizing The Graphical Method

In order to formalize the method we've shown above, we will require a few new definitions.

Definition Let $r \in \mathbb{R}$, $r \geq 0$ be a non-negative scalar and let $\mathbf{x}_0 \in \mathbb{R}^n$ be a point in $\mathbb{R}^n$. Then the set:

$$B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n | \; ||\mathbf{x} - \mathbf{x}_0|| \leq r\}$$

is called the *closed ball of radius* r *centered at point* $\mathbf{x}_0$ *in* $\mathbb{R}^n$.

In $\mathbb{R}^2$, a closed ball is just a disk and its circular boundary centered at $\mathbf{x}_0$ with radius r. In $\mathbb{R}^3$, a closed ball is a solid sphere and its spherical centered at $\mathbf{x}_0$ with radius r. Beyond three dimensions, it becomes difficult to visualize what a closed ball looks like.

We can use a closed ball to define the notion of boundedness of a feasible region:

Definition Let $S \subseteq \mathbb{R}^n$. Then the set S is *bounded* if there exists an $\mathbf{x}_0 \in \mathbb{R}^n$ and finite $r \geq 0$ such that S is totally contained in $B_r(\mathbf{x}_0)$; that is, $S \subset B_r(\mathbf{x}_0)$.

Definition is illustrated in Figure. The set S is shown in blue while the ball of radius r centered at $\mathbf{x}_0$ is shown in gray.

We can now define an algorithm for identifying the solution to a linear programing problem in two variables with a *bounded* feasible region (see Algorithm 1):

The example linear programming problem presented in the previous section has a single optimal solution. In general, the following outcomes can occur in solving a linear programming problem:

(1) The linear programming problem has a unique solution. (We've already seen this.)
(2) There are infinitely many alternative optimal solutions.
(3) There is no solution and the problem's objective function can grow to positive infinity for maximization problems (or negative infinity for minimization problems).
(4) There is no solution to the problem at all.

Case 3 above can only occur when the feasible region is unbounded; that is, it cannot be surrounded by a ball with finite radius. We will illustrate each of these possible outcomes in the next four sections. We will prove that this is true in a later chapter.

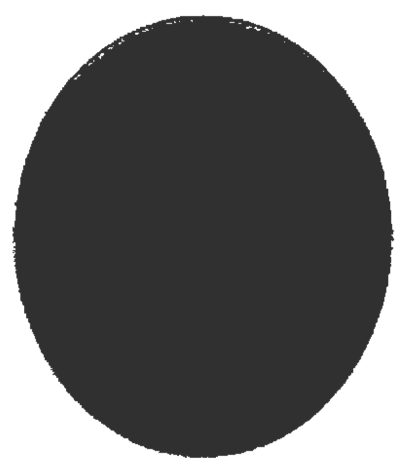


Figure A Bounded Set: The set S (in blue) is bounded because it can be entirely contained inside a ball of a finite radius r and centered at some point $\mathbf{x}_0$. In this example, the set S is in $\mathbb{R}^2$. This figure also illustrates the fact that a ball in $\mathbb{R}^2$ is just a disk and its boundary.

Algorithm for Solving a Linear Programming Problem Graphically
Bounded Feasible Region, Unique Solution

(1) Plot the feasible region defined by the constraints.
(2) Plot the level sets of the objective function.
(3) For a maximization problem, identify the level set corresponding the greatest (least, for minimization) objective function value that intersects the feasible region. This point will be at a corner.
(4) The point on the corner intersecting the greatest (least) level set is a solution to the linear programming problem.

Algorithm 1. Algorithm for Solving a Two Variable Linear Programming Problem Graphically–Bounded Feasible Region, Unique Solution Case

Example. Suppose the toy maker in Example finds that it can sell planes for a profit of \$18 each instead of \$7 each. The new linear programming problem becomes:

$$\begin{cases} \max\ z(x_1, x_2) = 18x_1 + 6x_2 \\ s.t.\ 3x_1 + x_2 \leq 120 \\ \quad x_1 + 2x_2 \leq 160 \\ \quad x_1 \leq 35 \\ \quad x_1 \geq 0 \\ \quad x_2 \geq 0 \end{cases}$$

Applying our graphical method for finding optimal solutions to linear programming problems yields the plot shown in Figure. The level curves for the function $z(x_1, x_2) = 18x_1 + 6x_2$ are *parallel* to one face of the polygon boundary of the feasible region. Hence, as we move further up and to the right in the direction of the gradient (corresponding to larger and larger values of $z(x_1, x_2)$) we see that there is not *one* point on the boundary of the feasible region that intersects that level set with greatest value, but instead a side of the polygon boundary described by the line $3x_1 + x_2 = 120$ where $x_1 \in [16, 35]$. Let:

$$S = \{(x_1, x_2) | 3x_1 + x_2 \leq 120,\ x_1 + 2x_2 \leq 160,\ x_1 \leq 35,\ x_1, x_2 \geq 0\}$$

that is, S is the feasible region of the problem. Then for any value of $x_1^* \in [16, 35]$ and any value x_2^* so that $3x_1^* + x_2^* = 120$, we will have $z(x_1^*, x_2^*) \geq z(x_1, x_2)$ for all $(x_1, x_2) \in S$. Since there are infinitely many values that x_1 and x_2 may take on, we see this problem has an infinite number of alternative optimal solutions.

Based on the example in this section, we can modify our algorithm for finding the solution to a linear programming problem graphically to deal with situations with an infinite set of alternative optimal solutions (see Algorithm 2):

Algorithm for Solving a Linear Programming Problem Graphically
Bounded Feasible Region

(1) Plot the feasible region defined by the constraints.
(2) Plot the level sets of the objective function.
(3) For a maximization problem, identify the level set corresponding the greatest (least, for minimization) objective function value that intersects the feasible region. This point will be at a corner.
(4) The point on the corner intersecting the greatest (least) level set is a solution to the linear programming problem.
(5) **If the level set corresponding to the greatest (least) objective function value is parallel to a side of the polygon boundary next to the corner identified, then there are infinitely many alternative optimal solutions and any point on this side may be chosen as an optimal solution.**

Algorithm 2. Algorithm for Solving a Two Variable Linear Programming Problem Graphically–Bounded Feasible Region Case

Exercise. Modify the linear programming problem from Exercise to obtain a linear programming problem with an infinite number of alternative optimal solutions. Solve the

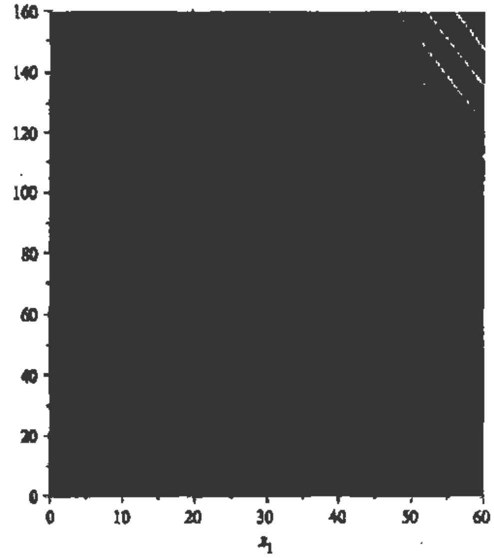


Figure. An example of infinitely many alternative optimal solutions in a linear programming problem. The level curves for $z(x_1, x_2) = 18x_1 + 6x_2$ are *parallel* to one face of the polygon boundary of the feasible region. Moreover, this side contains the points of greatest value for $z(x_1, x_2)$ inside the feasible region. Any combination of (x_1, x_2) on the line $3x_1 + x_2 = 120$ for $x_1 \in [16, 35]$ will provide the largest possible value $z(x_1, x_2)$ can take in the feasible region S.

new problem and obtain a description for the set of alternative optimal solutions. [Hint: Just as in the example, x_1 will be bound between two value corresponding to a side of the polygon. Find those values and the constraint that is binding. This will provide you with a description of the form for any $x_1^* \in [a, b]$ and x_2^* is chosen so that $cx_1^* + dx_2^* = v$, the point (x_1^*, x_2^*) is an alternative optimal solution to the problem. Now you fill in values for a, b, c, d and v.]

Problems with No Solution

Recall for *any* mathematical programming problem, the feasible set or region is simply a subset of $\mathbb{R}^n$. If this region is empty, then there is no solution to the mathematical programming problem and the problem is said to be *over constrained.* We illustrate this case for linear programming problems with the following example.

Example. Consider the following linear programming problem:

$$\begin{cases} \max \ z(x_1, x_2) = 3x_1 + 2x_2 \\ s.t. \ \dfrac{1}{40}x_1 + \dfrac{1}{60}x_2 \leq 1 \\ \quad \dfrac{1}{50}x_1 + \dfrac{1}{50}x_2 \leq 1 \\ \quad x_1 \geq 30 \\ \quad x_2 \geq 20 \end{cases}$$

The level sets of the objective and the constraints are shown in Figure.

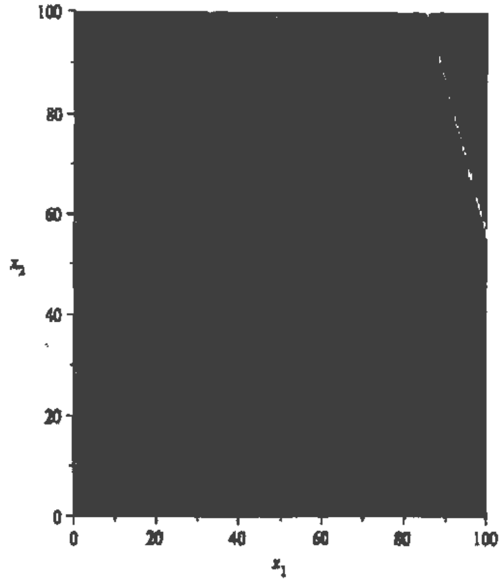


Figure. A Linear Programming Problem with no solution. The feasible region of the linear programming problem is empty; that is, there are no values for x_1 and x_2 that can simultaneously satisfy all the constraints. Thus, no solution exists.

The fact that the feasible region is empty is shown by the fact that in Figure there is no blue region--i.e., all the regions are gray indicating that the constraints are not satisfiable.

Based on this example, we can modify our previous algorithm for finding the solution to linear programming problems graphically (see Algorithm 3):

Algorithm for Solving a Linear Programming Problem Graphically
Bounded Feasible Region

(1) Plot the feasible region defined by the constraints.
(2) **If the feasible region is empty, then no solution exists.**
(3) Plot the level sets of the objective function.
(4) For a maximization problem, identify the level set corresponding the greatest (least, for minimization) objective function value that intersects the feasible region. This point will be at a corner.
(5) The point on the corner intersecting the greatest (least) level set is a solution to the linear programming problem.
(6) **If the level set corresponding to the greatest (least) objective function value is parallel to a side of the polygon boundary next to the corner identified, then there are infinitely many alternative optimal solutions and any point on this side may be chosen as an optimal solution.**

Algorithm 3. Algorithm for Solving a Two Variable Linear Programming Problem Graphically–Bounded Feasible Region Case

6.7 Problems with Unbounded Feasible Regions

Again, we'll tackle the issue of linear programming problems with unbounded feasible regions by illustrating the possible outcomes using examples.

Example. Consider the linear programming problem below.

$$\begin{cases} \max\ z(x_1, x_2) = 2x_1 - x_2 \\ s.t.\ x_1 - x_2 \leq 1 \\ \quad 2x_1 + x_2 \geq 6 \\ \quad x_1, x_2 \geq 0 \end{cases}$$

The feasible region and level curves of the objective function are shown in Figure. The

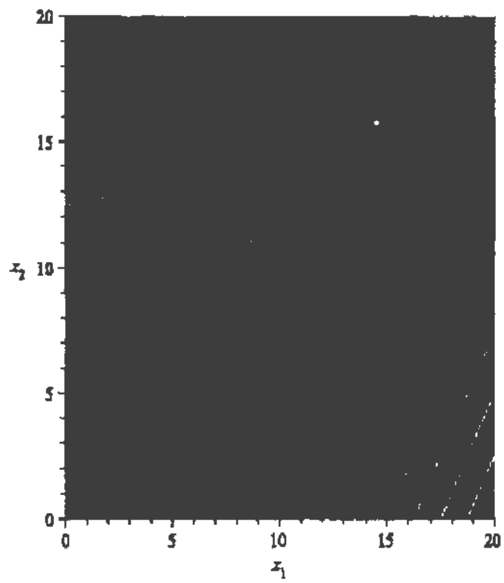


Figure. A Linear Programming Problem with Unbounded Feasible Region: Note that we can continue to make level curves of $z(x_1, x_2)$ corresponding to larger and larger values as we move down and to the right. These curves will continue to intersect the feasible region for any value of $v = z(x_1, x_2)$ we choose. Thus, we can make $z(x_1, x_2)$ as large as we want and still find a point in the feasible region that will provide this value. Hence, the optimal value of $z(x_1, x_2)$ subject to the constraints $+\infty$. That is, the problem is unbounded.

feasible region in Figure is clearly unbounded since it stretches upward along the x_2 axis infinitely far and also stretches rightward along the x_1 axis infinitely far, bounded below by the line $x_1 - x_2 = 1$. There is no way to enclose this region by a disk of finite radius, hence the feasible region is not bounded.

We can draw more level curves of $z(x_1, x_2)$ in the direction of increase (down and to the right) as long as we wish. There will always be an intersection point with the feasible region because it is infinite. That is, these curves will continue to intersect the feasible region for any value of $v = z(x_1, x_2)$ we choose. Thus, we can make $z(x_1, x_2)$ as large as we want and still find a point in the feasible region that will provide this value. Hence, the largest value

$z(x_1, x_2)$ can take when (x_1, x_2) are in the feasible region is $+\infty$. That is, the problem is unbounded.

Just because a linear programming problem has an unbounded feasible region does not imply that there is not a finite solution. We illustrate this case by modifying example.

Example. (Continuation of Example). Consider the linear programming problem from Example with the new objective function: $z(x_1, x_2) = (1/2)x_1 - x_2$. Then we have the new problem:

$$\begin{cases} \max \ z(x_1, x_2) = \dfrac{1}{2}x_1 - x_2 \\ s.t. \ x_1 - x_2 \leq 1 \\ \quad 2x_1 + x_2 \geq 6 \\ \quad x_1, x_2 \geq 0 \end{cases}$$

The feasible region, level sets of $z(x_1, x_2)$ and gradients are shown in Figure. In this case note, that the direction of increase of the objective function is *away* from the direction in which the feasible region is unbounded (i.e., downward). As a result, the point in the feasible region with the largest $z(x_1, x_2)$ value is $(7/3, 4/3)$. Again this is a vertex: the binding constraints are $x_1 - x_2 = 1$ and $2x_1 + x_2 = 6$ and the solution occurs at the point these two lines intersect.

