Institute of **Open** and **Distance Education**

Business Mathematics

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1BCOM5

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DR. C.V.RAMAN UNIVERSITY

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1BCOM5

Business Mathematics

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UNIT I

LESSON

1

RATIO AND PROPORTION

CONTENTS

1.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Explain the concept of ratio
- Know about the proportion
- Methods of finding ratio and proportion
- Understand the gaining ratio
- Learn about the sacrificing ratio

1.1 INTRODUCTION

In certain situations, comparison by division makes better sense than comparison by taking the difference. The comparison by division is called as the ratio of the two numbers. In this lesson, we will introduce the concept of ratios and proportions.

Ratio and proportions are very extensively used in many branches of knowledge like Physics, Chemistry, Mathematics and also business problems. In business problems related to cost-volume profit analysis or break even analysis and others, the techniques of ratio and proportions are extensively used.

A ratio compares two numbers. Just as you cannot compare apples and oranges, so the numbers you are comparing must have the same units.

For example, you cannot form the ratio of 2 feet to 4 meters because the two numbers are expressed in different units - feet vs. meters.

The equality of two ratios (fractions) is called proportion. If a : b = c : d, we write a : b :: c : d and we say that a, b, c, d are in proportion. In this lesson, we will also discuss gaining ratio and sacrifice ratio.

1.2 CONCEPTS OF RATIOS

Any two quantities of the same kind compared by dividing one by the other are said to be in a ratio.

Example: If a man earns ₹ 5,000 and saves ₹ 500 then the ratio of savings to earnings is:

$$\frac{\text{Savings}}{\text{Earnings}} = \frac{500}{5000} = \frac{1}{10}$$

i.e., Savings : Earnings = 1:10.

A simple definition of ratio is $\frac{p}{q}$ (here $q \neq 0$)

Antecedent and Consequent

If $\frac{p}{q}$ is a ratio then p is called the antecedent and q is called the consequent.

Example: If $\frac{2}{3}$ is a ratio, 2 is the antecedent and 3 is the consequent.

Equality of Ratios

Two ratios $\frac{a}{b}$ and $\frac{c}{d}$ where $b \neq 0, d \neq 0$ are said to be equal if, $ad = bc$.

i.e., $\quad \frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

or $\quad a : b = c : d \Leftrightarrow ad = bc$

Example: $\frac{2}{3}$ and $\frac{8}{12}$ are equal.

$\because \quad 2 \times 12 = 24 = 3 \times 8$

1.2.1 Types of Ratios

Here we will discuss about different types of ratios.

Compounded Ratio

For two or more ratios, if we take antecedent as product of antecedents of the ratios and consequent as product of consequents of the ratios, then the ratio thus formed is called mixed or compound ratio. As, compound ratio of $m : n$ and $p : q$ is $mp : nq$.

In other words, when two or more ratios multiply termwise; the ratio thus obtained is called compound ratio.

Example: The compounded ratio of the two ratios $a : b$ and $c : d$ is the ratio $ac : bd$, and that of $a : b$, $c : d$ and $e : f$ is the ratio, $ace : bdf$.

For ratios $m : n$ and $p : q$; the compound ratio is $(m \times p) : (n \times q)$.

For ratio $m : n, p : q$ and $r : s$; the compound ratio is $(m \times p \times r) : (n \times q \times s)$.

Duplicate Ratio

Duplicate ratio is the ratio of two equal ratios.

Example: The duplicate ratio of the ratio $x : y$ is the ratio $x^2 : y^2$.

In other words, the duplicate ratio of the ratio $m : n$ = Compound ratio of $m : n$ and $m : n$

$$= (m \times m) : (n \times n)$$
$$= m^2 : n^2$$

Therefore, the duplicate ratio of $4 : 7 = 4^2 : 7^2 = 16 : 49$

Triplicate Ratio

The triplicate ratio is the compound ratio of three equal ratios.

The triplicate ratio of the ratio $a : b$ is the ratio $a^3 : b^3$.

In other words, the triplicate ratio of the ratio $m : n$ = Compound ratio of $m : n$, $m : n$ and $m : n$

$$= (m \times m \times m) : (n \times n \times n)$$
$$= m^3 : n^3$$

Therefore, the triplicate ratio of $4 : 7 = 4^3 : 7^3 = 64 : 343$.