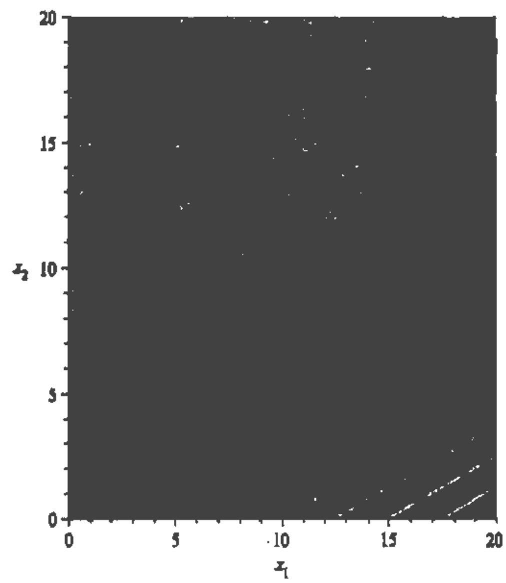


Figure. A Linear Programming Problem with Unbounded Feasible Region and Finite Solution: In this problem, the level curves of $z(x_1, x_2)$ increase in a more "southernly" direction that in Example -that is, *away* from the direction in which the feasible region increases without bound. The point in the feasible region with largest $z(x_1, x_2)$ value is $(7/3, 4/3)$. Note again, this is a vertex.

Based on these two examples, we can modify our algorithm for graphically solving a two variable linear programming problems to deal with the case when the feasible region is unbounded.

Algorithm for Solving a Two Variable Linear Programming Problem Graphically

(1) Plot the feasible region defined by the constraints.
(2) If the feasible region is empty, then no solution exists.
(3) If the feasible region is unbounded goto Line 8. Otherwise, Goto Line 4.
(4) Plot the level sets of the objective function.
(5) For a maximization problem, identify the level set corresponding the greatest (least, for minimization) objective function value that intersects the feasible region. This point will be at a corner.
(6) The point on the corner intersecting the greatest (least) level set is a solution to the linear programming problem.
(7) **If the level set corresponding to the greatest (least) objective function value is parallel to a side of the polygon boundary next to the corner identified, then there are infinitely many alternative optimal solutions and any point on this side may be chosen as an optimal solution.**
(8) (The feasible region is unbounded): Plot the level sets of the objective function.
(9) If the level sets intersect the feasible region at larger and larger (smaller and smaller for a minimization problem), then the problem is unbounded and the solution is $+\infty$ ($-\infty$ for minimization problems).
(10) Otherwise, identify the level set corresponding the greatest (least, for minimization) objective function value that intersects the feasible region. This point will be at a corner.
(11) The point on the corner intersecting the greatest (least) level set is a solution to the linear programming problem. **If the level set corresponding to the greatest (least) objective function value is parallel to a side of the polygon boundary next to the corner identified, then there are infinitely many alternative optimal solutions and any point on this side may be chosen as an optimal solution.**

Algorithm . Algorithm for Solving a Linear Programming Problem Graphically–Bounded and Unbounded Case

Exercise:

Use the graphical method to solve each of the following LP problems.

1. A wheat and barley farmer has 168 hectare of ploughed land, and a capital of E2000. It costs E14 to sow one hectare wheat and E10 to sow one hectare of barley. Suppose that his profit is E80 per hectare of wheat and E55 per hectare of barley. Find the optimal number of hectares of wheat and barley that must be ploughed in order to maximize profit? What is the maximum profit? [80,88], Profit E11 240

2. An company manufactures two electrical products: air conditioners and large fans. The assembly process for each is similar in that both require a certain amount of wiring and drilling. Each air conditioner takes 3 hours of wiring and 2 hours of drilling. Each fan must go through 2 hours of wiring and 1 hour of drilling. During the next production period, 240 hours of wiring time are available and up to 140 hours of drilling time may be used. Each air conditioner sold yields a profit of E25. Each fan assembled may be sold for a profit of E15. Formulate and solve this linear programming mix situation to find the best combination of air conditioners and fans that yields the highest profit. [40 air conditioners, 60 fans, profit E1900]

3. A manufacturer of lightweight mountain tents makes a standard model and an expedition model for national distribution. Each standard tent requires 1 labour hour from the cutting department and 3 labour hours from the assembly department. Each expedition tent requires 2 labour hours from the cutting department and 4 labour hours from the assembly department. The maximum labour hours available per day in the cutting department and the assembly department are 32 and 84 respectively. If the company makes a profit of E50 on each standard tent and E80 on each expedition tent, use the graphical method to determine how many tents of each type should be manufactured each day to maximize the total daily profit? [E1480]

4. A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labour hours from the cutting department and 0.8 labour hours from the assembly department. Each four-person boat requires 1.8 labour hours from the cutting department and 1.2 labour hours from the assembly department. The maximum labour hours available per month in the cutting department and the assembly department are 864 and 672 respectively. The company makes a profit of E25 on each two-person boat and E40 on each four-person boat. Use the graphical method to find the maximum profit. [E21 600]

5. LESCO Engineering produces chairs and tables. Each table takes four hours of labour from the carpentry department and two hours of labour from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of E70 and each chair a profit of E50. How many chairs and tables should be made in order to maximize profit? [40,30], P = E410

6. A company manufactures two products X and Y. Each product has to be processed in three departments: welding, assembly and painting. Each unit of X spends 2 hours in the welding department, 3 hours in assembly and 1 hour in painting. The corresponding times for a unit of Y are 3,2 and 1 respectively. The man-hours available in a month are 1500 for the welding department, 1500 in assembly and 550 in painting. The contribution to profits and fixed

overheads are E100 for product X and E120 for product Y. Formulate the appropriate linear programming problem and solve it graphically to obtain the optimal solution for the maximum contribution. [150, 400], P = 63000

7. Suppose a manufacturer of printed circuits has a stock of 200 resistors, 120 transistors and 150 capacitors and is required to produce two types of circuits.

   Type A requires 20 resistors, 10 transistors and 10 capacitors.

   Type B requires 10 resistors, 20 transistors and 30 capacitors.

   If the profit on type A circuits is E5 and that on type B circuits is E12, how many of each circuit should be produced in order to maximize profit? [6,3], P = 66

8. A small company builds two types of garden chairs.

   Type A requires 2 hours of machine time and 5 hours of craftsman time.

   Type B requires 3 hours of machine time and 5 hours of craftsman time.

   Each day there are 30 hours of machine time available and 60 hours of craftsman time. The profit on each type A chair is E60 and on each type B chair is E84. Formulate the appropriate linear programming problem and solve it graphically to obtain the optimal solution that maximizes profit. [6,6], P = 864

9. Namboard produces two gift packages of fruit. Package A contains 20 peaches, 15 apples and 10 pears. Package B contains 10 peaches, 30 apples and 12 pears. Namboard has 40 000 peaches, 60 000 apples and 27 000 pears available for packaging. The profit on package A is E2.00 and the profit on B is E2.50. Assuming that all fruit packaged can be sold, what number of packages of types A and B should be prepared to maximize the profit? [750 type A, 1625 type B]

10. A factory manufactures two products, each requiring the use of three machines. The first machine can be used at most 70 hours; the second machine at most 40 hours; and the third machine at most 90 hours. The first product requires 2 hours on Machine 1, 1 hour on Machine 2, and 1 hour on Machine 3; the second product requires 1 hour each on machines 1and 2 and 3 hours on Machine 3. If the profit in E40 per unit for the first product and E60 per unit for the second product, how many units of each product should be manufactured to maximize profit? [24,22, P = 2280]

Exercises : Minimization problems

1. A house wife wishes to mix together two kinds of food, I and II, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below;

	Vitamin A	Vitamin B	Vitamin C
Food I	1	2	3
Food II	2	2	1

One Kg of food I costs E6 and one Kg of food II costs E10. Formulate the above problem as a linear programming problem and find the least cost of the mixture which will produce the diet. [2,4, cost = E52]

2. A chicken farmer can buy a special food mix A at 20c per Kg and special food mix B at 40c per Kg. Each Kg of mix A contains 3000 units of nutrient N_1 and 1000 units of nutrient N_2; each Kg of mix B contains 4000 units of nutrient N_1 and 4000 units of nutrient N_2. If the minimum daily requirements for the chickens collectively are 36000 units of nutrient N_1 and 20000 units of nutrient N_2, how many pounds of each food mix should be used each day to minimize daily food costs while meeting (or exceeding) the minimum daily nutrient requirements? What is the minimum daily cost? [8kg of mix A, 3 kg of mix B; C = E2.80 per day]

3. A farmer can buy two types of plant food, mix A and mix B. Each cubic metre of mix A contains 20 kg of phosphoric acid, 30 kg of nitrogen, and 5 kg of potash. Each cubic metre of mix B contains 10 kg of phosphoric acid, 30 kg of nitrogen and 10 kg of potash. The minimum monthly requirements are 460 kg of phosphoric acid, 960 kg of nitrogen, and 220 kg of potash. If mix A costs E30 per cubic metre and mix B costs E35 per cubic metre, how many cubic metres of each mix should the farmer blend to meet the minimum monthly requirements at a minimal cost? What is the cost? [20 m^3, 12 m^3, E1020]

4. A city council voted to conduct a study on inner city community problems. A nearby university was contacted to provide sociologists and research assistants. Allocation of time and costs per week are given in the table. How many sociologists and how many research assistants should be hired to minimize the cost and meet the weekly labour-hour requirements? What is the weekly cost?

	LABOUR HOURS		MINIMUM LABOUR-HOURS NEEDED PER WEEK
	Sociologist	Research Assistant	
FIELDWORK	10	30	180
RESEARCH CENTRE	30	10	140
COSTS PER WEEK (E)	500	300	

5. A laboratory technician in a medical research centre is asked to formulate a diet from two commercially packaged foods, food A and food B, for a group of animals. Each kg of food A contains 8 units of fat, 16 units of carbohydrates, and 2 units of protein. Each Kg of food B contains 4 units of fat, 32 units of carbohydrate and 8 units of protein. The minimum daily requirements are 176 units of fat, 1024 units of carbohydrate, and 384 units of protein. If

food A costs 5c per Kg and food B costs 5c per Kg, how many kilograms of each food should be used to meet the minimum daily requirements at the least cost? What is the cost of this amount?

6. A can of cat food, guaranteed by the manufacturer to contain at least 10 units of protein, 20 units of mineral matter, and 6 units of fat, consists of a mixture of four different ingredients. Ingredient A contains 10 units of protein, 2 units of mineral matter, and $\frac{1}{2}$ unit of fat per 100g. Ingredient B contains 1 unit of protein, 40 units of mineral matter, and 3 units of fat per 100g. Ingredient C contains 1 unit of protein, 1 unit of mineral matter, and 6 units of fat per 100g. Ingredient D contains 5 units of protein, 10 units of mineral matter, and 3 units of fat per 100g. The cost of each ingredient is 3c, 2c, 1c, and 4c per 100g, respectively. How many grammes of each should be used to minimise the cost of the cat food, while still meeting the guaranteed composition?

7 Simplex Method

Chapter Includes:

1. Introduction
2. Standard Form
3. The Simplex Procedure
4. The Optimal Solution
5. Special Cases in the Simplex Procedure
6. The Minimisation Problem : Dual Problem
7. Transportation Model
8. The Simplex Method and Transportation Problems

7.1 Introduction

The Simplex method is based on an understanding of the algebra of the linear programming problem being solved. We begin by stating a general maximising linear programming problem involving n unknown (or decision) variables and m constraints as

$$\begin{aligned}
\text{Maximise:} \quad & z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{Subject to:} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
& x_1, x_2, \ldots, x_n \ge 0
\end{aligned} \tag{1}$$

or equivalently

$$\max \quad z = \sum_{i=1}^{n} c_i x_i$$

$$\text{subj. to:} \quad \sum_{i=1}^{n} a_{ki}x_i \le b_k \tag{2}$$

$$k = 1, 2, \ldots, m$$

$$x_i \ge 0, i = 1, 2, \ldots, n \tag{3}$$

We note in particular that all the constraints involve the $\le$ sign. We will use this type of maximising linear programming problem to introduce the Simplex method. Other types of inequalities as well as the minimising problem will be discussed later. The number m, of constraints, can be less, equal or even greater than n.

The Simplex method is similar to the graphical method in that it uses the extreme points of the feasible region to search for the solution. The main difference is that with the Simplex method, once the initial vertex has been chosen, movement from one vertex to another is in such a way that the value of the objective function improves with each move. Although there are $n + m$ variables in m equations, the solution of the problem concerns the n variables in the original constraints. If $m < n$ then some of the decision variables will have zero values.

Before we can employ the Simplex method we need to rewrite the problem in a standard form in which the constraints are equations rather than inequalities.

7.2 Standard Form

We consider the k-th constraint of the general linear programming problem (1)

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \le b_k \tag{4}$$

We convert the k-th inequality constraint to an equality constraint by introducing a new variable, $x_{n+k} \ge 0$, called a **slack variable**. The name of the variable derives from the fact that if the left hand side of the constraint is to balance with the right hand side of the constraint, then something has to be added to the left side.

If we do this for each of the m constraints we can write the standard form of the system (4) as

$$\begin{aligned} &\text{Maximise} && z = \textstyle\sum_{i=1}^{n} c_i x_i \\ &\text{Subject to:} && \textstyle\sum_{i=1}^{n} a_{ki} x_i + x_{n+k} = b_k \\ &&& k = 1, 2, \ldots, m \\ &&& x_i \geq 0, x_{n+k} \geq 0, \quad i = 1, 2, \ldots, n \end{aligned} \tag{5}$$

We can write the standard form of the linear programming problem as a set of matrix equations

$$\begin{aligned} z &= C\mathbf{x} \\ A\mathbf{x} &= \mathbf{b} \end{aligned} \tag{6}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & & \ldots & \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 \end{bmatrix} \tag{7}$$

and

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{8}$$

We note the following about the standard linear programming problem:

1. The objective function is unchanged. The slack variables can be included in the objective function with zero coefficients.
2. The m constraints of the new system are represented by m equations and there are now $n+m$ unknown variables (the solution variables plus the slack variables);
3. All the variables including the slack variables are nonnegative;
4. The right side values are nonnegative.

Definition 7.1 : *A set of variables x_i, together which satisfy the equality constraints $Ax = b$ are said to be* **basic variables**. *These basic variables form a* **basic solution** *or a* **basis**. *If all the basic variables are nonnegative then they form a* **basic feasible solution**. *We note that a basic feasible solution may not necessarily optimise the objective function.*

In relation to the graphical approach we point out that every basic feasible solution is an extreme point of the feasible region, and conversely, every extreme point is a basic feasible solution.

As we discuss the Simplex procedure we will use our prototype example of the paint mix problem presented by the linear programme (2), whose solution has been previously found using the graphical method.

The linear programming problem (2) is restated below with a slight change in the names of the decision variables (x_1 instead of S and x_2 instead of T) as

$$\begin{aligned} \text{Maximise:} \quad & P = x_1 + 1.5x_2 \\ \text{Subject to:} \quad & 2x_1 + 4x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 10 \\ & x_1 \geq 0\,, x_2 \geq 0 \end{aligned} \tag{9}$$

Example 7.1 :
Writing the linear programme (11) in standard form we obtain

$$\begin{aligned} \textit{Maximise:} \quad & P = x_1 + 1.5x_2 \\ \textit{Subject to:} \quad & 2x_1 + 4x_2 + x_3 = 12 \\ & 3x_1 + 2x_2 + x_4 = 10 \\ & x_1 \geq 0\,, x_2 \geq 0 \end{aligned} \tag{10}$$

where x_3 and x_4 are the slack variables.

Once we have written a linear programming in standard we are ready to solve it using the Simplex method.

7.3 The Simplex Procedure

The Simplex procedure involves the following steps:

Step 1: **Find the initial basic feasible solution**
The simplest choice of an initial feasible basic is to assume that none of the decision variables are basic. Hence we initially assume that the basic solution consists only of the slack variables.
i.e. Set $x_i = 0, i = 1, 2, \ldots, n$ and $x_{n+k} \neq 0, k = 1, 2, \ldots, m$. This choice of the initial solution means that we initially assume that objective functional value is zero. In terms of the graphical method we start at the origin so as to move along the best possible route to the optimal solution.

Example 7.2 : *If in the constraints of the standard form (12) we set $x_1 = 0$, $x_2 = 0$ we have the initial basic feasible solution $x_3 = 12$, $x_4 = 10$, and $P = 0$.*

Step 2: **Set up the initial Simplex tableau**
In this step we arrange the various matrices and vectors involved in the matrix form of the linear programming problem in the simplex tableau. This tableau contains all the information about the current basic variables and their corresponding values, the optimality status of the solution. The metbod then continues to use the principle of the Gauss-Jordan procedure to compute the next improved solution.

A typical initial simplex tableau has the form shown in Table 1

In the tableau.

1. The top row shows both the $n + m$ decision and slack variables $x_1, x_2, \ldots, x_{n+1}, \ldots, x_{n+m}$ as labels for the corresponding columns;

2. The coefficients of the constraints are shown in the middle rows;

3. The last row is the z-equation, showing the objective coefficients;

Basic	x_1	x_2	$\dots$	x_n	x_{n+1}	x_{n+2}	$\dots$	x_{n+m}	
x_{n+1}	a_{11}	a_{12}	$\dots$	a_{1n}	1	0	$\dots$	0	b_1
x_{n+2}	a_{21}	a_{22}	$\dots$	a_{2n}	0	1	$\dots$	0	b_2
$\vdots$	$\vdots$			$\vdots$	0			$\vdots$	$\vdots$
x_{n+m}	a_{m1}	a_{m2}	$\dots$	a_{mn}	0	0	$\dots$	1	b_m
z	$-c_1$	$-c_2$	$\dots$	$-c_n$	0	0	$\dots$	0	0

Table 1: The general simplex tableau.

4. The extreme left column shows the basic variables;

5. Each basic variable

   - appears in exactly one equation in which it has a coefficient 1. The column it labels has all zeros except in the row in which it is shown as a basic variable
   - has a value shown on the extreme right column.

Initially the negative coefficients in the z-equation are a result of writing the objective equation as

$$z - c_1x_1 - c_2x_2 - \dots - c_nx_n = 0 \tag{11}$$

so that z itself is treated like a variable. When the decision variables are initially set to zero, the initial value of z is also zero. The value of z will vary as the decision variables assume nonzero values. In particular for the maximising problem z will increase as any of the nonbasic variables with a negative entry in the z-row is increased.

Example 7.3 :

The initial Simplex tableau of our example is

Tableau 1:

Basic	x_1	x_2	x_3	x_4	*R.H.S.*
x_3	*2*	$\boxed{4}$	*1*	*0*	*12*
x_4	*3*	*2*	*0*	*1*	*10*
P	−1	−1.5	*0*	*0*	*0*

The initial basic variables are $x_3 = 12$ and $x_4 = 10$ which you can read from the extreme left and right columns of the tableau.

Step 3: **Test for optimality**

At any stage of the procedure you can check whether the current basic solution is optimal. This information is contained in the objective row of the tableau. If all the entries in the objective row are **nonnegative**, then the current basic solution is optimal. In particular all the columns associated with the basic solution will have zero coefficients in the objective row while the columns associated with the nonbasic variables will have positive coefficients.

For our example, in the last row of Tableau 1 we have the negative coefficients −1 and −1.5 corresponding to x_1 and x_2. Thus the present solution is not optimal.

Step 4: **Choose the variable to enter/leave the basic set**

First you decide which of the nonbasic variables will bring about the best improvement on the objective value if entered into the basic set. i.e. if it is increased from zero. The nonbasic variables corresponding to the negative coefficients in the objective row are candidates for entry into the basic set. The **entering variable** is the one associated with the column with the most negative coefficient in the z-row. This column is known as the **pivot column**.

Since this variable will become basic, one of the basic variables will have to become nonbasic (or will leave the basic set) and be reduced to zero. The **leaving variable** determined by the quotients of the right hand column and the pivot column. You first compute quotients of the right hand side and **positive** coefficients of the pivot column. Thus you compare only positive quotients. Once you have computed all positive quotients, you choose the row which has the least quotient. This is called the **pivot row**. The leaving variable is the one which corresponds to the pivot row on the left hand side of the tableau (among the basic variables).

The element at the intersection of the pivot row and pivot column is called the **pivot coefficient**. It is normally highlighted by circling it (we will highlight it by boldface type). It is necessary to identify this element in order to move on to the next step.

In our example the variables x_1 and x_2 are the candidates for entry into the basis. The most negative coefficient in the last row is -1.5 in the column labeled x_2. This is the pivot column and thus the entering variable is x_2. This variable is raised from zero a nonzero value which will be determined in the next step.

Now we need to decide which variable should leave the basis. If we divide the coefficients of the last column by corresponding coefficients in the x_2-column we obtain the quotients $\frac{12}{4} = 3$ and $\frac{10}{2} = 5$. The smallest quotient is 3, corresponding to x_3 in the extreme left column. Thus the x_3-row is the pivot row, the variable x_3 has to leave the basis as x_2 enters, and assumes the value 3. The pivot coefficient is 4. (in box)

We are now ready to proceed to the next step.

Step 5: **Update the Simplex tableau**

The next step is to update the tableau by reducing it using the Gauss reduction principle. By updating the tableau you are essentially determining the effect of the introduction of the new basic variable and discarding the one that has left the basis.

By Gauss reduction, you reduce the pivot coefficient to one and all other coefficients in that column to zero. Let us illustrate this using our example once again.

In updating the tableau we first divide the x_3-row by 4 to reduce the pivot coefficient to 1. The coefficients 2 and -1.5 in the pivot column should be reduced to 0 by either adding or subtracting a suitable multiple of the pivot row. i.e. R_2 becomes $R_2 - 2R_1, R_3$ becomes $R_3 + 1.5R_1$. Performing these Gaussian operations leads to the tableau

Tableau 2

	x_1	x_2	x_3	x_4	
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$	0	3
x_4	**2**	0	$-\frac{1}{2}$	1	4
P	$-\frac{1}{4}$	0	$\frac{3}{8}$	0	4.5

Thus the objective functional value has improved from 0 to 4.5 as x_2 is raised from zero to 3. Note *that* $4.5 = 1.5 \times 3$, *the contribution made by* x_2 *in the objective.* Step 6: **Repeat Steps 3 - 5** Test for optimality and pivot again until the optimal solution is obtained or some other conclusion is made of the problem.

Once again we test whether the current solution is optimal. Looking at the last row of the Tableau 2 above we see that there is still a negative coefficient, so the solution is not optimal. We repeat the last three steps of the Simplex procedure. Since $-\frac{1}{4}$ *in the objective row is the only negative coefficient, the corresponding column is the pivot column and* x_1 *enters the basis. The leaving variable is obtained by comparing the quotients* $\frac{3}{\frac{1}{2}} = 6$ *and* $\frac{4}{2} = 2$. *Hence* x_4 *should leave the basis and give way to* x_1. *The pivot coefficient is 2, at the intersection of the pivot row and pivot column.*

Updating the tableau by Gauss reduction leads to the tableau

Tableau 3

	x_1	x_2	x_3	x_4	
x_2	0	1	$\frac{3}{8}$	$-\frac{1}{4}$	2
x_1	1	0	$-\frac{1}{4}$	$\frac{1}{2}$	2
P	0	0	$\frac{5}{16}$	$\frac{1}{8}$	5

Since there are no more negative coefficients in the objective row of Tableau 3 we conclude that the current solution is optimal.

7.4 The Optimal Solution

Once the optimality test is met (i.e. all coefficients in the objective row are nonnegative), we can extract the solution from the final tableau. The optimal solution consists of the basic decision variables appearing in the extreme left column. The corresponding values appear in the extreme right column. Any decision variable which is not in the basic set has a zero value.

Referring this to our particular example, the extreme columns of the final tableau give the solution

$$x_2 = 2, \quad x_1 = 2 \quad P = 5$$

which agrees with what we obtained earlier using the graphical approach.

We note what was mentioned earlier about the coefficients of the basic variables in the last row and the existence of a unit matrix in the tableau.

If we relate this procedure to the geometrical solution we observe the following movements: From the origin the search for the solution moved to the vertex $(0, 2)$ then to $(2, 2)$.

We will now solve the following linear programming problem to illustrate the implementation of the complete Simplex algorithm.

Example 7.4 :

Consider the following linear programming problem in standard form

$$\begin{aligned} \textit{Maximise} \quad & z = 120x_1 + 100x_2 \\ \textit{Subj. to} \quad & 2x_1 + 2x_2 + x_3 = 8 \\ & 5x_1 + 3x_2 + x_4 = 15 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The initial tableau is shown in Tableau 1. The initial basic feasible solution is obtained by setting $x_1 = x_2 = 0$, *so that* $x_3 = 8$ *and* $x_4 = 15$.

Tableau 1

	x_1	x_2	x_3	x_4	
x_3	2	2	1	0	8
x_4	$\boxed{5}$	3	0	1	15
z	−120	−100	0	0	0

The initial solution is not optimal since there are negative coefficients in the z-row.
The entering variable is x_1 *corresponding to the most negative coefficient,* −120.
The required quotients to determine the leaving variable are $\frac{8}{2} = 4$, *and* $\frac{15}{3} = 3$ *of which 3 is smaller. Hence* x_4 *is the leaving variable. The pivot element is therefore 5.*

In pivoting, x_1 *now replaces* x_4 *in the extreme left column. The new* x_1*-row entries are obtained by dividing by 5. The variable* x_3 *remains in the basic solution but the coefficients in the corresponding row are obtained by carrying out a row operations on* x_3*-row and z-row so that the* x_1*-column entries are zero except for the pivot coefficient.*

The new tableau is shown in Tableau 2.

Tableau 2

	x_1	x_2	x_3	x_4	
x_3	0	$\boxed{\frac{4}{5}}$	1	$-\frac{2}{5}$	2
x_1	1	$\frac{3}{5}$	0	$\frac{1}{5}$	3
z	0	−28	0	24	360

The new basic feasible solution, $x_1 = 3$, $x_2 = 0$, *is not optimal. Hence we continue pivoting. The next variable to enter the basis is* x_2. *The leaving variable is* x_3 *(quotients are 5/2 and 5) Pivoting on* $\frac{4}{5}$ *leads to the new tableau shown in Tableau 3.*

Tableau 3

	x_1	x_2	x_3	x_4	
x_2	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$
z	0	0	35	10	430

Since all the entries in the z-row of the last tableau, Tableau 3 are positive, we deduce that optimality has been reached.

The solution is obtained from reading the first and last columns of the tableau. On the first column, the basic variables are x_1 *and* x_2. *The corresponding values in the last column are* $\frac{3}{2}$ *and* $\frac{5}{2}$. *The maximum value of z is 430, corresponding to z in the last column.*

Hence the optimal solution is

$$x_1 = \frac{5}{2}, \quad x_2 = \frac{3}{2}, \quad z = 430$$

Example 7.5 :

$$\begin{aligned} \textit{Maximize} \quad & P = 70x_1 + 50x_2 + 35x_3 \\ \textit{subject to} \quad & 4x_1 + 3x_2 + x_3 \le 240 \\ & 2x_1 + x_2 + x_3 \le 100 \\ & -4x_1 + x_2 \le 0 \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

Solution:

We add the slack variables x_4, x_5 and x_6 to convert the problem to standard form.

$$\begin{aligned} \textit{Maximize} \quad & P = 70x_1 + 50x_2 + 35x_3 \\ \textit{subject to} \quad & 4x_1 + 3x_2 + x_3 + x_4 = 240 \\ & 2x_1 + x_2 + x_3 + x_5 = 100 \\ & -4x_1 + x_2 + x_6 = 0 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \ge 0 \end{aligned}$$

The initial Tableau is shown below

Tableau 1

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	4	3	1	1	0	0	240
x_5	[2]	1	1	0	1	0	100
x_6	−4	1	0	0	0	1	0
P	−70	−50	−35	0	0	0	0

The initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$ so that $x_4 = 240$, $x_5 = 100$ and $x_6 = 0$. This solution is not optimal since there are negative coefficients in the last row containing P. The entering variable is x_1 corresponding to the most negative coefficient, -70. The quotients are $\frac{100}{2} = 50$ and $\frac{240}{4} = 60$ of which 50 is the smallest (note that we don't consider the quotient $\frac{0}{-4} = 0$). Thus x_5 is the leaving variable and the pivot element is 2.

Dividing row 2 by 2 gives

	Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
R_1 :	x_4	4	3	1	1	0	0	240
R_2 :	x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	50
R_3 :	x_6	−4	1	0	0	0	1	0
R_4 :	P	−70	−50	−35	0	0	0	0

To obtain Tableau 2 we perform the following row operations

$$-4R_2 + R_1, \quad 4R_2 + R_3, \quad 70R_2 + R_4$$

This gives

Tableau 2

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	[1]	−1	1	−2	0	40
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	50
x_6	0	3	2	0	2	1	200
P	0	−15	0	0	35	0	3500

Since -15 is the only negative coefficient in the P-row, it follows that the entering variable is x_2. The quotients required to determine the leaving variable are $\frac{200}{3} = 66\frac{2}{3}$, $\frac{50}{\frac{1}{2}} = 100$, $\frac{40}{1} = 40$ of which 40 is the smallest. Thus x_4 is the leaving variable and 1 is the pivot coefficient.

To obtain Tableau 4 we perform the following row operations

$$-\frac{1}{2}R_1 + R_2, \quad -3R_1 + R_3, \quad 15R_1 + R_4$$

This gives

Tableau 3

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	−1	1	−2	0	40
x_1	1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	30
x_6	0	0	$\boxed{5}$	−3	8	1	80
P	0	0	−15	15	5	0	4100

Looking at the coefficient of the P-row in Tableau 3 we note that the only negative coefficient is -15. Thus, x_3 is the entering variable. The quotients are $\frac{80}{5} = 16$, $\frac{30}{1} = 30$. Thus, x_6 is the leaving variable and 5 is the pivot coefficient.

Dividing row 3 by 5 gives

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	−1	1	−2	0	40
x_1	1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	30
x_3	0	0	1	$-\frac{3}{5}$	$\frac{8}{5}$	$\frac{1}{5}$	16
P	0	0	−15	15	5	0	4100

To obtain Tableau 4 we perform the following row operations

$$R_3 + R_1, \quad -R_3 + R_2, \quad 15R_3 + R_4$$

Tableau 4

Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	0	$\frac{2}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	56
x_1	1	0	0	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{1}{5}$	14
x_3	0	0	1	$-\frac{3}{5}$	$\frac{8}{5}$	$\frac{1}{5}$	16
P	0	0	0	6	39	3	4340

Since all the entries in the P-row of Tableau 4 are positive, we deduce that the optimal solution has been reached. The solution is

$$x_1 = 14, \quad x_2 = 56, \quad x_3 = 16 \quad \textit{and} \quad \textit{maximum } P = 4340$$

7.5 Special Cases in the Simplex Procedure

Certain situations may arise which do not comply to the assumptions so far made in implementing the Simplex procedure. Some of these situations and how they are handled are discussed below.

It may happen that during pivoting there is a tie in the entering and leaving variables; i.e. the most negative coefficient of the z-equation appears under more than one variable; or the smallest quotient corresponds to more than one variable. Normally the tie is broken (called tie breaking) by making an arbitrarily selection of the entering or leaving variables among those that qualify.

7.5.2 Unbounded Solution

Unboundedness describes linear programs that do not have finite solutions.Under very rare occasions in the Simplex method it may turn out that every coefficient in the pivot column is either zero or negative (called the unbounded solution situation). Hence there would be no way of computing a leaving variable. In this case, it may be necessary to check if there has been no computational errors or else z would be unbounded.

7.6 The Minimisation Problem : Dual Problem

We have so far discussed the Simplex method as applied to solving a maximizing linear programming problem. One way to solve a minimisation problem is to solve an equivalent maximising problem called the **dual problem**. The theory of duality simply states that every linear programming problem can be written in two forms: the **primal** form and the **dual** form. The original problem is called the **primal problem**. The objective of a dual problem is opposite that of the given primal problem. Thus a primal minimisation problem has a dual maximisation problem. The same holds for a maximisation problem. That is, a primal maximisation problem has a dual minimisation problem.

Sometimes it is easier to solve the dual problem than it is to solve the primal problem. The relationship between the two types of problems is given in the following statement.

There is an important result called the Von Neumann duality principle which relates the optimal value of the dual problem to that of the primal problem. The statement of the result is that **the optimal solution of a primal linear programming problem, if it exists, has the same value at the optimal solution of the dual problem.** Thus the optimal value determined for the dual problem is the same optimal value for the primal problem.

7.6.1 Solving a Minimization Problem

A minimization problem is in standard form if the objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

is to be minimized subject to the constraints.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. To solve this problem we use the following steps.

1. Form the **augmented matrix** for the given system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ & & & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ c_1 & c_2 & \cdots & c_n & \vdots & 1 \end{bmatrix}$$

2. Form the **transpose** of this matrix.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & \vdots & c_1 \\ a_{12} & a_{22} & \cdots & a_{m2} & \vdots & c_2 \\ & & & & \vdots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} & \vdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ b_1 & b_2 & \cdots & b_m & \vdots & 1 \end{bmatrix}$$

3. Form the dual maximization problem corresponding to this standard matrix. That is, find the maximum of the objective function given by

$$w = b_1y_1 + b_2y_2 + \ldots + b_my_n$$

subject to the constraints.

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m &\leq c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m &\leq c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m &\leq c_n \end{aligned}$$

where $y_1 \geq 0$, $y_2 \geq 0$,... and $y_m \geq 0$ and $y_m \geq 0$

4. Apply the **Simplex Method** to the dual maximization problem. The maximization value of w will be the minimum value of z. Moreover, the values of x_1, x_2, ... and x_n will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

We will illustrate the concept of duality by way of a minimization linear programming problem.

7.6.2 Constructing the Dual Problem

Example 7.6 :

Consider the minimization problem

$$\begin{aligned} \textit{Minimize:} \quad & C = 16x_1 + 45x_2 \\ \textit{Subject to:} \quad & 2x_1 + 5x_2 \geq 50 \\ & x_1 + 3x_2 \geq 27 \\ & x_1, x_2 \geq 0 \end{aligned} \qquad (12)$$

The following steps are involved in constructing the dual problem from a given primal problem.

1. Construct a special augmented matrix from the constraints coefficients of the primal problem without introducing slack/surplus variables and append the objective coefficients.

$$\begin{aligned} 2x_1 + 5x_2 &\geq 50 \\ x_1 + 3x_2 &\geq 27 \\ 16x_1 + 45x_2 &= C \end{aligned} \qquad A = \left[\begin{array}{cc|c} 2 & 5 & 50 \\ 1 & 3 & 27 \\ \hline 16 & 45 & 1 \end{array}\right]$$

2. Obtain the transpose of the augmented matrix

$$A^T = \left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right]$$

3. Write out the dual problem from the transpose matrix. This new problem will always be a maximization problem with $\leq$ problem constraints. To avoid confusion, we shall use different variables in this new problem:

$$\left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right] \qquad \begin{aligned} 2y_1 + y_2 &\leq 16 \\ 5y_1 + 3y_2 &\leq 45 \\ 50y_1 + 27y_2 &= P \end{aligned}$$

The dual of the minimization problem is the following maximization problem:

$$\begin{aligned} \text{Maximize} \quad & P = 50y_1 + 27y_2 \\ \text{Subject to} \quad & 2y_1 + y_2 \leq 16 \\ & 5y_1 + 3y_2 \leq 45 \\ & y_1 \geq 0, \quad y_2 \geq 0 \end{aligned}$$

4. Solve the dual problem in the usual way.

Note the following changes when constructing the dual problem, in addition to the change of notation:

1. The objective becomes the opposite of that of the primal problem.
2. $\leq$ signs become $\geq$ and vice versa.
3. There are as many decision variables in the dual problem as there are constraints in the primal problem.
4. There are as many constraints in the dual problem as there are decision variables in the primal problem.
5. The objective coefficients of primal problem become the right side (resource) values of the dual problem.
6. The right side (resource) values of the primal problem become the objective coefficients of the dual problem.

Example 7.7 :
Form the dual problem of

$$\begin{array}{ll} \text{Minimize} & C = 40x_1 + 12x_2 + 40x_3 \\ \text{Subject to} & 2x_1 + x_2 + 5x_3 \geq 20 \\ & 4x_1 + x_2 + x_3 \geq 30 \\ x_1, x_2, x_3 \geq 0 & \end{array}$$

Step 1. Form the matrix A

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 5 & 20 \\ 4 & 1 & 1 & 30 \\ \hline 40 & 12 & 40 & 1 \end{array}\right]$$

Step 2. Find the transpose of A, A^T.

$$A^T = \left[\begin{array}{cc|c} 2 & 4 & 40 \\ 1 & 1 & 12 \\ 5 & 1 & 40 \\ \hline 20 & 30 & 1 \end{array}\right]$$

Step 3. State the dual problem.

$$\begin{array}{ll} \text{Maximize} & P = 20y_1 + 30y_2 \\ \text{Subject to} & 2y_1 + 4y_2 \leq 40 \\ & y_1 + y_2 \leq 12 \\ & 5y_1 + y_2 \leq 40 \\ & y_1,\ y_2 \geq 0 \end{array}$$

In the next example we solve a minimization problem by solving its dual.

Example 7.8 :
Find the minimum value of

$$C = 3x_1 + 2x_2 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{array}{ccccc} 2x_1 & + & x_2 & \geq & 6 \\ x_1 & + & x_2 & \geq & 4 \end{array}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution:

The augmented matrix corresponding to this minimization problem is

$$\left[\begin{array}{cccc} 2 & 1 & \vdots & 6 \\ 1 & 1 & \vdots & 4 \\ \cdots & \cdots & \vdots & \cdots \\ 3 & 2 & \vdots & 1 \end{array}\right]$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\begin{bmatrix} 2 & 1 & \vdots & 3 \\ 1 & 1 & \vdots & 2 \\ \cdots & \cdots & \vdots & \cdots \\ 6 & 4 & \vdots & 1 \end{bmatrix}$$

This implies that the dual maximization problem is as follows.

Dual maximization problem: *Find the maximum value of*

$$P = 6y_1 + 4y_2$$

Subject to the constraints

$$\left.\begin{array}{rcrcr} 2y_1 & + & y_2 & \leq & 3 \\ y_1 & + & y_2 & \leq & 2 \end{array}\right\}$$

where $y_1 \geq 0$ and $y_2 \geq 0$.

After writing the dual problem in standard form we obtain the initial tableau

Tableau 1

Basic	y_1	y_2	y_3	y_4	
y_3	[2]	1	1	0	3
y_4	1	1	0	1	2
P	−6	−4	0	0	0

We see from the tableau that the pivot column is the y_1-column. The quotients are $\frac{3}{2}$ and $\frac{2}{1} = 1$. Hence the y_3-row is the pivot row. Thus y_1 is the entering variable which replaces y_3, the leaving variable. The pivot element at the intersection of the pivot row and pivot column is 2. To update the tableau we

Performing the Gauss reductions we obtain Tableau 2 given below.

Tableau 2

Basic	y_1	y_2	y_3	y_4	
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
y_4	0	[$\frac{1}{2}$]	$-\frac{1}{2}$	1	$\frac{1}{2}$
P	0	−1	3	0	9

We deduce that the current solution is not optimal. (Why?) Updating once more we obtain Tableau 3 given below

Tableau 3

Basic	y_1	y_2	y_3	y_4	
y_1	1	0	1	-1	1
y_2	0	1	-1	2	1
P	0	0	2	2	10

The current solution is optimal since all the coefficients in the last row are nonnegative.

7.6.3 Reading the Solution of the Primal Problem

We are now going to extract the solution of the primal problem from the final simplex tableau of the dual problem. The optimal objective value is

$$P = C = 10$$

Since the above final tableau is for the dual problem, we recall that in transposing the primal problem the objective coefficients of the original variables became the right-hand values of the constraints. This means that each original variable now corresponds to a slack variable. Thus we do not read the solution in the same way as for the primal simplex tableau.The optimal values of the original variables correspond to the slack variables in the final tableau of the dual problem.

For our particular example, the decision variables x_1 and x_2 of the primal problem correspond to the slack variables of the dual problem and their values are the corresponding coefficients in the last row of the final simplex tableau. Thus the solution is contained in the y_3 and y_4 columns and is

$$x_1 = 2, \qquad x_2 = 2$$

and the objective value is in the usual column i.e

$$y_1 = 1, \qquad y_2 = 1$$

Note that if we substitute the basic variables of the dual problem in the dual objective function we have

$$P = 6y_1 + 4y_2 = (6)(1) + (4)(1) = 10.$$

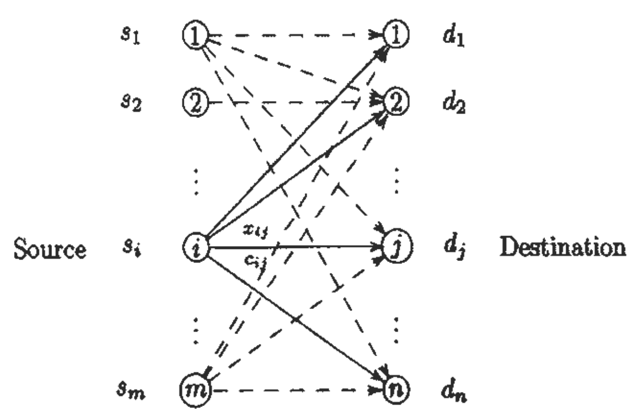
We get the same objective value if we substitute $x_1 = 2, \quad x_2 = 2$ into the primal objective function

$$C = 3x_1 + 2x_2 = (3)(2) + (2)(2) = 10$$

This verified the Von Neumann Optimality Principle.

7.7 Transportation Model

Transportation models deal with the determination of a minimum-cost plan for transporting a commodity from a number of sources to a number of destinations. To be more specific, let there be m *sources* (or *origins*) that produce the commodity and n ***destinations*** (or ***sinks***) that demand the commodity. At the i-th source, $i = 1, 2, \cdots, m$, there are s_i units of the commodity available. The demand at the j-th destination, $j = 1, 2, \cdots n$, is denoted by d_j. The cost of transporting one unit of the commodity from the i-th source to the j-th destination is c_{ij}. Let x_{ij}, $1 \le i \le m, 1 \le j \le n$, be the numbers of the commodity that are being transported from the i-th source to the j-th destination. Our problem is to determine those x_{ij} that will minimize the overall transportation cost. An optimal solution x_{ij} to the problem is called a ***transportation plan.***



We note that at the i-th source, we have the i-th source equation

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad 1 \le i \le m,$$

while at the j-th destination, we have the j-th destination equation

$$\sum_{i=1}^{m} x_{ij} = d_j, \qquad 1 \le j \le n.$$

Notice that if the total demand equals the total supply, then we have the following *balanced transportation equation:*

$$\sum_{i=1}^{m} s_i = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} d_j$$

and the model is said to be *balanced.*

In the case of an unbalanced model, i.e. the total demand is not equal to the total supply, we can always add dummy source or dummy destination to complement the difference. In the following, we only consider balanced transportation models. They can be written as the following linear programming problem:

$$\min \qquad x_0 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{n} x_{ij} = s_i & 1 \le i \le m, \\ \sum_{i=1}^{m} x_{ij} = d_j & 1 \le j \le n, \\ x_{ij} \ge 0 & 1 \le i \le m, 1 \le j \le n, \end{cases} \tag{13}$$

where $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

Notice that there are mn variables but only $m+n$ equations. To initiate the simplex method, we have to add $m+n$ more artificial variables and solving the problem by the simplex method seems to be a very tedious task even for moderate values of m and n. However, the transportation models possess some important properties that make the calculation easier to be handled.

Using the vector notations

$$\mathbf{x} = [x_{11}, x_{12}, x_{13}, \cdots, x_{1n}, x_{21}, \cdots, x_{2n}, \cdots, x_{m1}, \cdots, x_{mn}]^T,$$
$$\mathbf{c} = [c_{11}, c_{12}, c_{13}, \cdots, c_{1n}, c_{21}, \cdots, c_{2n}, \cdots, c_{m1}, \cdots, c_{mn}]^T,$$
$$\mathbf{b} = [s_1, s_2, \cdots, s_m, d_1, d_2, \cdots, d_n]^T,$$

the transportation model can be stated as the following linear programming problem:

$$\min \qquad x_0 = \mathbf{c}^T\mathbf{x}$$

$$\text{subject to} \begin{cases} A\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \ge \mathbf{0}. \end{cases} \tag{14}$$

where the technology matrix A of the model is of the form:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & & & & & & & & & & & & 0 \\ & & & & 1 & 1 & \cdots & 1 & & & & & & & & \\ & & & & & & & & \ddots & & & & & & & \\ & & & & & & & & & 1 & 1 & \cdots & 1 & & & \\ & & & & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & & & 1 & 1 & \cdots & 1 \\ 1 & & & & 1 & & & & 1 & & & & 1 & & & \\ & 1 & & & & 1 & & & & 1 & & & & 1 & & \\ & & \ddots & & & & \ddots & & \cdots & & \ddots & & \cdots & & \ddots & \\ 0 & & & 1 & & & & 1 & & & & 1 & & & & 1 \end{bmatrix} \qquad (15)$$

Hence if we denote $\mathbf{a}_{ij}$ the $[(i-1)n+j]$-th column of the matrix A, then

$$\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j} \qquad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, n. \qquad (16)$$

Here as usual, $\mathbf{e}_i$ denotes the i-th unit vector. Next we are going to prove some algebraic properties of the matrix A.

Theorem 7.1. *The rank of the matrix A is equal to $m+n-1$.*

Proof. We first claim that $\text{rank}(A) \le m+n-1$. Let $\mathbf{s}_i$ be the i-th row of A (*source rows*) and $\mathbf{d}_j$ be the $(m+j)$-th row of A (*destination rows*). Then it is clear from (6.3) that

$$\sum_{i=1}^{m} \mathbf{s}_i - \sum_{j=1}^{n} \mathbf{d}_j = \mathbf{0}.$$

Hence the rows $\mathbf{s}_i$ and $\mathbf{d}_j$ are linearly dependent. Thus $\text{rank}(A) < m+n$.

Next we prove that $\text{rank}(A) \ge m+n-1$ by constructing a nonsingular $(m+n-1)$-by-$(m+n-1)$ submatrix of A. Suppose we take the n-th, the $2n$-th, the $3n$-th, $\cdots$, the mn-th columns of A together with the 1-st, the the 2-nd, the 3-rd, $\cdots$, the $(n-1)$-th columns of A. This resulting matrix is of order $(m+n)$ by $(m+n-1)$. If we delete the last row of the matrix, then we obtain the following $(m+n-1)$-by-$(m+n-1)$ matrix:

$$D = \begin{bmatrix} 1 & 0 & & 0 & 1 & 1 & \cdots & 1 \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}.$$

Since D is a triangular matrix, $\det D = 1$. Therefore D is non-singular and $\text{rank}(A) \ge \text{rank}(D) = m+n-1$. Thus we conclude that $\text{rank}(A) = m+n-1$, which is equivalent to saying that one of the equations in (7.1) is redundant.

Thus a basic solution to (7.1) has at most $m+n-1$ nonzero entries.

Theorem 7.2. *Every minor of A can only have one of the values 1, -1 or 0. More precisely, given any A_k, a k-by-k submatrix of A, we have $\det A_k = \pm 1$ or 0.*

Proof. Notice first that every column of A has exactly two 1's, thus any column of A_k has either two 1's, only one 1 or exactly no 1. If A_k contains a column that has no 1, then clearly $\det A_k = 0$ and we are done. Thus we may assume that every column of A_k contains at least one 1. There are two cases to be considered. The first case is where every column of A_k contains two 1's. Then one of the 1's must come from the source rows and the other one must come from the destination rows. Hence subtracting the sum of all source rows from the sum of all destination rows in A_k will give us the zero vector. Thus the row vectors of A_k are linearly dependent. Hence $\det A_k = 0$. It remains to consider the case where at least one column of A_k contains exactly one 1. By expanding A_k with respect to this column, we have

$$\det A_k = \pm \det A_{k-1}$$

where the sign depends on the indices of that particular 1. Now the theorem is proved by repeating the argument to A_{k-1}.

Definition 7.2 : A matrix A is said to be *totally unimodular* if every minor of A is either 1, -1 or 0.

Thus the coefficient matrix of a transportation problem is totally unimodular.

7.8 The Simplex Method and Transportation Problems

Let us first prove that transportation models always have optimal solution. In fact, given problem (13) , if we put

$$x_{ij} = \frac{s_i d_j}{\alpha}, \quad 1 \le i \le m, 1 \le j \le n,$$

where

$$\alpha = \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then it is easy to check that it is a solution to $A\mathbf{x} = \mathbf{b}$. Hence transportation problems always have a feasible solution. Since all x_{ij} and c_{ij} are nonnegative, $x_0 \ge 0$. In particular, the objective function is bounded from below. Hence it follows that a transportation problem must have an optimal solution.

Let us see what happens if (13) is solved by simplex method. Since $\operatorname{rank} A = m + n - 1$, a basic optimal solution to (13) have only $m + n - 1$ basic variables, i.e. no more than $m + n - 1$ of the x_{ij} in the solution are different from zero. To solve (13) by simplex method, we first change it into standard form by adding $m + n$ artificial variables to (14) . Then we have

$$\begin{array}{ll} \min & x_0 = \mathbf{c}^T\mathbf{x} + M\mathbf{1}^T\mathbf{x}_a \\ \text{subject to} & \begin{cases} [\,A, I\,]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_a\end{bmatrix} = \mathbf{b}, \\ \mathbf{x}, \mathbf{x}_a \ge 0. \end{cases} \end{array} \tag{17}$$

Here $\mathbf{x}_a$ is the artificial variable. Since basic feasible solution exists, the artificial variables for the problem can always be driven to zero in phase I (or else the problem has no optimal solution, a contradiction). Since (14) and (17) have the same optimal solution, basic optimal solution to (17) can have no more than $m+n-1$ non-zero variables, i.e. a basic optimal solution to (17) must have at least one artificial variable in the basis at zero level. (Recall that artificial variables at zero level in Phase II indicate redundancy).

Suppose that we have found by some means a basic feasible solution to (17) which is also a feasible solution to (14) , (i.e. we are in phase II). Let B be the basic matrix (of order $m + n$) of $[A, I]$, then B contains $m + n - 1$ columns of A and one artificial vector $\mathbf{q}$ with the corresponding artificial variable at zero level. Therefore we may consider the $m+n-1$ linearly independent column vectors of A in B as a set of basis vectors for (14) . The collection of these $m + n - 1$ vectors will be denoted by $\mathbf{a}^B_{\alpha\beta}$ and the corresponding basic variables will be denoted by $x^B_{\alpha\beta}$. More precisely, if $B = [\mathbf{a}^B_{\alpha\beta}, \mathbf{q}]$ is basic matrix for (17) , we then define $B = [\mathbf{a}^B_{\alpha\beta}]$ as a basic matrix for (14) .

We observe that any column vector $\mathbf{a}_{ij}$ of A is just a linear combination of vectors of B, i.e.

$$\mathbf{a}_{ij} = \sum_{\alpha\beta} y_{(\alpha\beta)(ij)} \mathbf{a}^B_{\alpha\beta}$$

where $\sum_{\alpha\beta}$ means summation over all vectors in the basis. We recall that (18) is just the change of basis equation (2.24):

$$\mathbf{a}_{ij} = B\mathbf{y}_{ij}, \tag{19}$$

where B is $(m+n)$-by-$(m+n-1)$ and contains the columns $\mathbf{a}^B_{\alpha\beta}$. Thus in the language of simplex method, $y_{(\alpha\beta)(ij)}$ are just the entries in the simplex tableau at the current iteration. Now we prove the two most important properties of transportation models.

Theorem 7.3. *The coefficients $y_{(\alpha\beta)(ij)}$ can only take the values $1, -1$ or 0.*

Proof. Let R_i be the $(m+n-1)$-by-$(m+n-1)$ matrix obtained from B in (19) by deleting the ith row of B. By (19), $R_i\mathbf{y}_{ij}$ is the same as $\mathbf{a}_{ij}$ with the ith entry removed. Hence by (16), we see that

$$R_i\mathbf{y}_{ij} = \mathbf{e}_{m-1+j}.$$

Thus

$$\mathbf{y}_{ij} = R_i^{-1}\mathbf{e}_{m+1-j} = \frac{1}{\det R_i}(\operatorname{adj} R_i)\mathbf{e}_{m-1+j},$$

where $\operatorname{adj} R_i$ is the adjoint of R_i. Note that R_i is obtained by taking $(m+n-1)$ columns and $(m+n-1)$ rows of A, hence is a submatrix of A. This also follows from the fact that R_i is a submatrix of B and B is a submatrix of A. Since B is a basic matrix, R_i has full rank. Thus, by Theorem 6.2, we have $\det R_i = \pm 1$. Since the entries of $\operatorname{adj} R_i$ are just minors of R_i and hence of A, their values can only be ± 1 or 0. Thus we see that $\mathbf{y}_{ij} = \pm 1$ or 0.

Thus (18) becomes

$$\mathbf{a}_{ij} = \sum_{\alpha\beta} (\pm 1)\mathbf{a}^B_{\alpha\beta},$$

where we have omitted those $\mathbf{a}^B_{\alpha\beta}$ with $y_{(\alpha\beta)(ij)} = 0$ in the summation. We note that the conclusion of Theorem 7.3. holds for any linear programming problem where its coefficient matrix is totally unimodular.

Theorem 7.4. (The Stepping Stones Theorem). *Let $B = \{\mathbf{a}_{\alpha\beta}\}$ be a set of $(m+n-1)$ linearly independent columns of A. Then for all column vector $\mathbf{a}_{ij}$ of A, $1 \le i \le m$, $1 \le j \le n$, we have*

$$\mathbf{a}_{ij} = \mathbf{a}_{ii_1} - \mathbf{a}_{i_2i_1} + \mathbf{a}_{i_2i_3} - \mathbf{a}_{i_4i_3} + \cdots + (-1)^k \mathbf{a}_{i_kj}, \tag{20}$$

where $\mathbf{a}_{ii_1}, \mathbf{a}_{i_li_{l+1}}, \mathbf{a}_{i_kj}$ are in B for $l = 1, \cdots, k-1$. Moreover, the expression (20) is unique.

Proof. Since rank $A = m+n-1$, all column vectors of A can be written as a linearly combinations of vectors in B. Moreover by Theorem 6.3, we have

$$\mathbf{a}_{ij} = \sum_{\alpha\beta}(\pm 1)\mathbf{a}_{\alpha\beta} = \sum_{\alpha\beta\in I^+} \mathbf{a}_{\alpha\beta} - \sum_{\gamma\delta\in I^-} \mathbf{a}_{\gamma\delta}$$

where $I^{\pm}$ are index sets depending on the $\mathbf{a}_{ij}$. By (16), this becomes

$$\mathbf{e}_i + \mathbf{e}_{m+j} = \sum_{\alpha\beta\in I^+} \mathbf{e}_\alpha + \sum_{\alpha\beta\in I^+} \mathbf{e}_{m+\beta} - \sum_{\gamma\delta\in I^-} \mathbf{e}_\gamma - \sum_{\gamma\delta\in I^-} \mathbf{e}_{m+\delta}.$$

From this expression, it is clear that there exists an $1 \le i_1 \le n$ such that $(ii_1) \in I^+$. Subtracting $\mathbf{a}_{ii_1}$ from both sides, we get

$$\mathbf{e}_{m+j} - \mathbf{e}_{m+i_1} = \sum_{\alpha\beta \in I_1^+} \mathbf{e}_\alpha + \sum_{\alpha\beta \in I_1^+} \mathbf{e}_{m+\beta} - \sum_{\gamma\delta \in I^-} \mathbf{e}_\gamma - \sum_{\gamma\delta \in I^-} \mathbf{e}_{m+\delta},$$

where $I_1^+ = I^+ \setminus \{(ii_1)\}$. Now if $i_1 = j$, we are done. If not, then from the expression, it is clear that there exists an $1 \le i_2 \le m$ such that $(i_2 i_1) \in I^-$. Subtracting $\mathbf{a}_{i_2 i_1} = \mathbf{e}_i - \mathbf{e}_{m+i_1}$ from both sides, we get

$$\mathbf{e}_{m+j} + \mathbf{e}_{i_2} = \sum_{\alpha\beta \in I_1^+} \mathbf{e}_\alpha + \sum_{\alpha\beta \in I_1^+} \mathbf{e}_{m+\beta} - \sum_{\gamma\delta \in I_2^-} \mathbf{e}_\gamma - \sum_{\gamma\delta \in I_2^-} \mathbf{e}_{m+\delta},$$

where $I_2^- = I^- \setminus \{(i_2 i_1)\}$. Equation (20) now follows by repeating the arguments again until I_l^- is empty. Since B is a basis, it is clear that (20) is unique.

In the following, we consider how to iterate from one simplex tableau to the next.

Update of the solution $\mathbf{x}$.
Let $\mathbf{a}_{st}$ be the entering vector and $\mathbf{a}_{uv}^B$ be the leaving vector. Then the solution x_{ij} are updated according to

$$\begin{cases} \hat{x}_{\alpha\beta}^B = x_{\alpha\beta}^B - x_{uv}^B \dfrac{y_{(\alpha\beta)(st)}}{y_{(uv)(st)}} & \text{if } (\alpha\beta) \neq (uv) \\ \hat{x}_{uv} = \dfrac{x_{uv}^B}{y_{(uv)(st)}} \end{cases} \tag{21}$$

This equation is to be compared with the updating rule in simplex method:

$$\begin{cases} \hat{x}_{B_i} = x_{B_i} - x_{B_r} \dfrac{y_{ij}}{y_{rj}} & \text{if } i \neq r \\ \hat{x}_{B_r} = \dfrac{x_{B_r}}{y_{rj}} \end{cases}$$

But by Theorem 7.3, the pivot element $y_{(uv)(st)}$ will always be equal to 1 and that the other $y_{(\alpha\beta)(st)} = \pm 1$ or 0. Therefore, we see that (6.9) can be rewritten as

$$\begin{cases} \hat{x}_{\alpha\beta}^B = x_{\alpha\beta}^B \quad \text{or} \quad \hat{x}_{\alpha\beta}^B = x_{\alpha\beta}^B \pm x_{uv}^B \\ \hat{x}_{uv}^B = x_{uv}^B \end{cases} \tag{22}$$

The property (6.10) is usually referred to as *integer property*. It shows that if the starting basic feasible solution $\mathbf{x}$ is an integral vector (i.e. all entries are integer), then at each subsequent iteration, the solution $\mathbf{x}$ is also an integral vector. In particular, the optimal solution $\mathbf{x}^*$ is also an integral vector.

We remark that the integer property of transportation problems is derived from the fact that all entries of $\mathbf{y}_{ij}$ can either be 1, -1 or 0. Thus by recalling Theorem 6.3, we see that if the coefficient matrix of a linear programming problem is totally unimodular, then the problem will have the integer property.

Update of tableau entries $\mathbf{y}_{ij}$.
For usual simplex method, the tableau entries $\mathbf{y}_j$ are updated by the elementary row operations:

$$\begin{cases} \hat{y}_{B_i} = y_{B_i} - \dfrac{y_{ij}}{y_{rj}} & \text{if } i \neq r \\ \hat{y}_{B_r} = \dfrac{y_{B_r}}{y_{rj}} \end{cases}$$

In our notations, we then have

$$\begin{cases} \hat{y}_{(\alpha\beta)(ij)} = y_{(\alpha\beta)(ij)} - \dfrac{y_{(uv)(ij)}}{y_{(uv)(st)}} & \text{if } (\alpha\beta) \neq (uv) \\ \hat{y}_{(uv)(ij)} = \dfrac{y_{(uv)(ij)}}{y_{(uv)(st)}} \end{cases}$$

Since the pivot element $y_{(uv)(st)}$ is always equal to 1, we have

$$\begin{cases} y_{(\alpha\beta)(ij)} = y_{(\alpha\beta)(ij)} - y_{(uv)(ij)} & \text{if } (\alpha\beta) \neq (uv) \\ y_{(uv)(ij)} = y_{(\alpha\beta)(ij)} \end{cases}$$

Computation of $z_{ij} - c_{ij}$.

Recall that in the simplex method,

$$z_j - c_j = \mathbf{c}_{B_i}^T \mathbf{y}_j - c_j = \sum_{x_i \in B} c_{B_i} y_{ij} - c_j.$$

Hence in our notations, we have

$$z_{ij} - c_{ij} = \sum_{x_{(\alpha\beta)} \in B} y_{(\alpha\beta)(ij)} c_{\alpha\beta}^B - c_{ij}. \tag{23}$$

Because of the simple algebraic structure of the transportation models, it is not necessary to use the simplex tableau, which is of size $(m+n+1)$ by $(mn+n+n+1)$, to hold all necessary information. In the following, we will construct a different tableau, called the ***transportation tableau***, that can hold the same pieces of information and yet is easy to be handled. For the transportation model in (13) , its transportation tableau consists of m by n boxes and is of the form

						Supply
	c_{11} x_{11}	c_{12} x_{12}	$\cdots$ $\cdots$	$\cdots$ $\cdots$	c_{1n} x_{1n}	s_1
	c_{21} x_{21}	c_{22} x_{22}	$\cdots$ $\cdots$	$\cdots$ $\cdots$	c_{2n} x_{2n}	$\vdots$
	$\vdots$ $\vdots$	$\vdots$ $\vdots$	c_{ij} x_{ij}	$\vdots$ $\vdots$	$\vdots$ $\vdots$	s_i
	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$
	c_{m1} x_{m1}	c_{m2} x_{m2}	$\cdots$ $\cdots$	$\cdots$ $\cdots$	c_{mn} x_{mn}	s_m
Demand	d_1	$\cdots$	d_j	$\cdots$	d_n	

In the transportation tableau, nonbasic variables (i.e. those $\mathbf{a}_{ij}$ not in the basis) are not written out explicitly.

Recall that simplex tableau contains the following information:

(i) The current solution in the $\mathbf{b}$ column.

(ii) The x_0 row contains the reduced cost coefficients $z_j - c_j$.

(iii) The transformed columns of A, denoted as usual by $\mathbf{y}_j$. They are related to the columns $\mathbf{a}_j$ of A by (2.20): $\mathbf{y}_j = B^{-1}\mathbf{a}_j$.

(iv) The current basic variables.

We will see that the transportation tableau can be manipulated easily to give us these necessary pieces of information. For one thing, according to our convention on the transportation tableau, those variables that are not listed in the tableau are nonbasic. Those variables that are listed are basic and their values are the values of the current basic feasible solution. Next we show by an example how to compute the current coefficient matrix $y_{(\alpha\beta)(ij)}$ and the corresponding reduced cost coefficient $z_{ij} - c_{ij}$.

Example. Let us consider a problem with eight variables x_{ij}, $1 \le i \le 2$, $1 \le j \le 4$. We then have the following 2-by-4 transportation tableau.

x_{11}	x_{12}		
	x_{22}	x_{23}	x_{24}

Since rank $A = 2 + 4 - 1 = 5$, there will be five basic variables in any basic feasible solutions of the problem. According to our convention, $x_{11}, x_{12}, x_{22}, x_{23}$ and x_{24} are the current basic variable. Thus $\mathbf{a}_{11}, \mathbf{a}_{12}, \mathbf{a}_{22}, \mathbf{a}_{23}$ and $\mathbf{a}_{24}$ are the basis vectors. The other three vectors are just linearly combinations of these five vectors. For example,

$$\mathbf{a}_{21} = \mathbf{a}_{22} - \mathbf{a}_{12} + \mathbf{a}_{11}.$$

Thus $y_{(22)(21)} = y_{(11)(21)} = 1$ and $y_{(12)(21)} = -1$. Therefore, according to (23),

$$z_{21} - c_{21} = c_{22} - c_{12} + c_{11} - c_{21}.$$

Similarly, we have

$$\mathbf{a}_{13} = \mathbf{a}_{12} - \mathbf{a}_{22} + \mathbf{a}_{23},$$

i.e. $y_{(12)(13)} = y_{(23)(13)} = 1$ and $y_{(22)(13)} = -1$. Therefore,

$$z_{13} - c_{13} = c_{12} - c_{22} + c_{23} - c_{13}.$$

Finally,

$$\mathbf{a}_{14} = \mathbf{a}_{12} - \mathbf{a}_{22} + \mathbf{a}_{24}$$

and hence $y_{(12)(14)} = y_{(24)(14)} = 1$ and $y_{(22)(14)} = -1$. Thus we have

$$z_{14} - c_{14} = c_{12} - c_{22} + c_{24} - c_{14}.$$

We remark that a loop is formed each time. For example, for x_{14}, we have the following loop.

x_{11}	x_{12}		
	x_{22}	x_{23}	x_{24}

The loop starts with a nonbasic variable, passes through a sequence of basic variables (stepping stones) and finally returns to the starting nonbasic variable. This is a consequence of Theorem 7.4. We note moreover that for each of these nonbasic variable, there exists one and only one such loop. In fact, by Theorem 7.4, if there are two such loops, then the nonbasic vector can be expressed in terms of two different linear combinations of vectors in the basis. This will be a contradiction to the linear independence of the basic vectors.

EXERCISE :

Solve the following LP problems using the Simplex Method

1.
$$\begin{aligned} \text{maximize } P = 70x_1 &+ 50x_2 \\ \text{subject to } 4x_1 + 3x_2 &\leq 240 \\ 2x_1 + x_2 &\leq 100 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$P = 4100, x_1 = 30, x_2 = 40$

2.
$$\begin{aligned} \text{maximize } P = 10x_1 &+ 5x_2 \\ \text{subject to } 4x_1 + x_2 &\leq 28 \\ 2x_1 + 3x_2 &\leq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$P = 80, x_1 = 6, x_2 = 4$

3.
$$\begin{aligned} \text{maximize } P = 70x_1 &+ 50x_2 + 35x_3 \\ \text{subject to } 4x_1 + 3x_2 + x_3 &\leq 240 \\ 2x_1 + x_2 + x_3 &\leq 100 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$P = 4550, x_1 = 0, x_2 = 70, x_3 = 30$

4.
$$\begin{aligned} \text{maximize } P = 2x_1 &+ x_2 \\ \text{subject to } 5x_1 + x_2 &\leq 9 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$P = 6, x_1 = 1, x_2 = 4$

5.
$$\begin{aligned} \text{maximize } P = 30x_1 &+ 40x_2 \\ \text{subject to } 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 7 \\ x_1 + 2x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$P = 260, x_1 = 2, x_2 = 5$

8 Compound Interest and Annuities

Chapter Includes:

1. Certain Types of Interest Rates
2. Basic Concept in finance
3. Time Value of Money
4. Future Value Vs. Present Value
5. Computing Present Value
6. Computing Future Value
7. Value with and without Compounding
8. Future Value with and without Compounding
9. Compound Value
10. Effective Interest Rate
11. Continuous Compounding
12. Annuity
13. Regular Annuity Vs. Annuity Due
14. Present Value of a growing annuity
15. Present Value of Perpetuity
16. Future value of a growing annuity
17. Debentures

8.1 Certain Types of Interest Rates

A. Credit Market Instruments

A good first step is to carefully define what we are going to measure. Interest rates apply to four types of credit market instruments:

Simple loan

- provides the borrower with an amount of funds (the principal).
- borrower then pays back the principal amount and interest in one lump sum at maturity.

Fixed-payment loan

- provides the borrower with an amount of principal
- the principal and interest are repaid with equal monthly payments for a certain period
- each monthly payment is a combination of principal and interest

Coupon bond

- purchased at some price
- entitles the owner to fixed interest payments annually (coupon payments) until maturity and a face value payment (or par value) at maturity
- characterized by the issuer, the maturity, and the coupon rate, which is multiplied by the face value to determine the coupon payment
- Note: your textbook focuses on annual payments, but in fact, almost all coupon bonds issued in the United States have semi-annual payments.

Discount bond (also known as a zero coupon bond)

- purchased at some price below its face value (or at a discount)
- entitles the owner to a face value payment at the maturity date.
- There are no interest payments, hence the name "zero coupon bond."

The simple loan and the discount bond both consist of only one cash flow while fixed-payment and coupon bonds have multiple cash flows life of the instrument. Both the amount and timing of cash flows are important when comparing financial instruments. To make an accurate comparison among instruments with differences in the amount and timing of cash flows we need to understand and calculate *present value* and *yield to maturity*.

B. Present and Future Value

I realize many of you are already familiar with present value from other accounting/finance courses, but let's review..

Present value is based on the fundamental reality that you are not indifferent between getting \$100 today versus waiting one year to receive \$100. Why? Well in financial markets, you could receive interest on that \$100 over the course of one year, and end up with more than \$100 at the end of the year. The cost of waiting is the simple interest rate; i.e. the interest rate on a simple loan. You lend me \$100 at an interest rate of 5% per year, then at the end of one year you will receive \$100 + (.05 x \$100) = \$100 x (1 + .05) = \$105. At the end of 2 years you will receive

$$\$105 \times (1+.05) = \$100 \times (1+.05) \times (1+.05) = \$100 \times (1+.05)^2 = \$110.25$$

In general, if the simple interest rate is i and the loans are made for n years you will receive:

$$\$100 \times (1+i)^n$$

The amount above is known as the **future value** of \$100 in n years.

So, working backwards, for any amount received in the future we need to *discount it to the present*. In other words, if you are getting \$100 in one year, how much less would you accept in order to get it today? The answer is the **present value** and will depend on the interest rate.

Suppose again the interest rate is 5%. If you will receive \$100 in one year, what is the present value? We want to solve the equation

PV x (1 + .05) = \$100 or

$$PV = \frac{\$100}{(1+.05)} = \$95.24$$

If you will receive \$100 in 3 years, what is the present value?

$$PV = \frac{\$100}{(1+.05)^3} = \$86.36$$

In general, for the PV of \$100, n years from now, with a simple interest rate of i, we use the formula

$$PV = \frac{\$100}{(1+i)^n}$$

Note that larger values for n and i imply smaller PV.

C. Yield to Maturity

Now that we understand present value, we have the tools to calculate the most important measure of interest rates, the yield to maturity. The **yield to maturity** is the interest rate that makes the discounted value of the future payments from a debt instrument equal to its current value (market price) today. Let's look at the yield to maturity for the 4 credit market instruments discussed above.

Simple Loan

This is the easiest case, because there is only one cash flow at the end of the loan to discount.

Example 1: Suppose the loan is for \$1500, for 1 year, with a simple interest rate of 6%.

The value of the loan today is \$1500. The future payments on the loan are \$1500(1+.06) = \$1590.

So the yield to maturity is the i that solves the equation

$$\$1500 = \frac{\$1590}{(1+i)}$$

Solving for i, (1+ i) = 1590/1500

i = .06 6%

For a simple loan, the yield to maturity is the same as the simple interest rate. Why? Because there is only one cash flow.

Fixed-Payment Loan

This case is more complicated due to the multiple payments through the life of the loan. Your textbook example uses a loan with annual payments on page 71. However, the most common forms of this type of loan are for monthly payments, like a mortgage, student loans or an auto loan. Loans with multiple payments during the year are a bit more complicated, as shown in the example below:

Example 2: Suppose you take out a \$15,000 car loan for 5 years, with monthly payments of \$300.

The value of the loan today is \$15,000. The future payments are \$300 payments over the next 60 months.

The yield to maturity is the i that solves the following equation:

$$\$15{,}000 = \frac{\$300}{\left(1+\frac{i}{12}\right)} + \frac{\$300}{\left(1+\frac{i}{12}\right)^2} + \frac{\$300}{\left(1+\frac{i}{12}\right)^3} + \ldots + \frac{\$300}{\left(1+\frac{i}{12}\right)^{60}}$$

Note that since payments are monthly for 5 years, there are a total of 5 x 12 = 60 payment periods. Also, the yield to maturity, i, is expressed on an annual basis, so i/12 represents the monthly discount rate. This assumes that interest charges compound annually instead of monthly. If interest charges compounded monthly, then the appropriate monthly discount rate is where

$$i = (1+i_m)^{12} - 1$$

or

$$i_m = (1+i)^{1/12} - 1$$

Your textbook is cavalier with this point, but the distinction is important. In this application, though, it makes very little numerical difference in the answer.

So how do we solve this for i? Well it is not easy, since there is no way to isolate i in this equation. It could be done by trial and error (trying values of i until the right-hand side of the equation is $15,000), but that is too time consuming. This problem is solved with the aid of a table, financial calculator, or spreadsheet programs that do this automatically. A financial calculator is not required for this course, so I provide loan or bond table when needed.

Consider the following loan table:

Monthly Payments on $15,000 Loan			
yield to maturity	maturity (years)		
	5	10	15
6.00%	$289.99	$166.53	$126.58
6.50%	$293.49	$170.32	$130.67
7.00%	**$297.02**	$174.16	$134.82
7.50%	**$300.57**	$178.05	$139.05
8.00%	$304.15	$181.99	$143.35

We are looking for a 5 year loan (shaded yellow), and a monthly payment of $300. Looking at the table above we see that at 7.5% yield to maturity, the payment is $300.57. So the yield to maturity is slightly under 7.5% (7.42% to be more precise).

Click below for the "high tech" ways to solve this example:

Financial Calculator: TI BA II+	Excel Spreadsheet

Coupon Bond

With the multiple interest payments involved, this case is similar to the fixed payment loan in its complexity. Again, your textbook example uses a coupon bond with annual coupon payments on page 72. However, all bonds issued in the United States have coupon payments semi-annually, or every 6 months, including Treasury notes, Treasury bonds, and corporate bonds. So the example below also uses 6-month payments.

***Example** 3*: Consider a 2-year Treasury note with a face value of $10,000, a coupon rate of 6%, and a price of $9750.

So the yield to maturity will solve the equation:
bond price = PV(future bond payments)

What are the future payments?
There are coupon payments every 6 months, and a face value payment at maturity.

What are the coupon payments?
The coupon payments are [face value x coupon rate]/2 = $10,000 x .06 x .5 = $300. Note that we divide by 2 because there are 2 coupon payments in a year.

So the payment schedule is

6 months	$300
1 year	$300
18 months	$300
2 years	$10,300

So the yield to maturity will solve the following equation:

$$\$9750 = \frac{\$300}{\left(1+\frac{i}{2}\right)} + \frac{\$300}{\left(1+\frac{i}{2}\right)^2} + \frac{\$300}{\left(1+\frac{i}{2}\right)^3} + \frac{\$10{,}300}{\left(1+\frac{i}{2}\right)^4}$$

Note that since payments are every 6 months for 2 years, there are a total of 2 x 2 = 4 payment periods. Also, the yield to maturity, i, is expressed on an annual basis, so *i/2* represents the 6 month discount rate. This assumes that interest charges compound annually instead of semiannually. If interest charges compounded semi-annually, then the appropriate discount rate is where

$$i = (1+i_2)^2 - 1$$

or

$$i_2 = (1+i)^{1/2} - 1$$

Your textbook is cavalier with this point, but the distinction is important. In this application, though, it makes very little numerical difference in the answer.

Like the fixed payment loan, this problem is solved with the aid of a table, financial calculator, or spreadsheet programs that do the trial-and-error calculations automatically. A financial calculator is not required for this course, so I provide loan or bond table when needed.

Consider the following bond table:

Bond Prices, 6% coupon rate, semi-annual payments			
yield to maturity	maturity (years)		
	2	5	10
5.50%	$10,093.49	$10,216.00	$10,380.68
6.00%	$10,000.00	$10,000.00	$10,000.00
6.50%	$9,907.63	$9,789.44	$9,636.52
7.00%	**$9,816.35**	$9,584.17	$9,289.38
7.50%	**$9,726.15**	$9,384.04	$8,957.78
8.00%	$9,637.01	$9,188.91	$8,640.97

We are looking for a 2 year bond (shaded yellow), and a price of $9750. Looking at the table above we see that at 7.5% yield to maturity, the price is $9726.15. So the yield to maturity is slightly under 7.5% (7.37% to be more precise).

Spreadsheets also have financial functions built in. Here is how to compute the answer to example 3 using Excel:

(1) Click on the cell where you want the answer displayed.

(2) In the "Insert" menu, choose "Function." (or click the function icon in your toolbar, [*fx*])

(3) Choose the "financial" function category, and choose "RATE" and a box pops up. (4) Now fill in the spaces in the box: Nper = 4, Pmt = 300, Pv = -9750, Fv = 10000, and ignore type

(5) The formula result displays at the bottom of the box, "Formula result = 3.683596826%." This is i/2, so multiply by 2 to get the annual yield to maturity of 7.37%

Looking at the bond table above, there are 3 important points to be made about the relationship between bond prices, maturity, and the yield to maturity:

- The yield to maturity equals the coupon rate ONLY when the bond price equals the face value of the bond.
- When the bond price is less than the face value (the bond sells at a discount), the yield to maturity is greater than the coupon rate. When the bond price is greater than the face value (the bond sells at a premium), the yield to maturity is less than the coupon rate.
- The yield to maturity is inversely related to the bond price. ***Bond prices and market interest rates move in opposite directions***. Why? As interest rates rise, new bonds will pay higher coupon rates than existing bonds. The prices of existing bonds fall in the secondary market, so the yield to maturity rises. ***This negative relationship between interest rate and value is true for all debt securities, not just coupon bonds.***

Discount (Zero coupon) Bond

Because discount bonds have only one payment at maturity, it yield to maturity is easy to calculate and is similar to that of a simple loan. Most discount bonds have a maturity of LESS than one year, so the example below looks at such a case:

Example 4: Consider a Treasury bill with 90 days to maturity, a price of $9875, and a face value of $10,000.

The current value is $9850, and the only future payment is $10,000 at maturity. However, we do not wait a year for this payment but only 90 days so we need to adjust the discounting for this.

The yield to maturity solves the following equation:

Solving for i,

$$\left(1+i\left(\frac{90}{365}\right)\right)=\frac{\$10{,}000}{\$9850}$$

$$i\left(\frac{90}{365}\right)=\frac{\$10{,}000}{\$9850}-1$$

$$i\left(\frac{90}{365}\right)=\frac{\$10{,}000-9850}{\$9850}$$

$$i=\frac{\$10{,}000-9850}{\$9850}\times\frac{365}{90}=6.18\%$$

This method is the convention in financial markets, known as the *bond equivalent basis*.

If you use a financial calculator you may come up with a different answer: If you do the following on your financial calculator,

10000 [FV]
9850 [+/-] [PV]
0 [PMT]
91/365 [N]
[CPT] [I/Y]

you come up with i = 6.32%. Which is greater than our lecture notes calculation of 6.18%. Why? Because you instructed you calculator to annualize i by compounding every 91 days. The calculator solved the equation:

$$\$9850=\frac{\$10{,}000}{(1+i)^{(90/365)}}$$

While the method above makes sense and is a legitimate measure of an interest rate, the method in the lecture notes, known as a *bond equivalent basis*, is what we use by tradition in financial markets.

In general, the yield to maturity is found by the formula

$$i=\frac{F-P}{P}\times\frac{365}{d}$$

where F is the face value, P is the bond price, and d is the days to maturity

D. Current Yield

The yield to maturity is the truest measure of the interest rate. However there are other measure out there developed for their computational convenience. It this day of cheap computing, it is easy to forget that calculators were not available until 1975 (and then cost

$200 for one that could just do arithmetic!). Bonds traded long before that, so traders used yield measures that approximated the yield to maturity but were easier to calculate.

The **current yield** is an approximation used for coupon bonds. It is simply the annual coupon payment divided by the price of the bond:

$$i_c = \frac{C}{P}$$

where C is the coupon payment and P is the bond price. This is obviously a lot simpler that the yield to maturity

The current yield is a better approximation

- for longer maturity bonds and
- when the price of the bond is close to its face value.

***Example** 5*: Consider a 2-year Treasury note with a face value of $10,000, a coupon rate of 6%, and a price of $9750.

the current yield is

$$i_c = \frac{600}{9750} = 6.15\%$$

Recall that the true yield to maturity, from example 3, is 7.37%. So in this example, the approximation is lousy because it is only a 2-year bond and it is selling at 25% below its face value.

E. Discount Yield

Also known as the **yield on a discount basis**, the **discount yield** is used by dealers to quote the interest rates on U.S. Treasury bills. Again, this a computationally convenient approximation of the yield to maturity.

$$i_{db} = \frac{F-P}{F} \times \frac{360}{d}$$

Compare this to the formula for the yield to maturity:

$$i = \frac{F-P}{P} \times \frac{365}{d}$$

Note there are two major differences:

(1) The yield to maturity takes the discount (F-P) as a proportion of the bond price, while the discount yield takes the discount as a proportion of the face value.

(2) The yield to maturity uses a 365-day year while the discount yield uses a 360-day year.

Both of these differences make the arithmetic easier in the case of the discount yield, but the also cause the discount yield to understate the true yield to maturity (F is always greater than P and 365 is always greater than 360). ***The discount yield will always be less than the yield to maturity for any zero coupon bond.***

Example 6: Consider a Treasury bill with 90 days to maturity, a price of \$9850, and a face value of \$10,000.

$$i_{db} = \frac{10{,}000 - 9850}{10{,}000} \times \frac{360}{90} = .015 \times 4 = 6\%$$

This is slightly less than the yield to maturity which is 6.18% (example 4).

Wow, this is a lot of stuff to think about. What next? I suggest if you want some more practice with calculating various interest rates, try the problems at the end of chapter 4, page 90. There are solutions in the back of the book for the odd numbered problems.

II. Other Measurement Issues

Understanding what the interest rate does and does tell you is as important as measuring the interest rate in the first place. Here are a couple of issues in interest rate measurement.

A. Interest Rates vs. Returns

The yield to maturity assumes that the bond is held until maturity. If that is not true, then fluctuations in the bond price (which occur with interest rate fluctuations) will affect the **return,** or the gain to the investor from holding this security. The return for holding a bond between periods t and t+1 is

$$RET = \frac{C + P_{t+1} - P_t}{P_t}$$

where Pt is the initial price and Pt+1 is the price at the end of the holding period.

We can rewrite this formula as

$$RET = \frac{C}{P_t} + \frac{P_{t+1} - P_t}{P_t}$$

The last term is **the rate of capital gain,** g, or the change in the bond price relative to the initial bond price. So a bond's return can be rewritten as

$$RET = i_c + g$$

A bond's return is identical to the yield to maturity if the holding period is identical to the time left to maturity.

B. Maturity and Bond Price Volatility

Any bond price moves in the opposite direction of interest rates, but what determines how much a bond's price fluctuates, or in other words, it volatility? Let's reconsider the bond table from part I, example 3:

Bond Prices, 6% coupon rate, semi-annual payments			
yield to maturity	maturity (years)		
	2	5	10
5.50%	$10,093.49	$10,216.00	$10,380.68
6.00%	**$10,000.00**	**$10,000.00**	**$10,000.00**
6.50%	$9,907.63	$9,789.44	$9,636.52
7.00%	$9,816.35	$9,584.17	$9,289.38
7.50%	$9,726.15	$9,384.04	$8,957.78
8.00%	**$9,637.01**	**$9,188.91**	**$8,640.97**

Look at each bond's price (the 2-year, 5-year, and 10-year bonds) as the yield to maturity rises from 6% to 8%. The prices fall for all of the bonds, but by different amounts. The price on the 2-year bond falls less than $400 or less than 4%. The price on the 10-year bond falls by more than $1300 or more than 13%. This brings to the principle bond characteristic that affects price volatility: ***Prices (and thus returns) are more volatile for long-term bonds than short-term bonds. In other words, long-term bonds have greater*** **interest-rate risk.**

Why is this the case? Intuitively, with a long-term bond, you are "locked in" to a coupon rate for a longer period of time. So if newer bonds are issued with lower coupon rates, your long-term bond becomes much more valuable. If new bonds have higher coupon rates, your long-term bond becomes much less valuable. For a bond with less than 1 year left until maturity, the change in interest rates will not matter that much. The consequences of changing interest rates are much more serious for bonds with longer times left until maturity.

C. Real vs. Nominal Interest Rates

Up until now, we have not accounted for the effects of inflation on the return or interest rate on a bond. While the owner of a bond is entitled to future payments, in an economy with inflation, the purchasing power of those payments is declining over time. It is pretty much a given that $10,000 in 2011 will buy less than $10,000 today.

The interest rate (yield to maturity) we calculate in Part I is specifically the **nominal interest rate**, which does not consider the impact of inflation. Instead, expected price changes are reflected in the **real interest rate**. The relationship between the real and nominal interest rate, known as the Fisher equation, is given by:

$$i = i_r + \pi^e$$

or

$$i_r = i - \pi^e$$

Where π^e is the expected inflation rate.

So the nominal interest rate is the sum of the real interest and the expected inflation rate. The real interest is a truer measure of the cost of borrowing. Lower real interest rates increase the incentive to borrow (while reducing the incentive to lend). Higher real interest rates decrease the incentive to borrow (while increasing the incentive to lend).

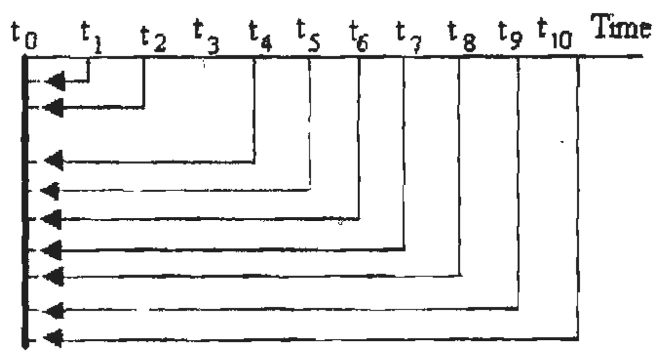
8.2 Basic Concept in finance

The functions of finance in an organization is interlinked with other managerial responsibilities and in many instances, the finance manager could also done the role of a managing director. For the smooth functioning as well as to achieve excellence, organizations have to concentrate on the financial impact of a decision and its consequences. This also helps the organization to aim at a desired competency level against its competitors.

- *In organizations, flow of money occurs at various points of time. In order to evaluate the worth of money, the financial managers need to look at it from a common platform, namely one time duration. This common platform enables a meaningful comparison of money over different time periods.*
- *An important principle in financial management is that the value of money depends on when the cash flow occurs – which implies Rs.100 now is worth more than Rs.100 at some future time.*

8.3 Time Value of Money

Common Platform for Equating Monetary Flows



8.3.1 The Time-Value of Money

Money like any other desirable commodity has a price. If you own money, you can, 'rent' it to someone else, say a banker, who can use it to earn income. This 'rent' is usually in the form of interest. The investor's return, which reflects the time-value of money, therefore indicates that there are investment opportunities available in the market. The return indicates that there is a

- risk-free rate of return rewarding investors for forgoing immediate consumption
- compensation for risk and loss of purchasing power.

- ***Risk:*** *An amount of Rs.100 now is certain, whereas Rs.100 receivable next year is less certain. This 'uncertainty' principle affects many aspects of financial management and is termed as risk value of money.*

- ***Inflation:*** *Under inflationary conditions, the value of money, expressed in terms of its purchasing power over goods and services, declines. Hence Rs.100 possessed now is not equivalent to Rs.100 to be received in the future.*

- ***Personal consumption preference:*** *Most of us have a strong preference for immediate rather than delayed consumption. As a result we tend to value the Rs.100 to be received now more than Rs.100 to be received latter.*

8.4 Future Value Vs. Present Value

Future value (FV) and present value (PV) adjust all cash flows to a common time. This is relevant when we want to compare the cash flows occurring at different periods of time. Either in terms of projects, performance or turnover, the cash flows accrue to the company at different stages. The evaluation of all these cash flows are true when they are all brought to the same base period.

8.5 Computing Present Value

In financial parlance, a value of currency is not kept idle. The amount, if invested would certainly bring additional returns in the future. This future expectation from the present investment is termed as the future value.

Let us assume x amount is invested now and the investor expects r% to accrue on the investment one year ahead. This is translated into present and future values as follows:

PV = Rs. x

FV = Rs. x + (r * x)

Computing Future Value – Example

Let us assume Rs.1,000 is invested now and the investor expects 5% to accrue on this investment one year ahead. This is translated into present and future values as follows:

PV = Rs.1,000

FV = Rs.1,000 + (.05 * 1,000) = Rs.1,050.

8.6 Computing Future Value

This can be restated as FV = PV * (1+r)

This relationship leads to the following concept of discounting the future value to arrive at the present value i.e.,

PV = FV / (1 + r)

This is the formula for equating the future value that is associated at the end of 1st year. Now the concept of time over a longer duration can be easily brought into the above equation, where 'n' defines the time duration after which the cash flows are expected.

Computing Present Value – Example

Let us assume that Rs.1,000 is to be received at the end of 1 year from now and the investor expects 5% rate of return on this investment.

Here FV = Rs.1,000

Hence the present value is computed as:

PV = FV / (1 + r)

= Rs.1000 / (1.05) = Rs.952.

8.7 Value with and without Compounding

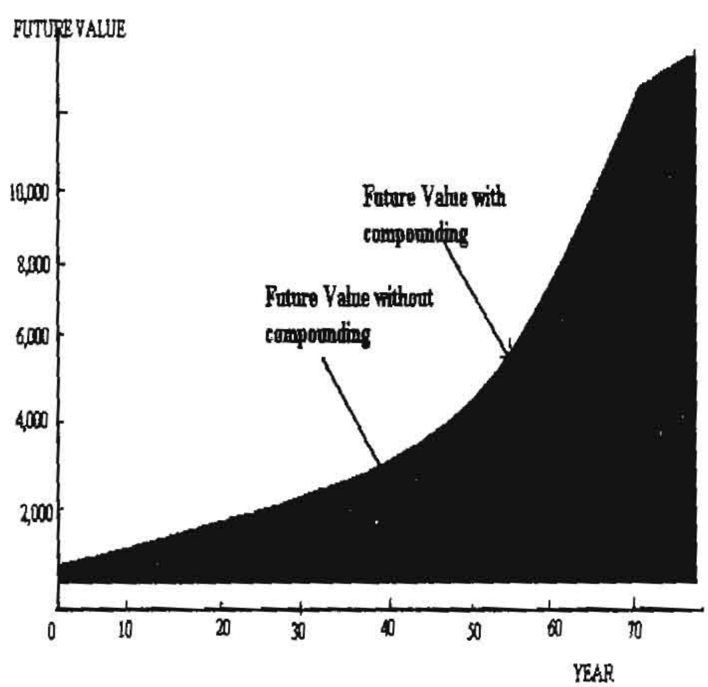
- *Interest without compounding is a simple interest formula i.e., Pnr/100 Where: P is the principle, n is the number of years and r is the interest rate.*
- *Interest with annual compounding adds the interest received earlier to the principle amount and increases the final amount that is received from the investment. Hence, the FV of an investment for a two year duration with annual compounding would be:*

  *FV = PV * (1+r)* (1+r) = PV * (1+r)^2.*
- *Hence Present Value is:*

  PV = FV / (1+r)^2.
- *This equation can be generalized for 'n' years as:*

  PV = FV / (1 + r)^n

8.8 Future Value with and without Compounding



8.9 Compound Value

In compounding, it is assumed that a certain sum accrues at the end of a time duration, which is again reinvested. In short, when a sum is invested in a year, it will yield interest and the interest is reinvested for the next year and so on till the time when withdrawal is made. The 3 year or 4 year bank deposit is a typical example of this annual interest compounding. Here:

FV = Principal + interest

FV = P(1+r)^n

The term (1+r)^n is the compound value factor (CVF) of a lump sum of Re.1, and it always has a value greater than 1 for positive r, indicating that CVF increases as r and n increase.

Compound Value – Example

Assume a lump sum of Rs.1,000 is deposited in a bank fixed deposit for 3 years for an interest rate of 10% per annum.

FV = Principal + interest

FV = P(1+r)^n

FV = 1000 x (1+.10)^3

= 1000 x 1.331

= Rs.1,331.

8.9.1 Compound In Less Than a Duration

- *Usually, it is common practice to compound the interest on a yearly basis. But, there are instances when compounding is done on a half-yearly, quarterly, monthly or a daily basis. The half-yearly interest rates indicate that interest is payable semiannually, i.e., interest is received r%/2 twice every year. When the principle of compounding is applied, this implies that the r%/2 received twice an year will yield an actual rate which is higher than the declared (r%) rate. This actual rate is called the effective annual rate.*

- *For instance, let us take an illustration of a banker declaring a 10% p.a. interest payable semiannually. This implies that at the end of the year the amount received for every one rupee will be 1 * (1+[10%/2]) * (1+[10%/2]) i.e., (1.05) * (1.05) = (1.05)^2 = 1.1025.*

- *The Effective interest rate is 10.25%*

8.10 Effective Interest Rate

The effective interest rate in the previous example was computed as 1.1025 - 1 = .1025 and in percentage terms it will be 10.25%. The effective rate of interest is hence 10.25% and not 10%. This can be expressed through the following formula:

$$FV = PV (1+ r/m)^{(m*n)}$$

where m is the number of times within a year interest is paid.

When half-yearly interest payments are made 'm' will be 12/6 i.e., 2. When quarterly interest payments are made 'm' will be 12/3 i.e., 4. When monthly compounding is done then 'm' will be 12/1 i.e., 12. Compounding on a daily basis, 'm' will be 365/1 i.e., 365. This is referred to as multi-period compounding.

8.11 Continuous Compounding

Sometimes compounding may be done continuously. For example, banks may pay interest continuously; they call it continuous compounding. It can be mathematically proved that the continuous compounding function will reduce to the following:

$$FV = PV \times \{e^x\}$$

When x = (r * n) and e is mathematically defined as equal to 2.7183.

Continuous Compounding – Example

The present value of an investment is Rs.1,000. At 10% p.a. interest rate at the end of 5 years, the future value of this investment with continuous compounding will be:

$$FV = 1{,}000 \times \{e^{.5}\} = Rs.1{,}648.72$$

When x = (r * n = .1 x 5 = .5) and e is mathematically defined as equa' to 2.7183.

Similarly, the present value of a future flow of Rs.100 at 10% p.a. interest rate to be received 5 years hence with continuous compounding will be

$$PV = FV / \{e^{.5}\} = 100 / \{e^{.5}\} = Rs.60.65.$$

8.12 Annuity

There can be a uniform cash flow accrual every year over a period of 'n' years.

This uniform flow is called "**Annuity**".

An annuity is a fixed payment (or receipt) each year for a specified number of years. The future compound value of an annuity as follows:

$$FV = A \{[(1+r)^n - 1]/ r\}$$

The term within the curly brackets {} is the compound value factor for an annuity of Re.1, and A is the annuity.

The present value of an annuity hence will be

$$PV = A \{[1 - 1/(1+r)^n]/r\}$$

Annuity – Example

The Future value of Rs.10 received every year for a period of 5 years at an assumed interest rate of 10% per annum will be

$$FV = 10 \{[(1+0.1)^5 - 1]/ 0.1\} = Rs.61.051$$

The Present value of Rs.100 to be received every year in the next five years at an assumed interest rate of 10% per annum will be

$$PV = 100\{[1 - 1/(1+0.1)^5]/0.1\} = Rs.379.08$$

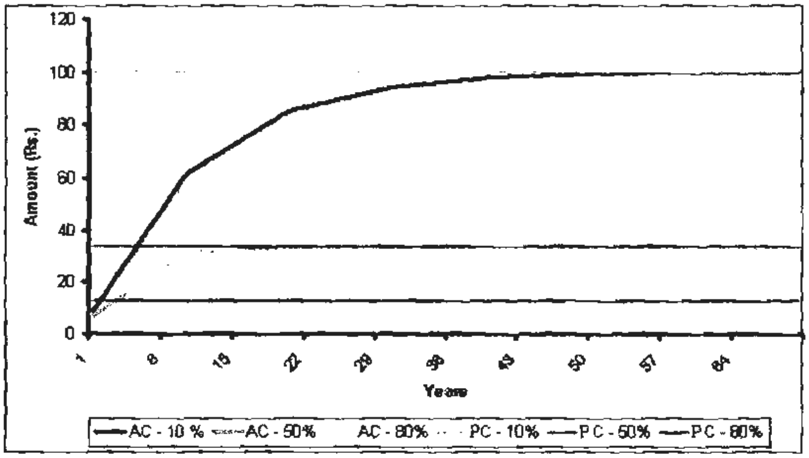
8.12.1 Resent Value of Perpetuity

Perpetuity is an annuity that occurs indefinitely. In perpetuity, time period, n, is so large (mathematically n approaches infinity) that the expression $(1+r)^n$ in the present value equation tends to become zero, and the formula for a perpetuity simply condenses into:

$$PV = A/r$$

where A is the annuity amount occurring indefinitely and r is the interest rate.

Present Value of an Annuity Vs. Perpetuity



AC - Annuity Compounding PC - Perpetuity Compounding Annuity / Perpetuity Value - Rs.10

8.13 Regular Annuity Vs. Annuity Due

- *When an annuity's cash payments are made at the end of each period, it is referred as regular annuity. On the other hand, the annual payments/receipt can also be made at the beginning of each period. This is referred to as annuity due.*

- *Lease is a contract in which lease rentals (payment) are to be paid for the use of an asset. Hire purchase contract involves regular payments (installments) for acquiring (owning) an asset. A series of fixed payments starting at the beginning of each period for a specified duration is called an annuity due.*

8.13.1 Annuity Due

The formula for computing value of an annuity due is:

FV = A[(1 + r) + (1+r)^2+ (1+r)^3 +....+ (1+r)^n-1]

FV = A {[(1+r)^(n-1) -1] / r}

Hence,

PV = A {[1 - 1/(1+r)^n]/r } * (1+r)

PV = A(PVRA,r)*(1+r)

Where PVAR is present value of regular annuity and r is the interest rate.

Annuity Due – Example

The future value of Rs.10 received in the beginning of each year for a 5 year duration at an assumed rate of 10% p.a. will be:

FV = 10 {[(1+0.1)^(5-1) -1] / 0.1} = Rs.46.41.

The present value of Rs.100 received in the beginning of each year for 5 years at an assumed interest rate of 10% p.a. will be:

PV = 100 {[1 - 1/(1+1.1)^5]/0.1 } x (1+0.1)= Rs.416.98.

8.13.2 Multi Period Annuity Compounding

The compound value of an annuity in case of the multi-period compounding is given as follows:

FV = A <{[(1+r/m)^(n x m)] -1 } /(r/m)>

PV = A <{1 -[1/(1+r/m)^(n x m)]} / (r/m)>

In all instances, the discount rate will be (r/m) and the time horizon will be equal to (n x m).

8.14 Present Value of a growing annuity

An annuity may not be a constant sum through the time duration, it may also grow at a rate of g% every year. This is referred as a growing annuity. When there is a growth for specific number of years, the present value of an annuity is stated using the following formula:

$$PV = A\left[\frac{1}{r-g} - \frac{1}{r-g} \times \frac{(1+g)^n}{(1+r)^n}\right]$$

Present Value Of A Growing Annuity – Example

An annuity of Rs.100 is expected to grow at a rate of 2% every year. Assuming the interest rate as 10% per annum the present value for this growing annuity for a 5 year duration will be:

PV = 100 x {(1/0.08)-[(1/0.08)*(1.02)^5/(1.1)^5]}

= Rs.393.07.

FUTURE VALUE OF A GROWING ANNUITY

Future value of a growing annuity can be defined by the following formula:

$$FV = A\left[\frac{(1+r)^n}{(r-g)} - \frac{(1+g)^n}{(r-g)}\right]$$

Future Value Of A Growing Annuity - Example

Future value of an annuity of Rs.10 growing at 2% every year with an assumed rate of interest at 10% for five years is computed as:

FV = 10 x {[1.1^5/0.08]-[1.02^5/0.08]}
= Rs.63.30

8.15 Present Value of Perpetuity

In financial decision-making there are number of situations where cash flows may grow at a compound rate. Here, the annuity is not a constant amount A but is subject to a growth factor 'g'. When the growth rate 'g' is constant, the formula can be simplified very easily. The calculation of the present value of a constantly growing perpetuity is given by the following equation:

PV = A/(1+r) + A(1+g)/(1+r)^2 + A(1+g)^2/(1+r)^3 + ...

This equation can be simplified as:

PV = A / (r - g)

8.16 Future value of a growing annuity

In financial decision-making there are number of situations where cash flows may grow at a compound rate. Here, the annuity is not a constant amount A but is subject to a growth factor 'g'. When the growth rate 'g' is constant, the formula can be simplified very easily. The calculation of the present value of a constantly growing perpetuity is given by the following equation:

PV = A/(1+r) + A(1+g)/(1+r)^2 + A(1+g)^2/(1+r)^3 + ...

This equation can be simplified as:

PV = A / (r - g)

Present Value Of A Growing Annuity Perpetuity – Example

The present value of an annuity of Rs.10 growing at 2% every year with an assumed rate of interest of 10% to perpetuity is:

PV = A / (r - g)

PV = 10 / (0.1 - 0.02) = Rs.125.

8.17 Debentures

Debentures are creditor ship securities representing long-term indebtedness of a company. A debenture is an instrument executed by the company under its common seal acknowledging indebtedness to some person or persons to secure the sum advanced. It is, thus, a security issued by a company against the debt. A public limited company is allowed to raise debt or loan through debentures after getting certificate of commencement of business if permitted by its Memorandum of Association. Companies Act has not defined the term debenture.

Debentures, like shares, are equal parts of loan raised by a company. Debentures are usually secured by the company by a fixed or floating debentures at periodical intervals, generally six months and the company agrees to pay the principal amount at the expiry of the stipulated period according to their terms of issue. Like shares, they are issued to the public at part, at a premium or at a discount. Debenture-holders are creditors of the company. They have no voting rights but their claims rank prior to preference shareholders and equity shareholders. Their exact rights depend upon the nature of debentures they hold.

The capital is not only raised through shares, it is sometimes raised through loans, taken in the form of debentures.

A debenture is a written acknowledgment of a debt taken by a company. It contains a contract for the repayment of principal sum by some specific date and payment of interest at a specified rate irrespective of the fact, whether the company has a profit or loss. Debenture holders are, therefore, creditors of the company. Of course, they do not have any right on the profits declared by the company. Like shares, debentures can also be sold in or purchased from the market and all the terms used for shares also apply in this case ; with the same meanings.

Let us take some examples.

Example : Find the income percent of a buyer on 10 % debentures of face value Rs.100, available in the market at Rs. 125.

Solution : Income on Rs. 125 is Rs. 10

$\therefore$ Income on Rs. 100 $= \frac{10}{1255} \times 100 = $ Rs. 8

$\therefore$ Income, in percents, on debentures = 8%

Example : Shama has 1000 shares of par value Rs. 10 each of a company and 200 debentures of par value Rs. 100 each. The company pays an annual dividend of 10% and an interest of 15% on debentures. Find the total income of Shama and rate of return on her investment.

Solution : Dividend on 1000 shares = Rs. $\left(\frac{1000 \times 10 \times 10}{100}\right)$ = Rs. 1000

Annual interest on 200 debentures = Rs. $\left(\frac{200 \times 100 \times 15}{100}\right)$ = Rs. 3000

$\therefore$ Total income of Shama = Rs. 4000

Total investment of Shama = Rs. $(1000 \times 10 + 200 \times 100)$ = Rs. 30000

$$\therefore \text{ Rate of return} = \left(\frac{4000 \times 100}{30000}\right)\% = 13.33\,\%$$

Debentures can be of following types:

Redeemable and Irredeemable Debentures

Redeemable debentures are those which can be redeemed or paid back at the end of a specified period mentioned on the debentures or within a specified period at the option of the company by giving notice to the debenture holders or by installments as per terms of issue. Irredeemable debentures are those which are repayable at any time by the company during its existence. No date of redemption is specified. the debenture holders cannot claim their redemption. However, they are due for redemption if the company fails to pay interest on such debentures or on winding up of the company. They are also called perpetual debentures.

Secured and Unsecured Debentures

Secured or mortgaged debentures carry either a fixed charge on the particular asset of the company or floating charge on all the assets of the company. Unsecured debentures, on the other hand, have no such charge on the assets of the company. They are also known as simple or naked debentures.

Registered and Bearer Debentures

Registered debentures are registered with the company. Name, address and particulars of holdings of every debenture holders are recorded on the debenture certificate and in the books of the company. At the time of transfer, a regular transfer deed duly stamped and properly executed is required. Interest is paid only to the registered debenture holders. Bearers debentures on the other hand, are transferred by more delivery without any notice to the company. Company keeps no record for such debentures. Debentures-coupons are attached with the debentures-certificate and interest can be claimed by the coupon-holder.

Convertible and Non-convertible Debentures

Convertible debentures are those which can be converted by the holders of such debentures into equity shares or preference shares, cannot be converted into shares. Now, a company can also issue partially convertible debentures under which only a part of the debenture amount can be converted into equity shares.

EXERCISE

1. What do mean by Inflation?
2. What is meant by Return?
3. What is the Interest without compounding? Write formula.
4. What do mean by Annuity?