Subduplicate Ratio

The subduplicate ratio $m : n$ is the ratio $\sqrt{m} : \sqrt{n}$. So, the subduplicate ratio of the ratio $m^2 : n^2$ is the ratio $m : n$.

Example: The subduplicate ratio of $25 : 81 = \sqrt{25} : \sqrt{81} = 5 : 9$.

Subtriplicate Ratio

The subtriplicate ratio $m : n$ is the ratio $\sqrt{m} : \sqrt{n}$. So, the subduplicate ratio of the ratio $3\sqrt{m} : 3\sqrt{n}$ is the ratio $m : n$.

Example: The subtriplicate ratio of $125 : 729 = 3\sqrt{125} : 3\sqrt{729} = 5 : 9$

Reciprocal Ratio

The reciprocal ratio of the ratio $m : n$ $(m \neq 0, n \neq 0)$ is the ratio $\dfrac{1}{m} : \dfrac{1}{n}$

For any ratio $x : y$, where $x, y \neq 0$, its reciprocal ratio $= \dfrac{1}{x} : \dfrac{1}{y} = y : x$

Similarly, we can say if the antecedent and consequent of a ratio be interchanged, the changed ratio is called the inverse ratio of the previous ratio.

Example: Reciprocal ratio of $7:13 = \dfrac{1}{7} : \dfrac{1}{13} = 13:7$

5 : 7 is the inverse ratio of 7 : 5

Ratio of Equalities

For a ratio, if the antecedent and consequent are equal, the ratio is called ratio of equality.

Example: 5 : 5 is the ratio of equalities.

Ratio of Inequalities

For a ratio, if the antecedent and consequent are unequal, the ratio is called ratio of inequality.

Example: 5 : 7 is the ratio of inequalities.

Ratio of Lesser Inequalities

For a ratio, if antecedent is less than the consequent, the ratio is called the ratio of lesser inequality.

Example: 7 : 9 is a ratio of lesser inequalities.

Ratio of Greater Inequalities

For a ratio, if antecedent is greater than the consequent, the ratio is called the ratio of greater inequality.

Example: 13 : 10 is a ratio of greater inequalities.

Notes:

1. If the ratio $x : y$, if $x = y$, we get ratio of equality. If $x \neq y$, we get ratio of inequality, $x > y$ gives ratio of greater inequality.

2. $y : x$ and $x : y$ are mutually inverse ratio to each other.

1.2.2 Properties of Ratio

(a) *Componendo:* If, $a:b = c:d$ then $(a+b):b = (c+d):d$, this property is called *Componendo.*

Proof: $\dfrac{a}{b} = \dfrac{c}{d}$ (given)

Add 1 on both sides.

$\therefore \qquad \dfrac{a}{b}+1 = \dfrac{c}{d}+1$

i.e., $\qquad \dfrac{a+b}{b} = \dfrac{c+d}{d}$

$\therefore \qquad a:b = c:d \Leftrightarrow \dfrac{a+b}{b} = \dfrac{c+d}{d}$

$\therefore \qquad (a+b):b = (c+d):d$

(b) *Dividendo:* If, $a:b = c:d$ then $(a-b):b = (c-d):d$.

This property is called *Dividendo.*

Proof: $\dfrac{a}{b} = \dfrac{c}{d}$ (given)

Subtracting 1 from both sides, we get

$$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

i.e., $\quad \dfrac{a-b}{b} = \dfrac{c-d}{d}$

$\therefore \quad (a-b):b = (c-d):d$

(c) **Componendo et Dividendo:** If, $\dfrac{a}{b} = \dfrac{c}{d}$ then, $(a+b):(a-b) = (c+d):(c-d)$.

This property is called *Componendo et Dividendo.*

Proof:

$$\frac{a}{b} = \frac{c}{d} \qquad \text{(given)} \qquad \qquad ...(1)$$

Add 1 to both sides of (1)

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

i.e., $\quad \dfrac{a+b}{b} = \dfrac{c+d}{d} \qquad \qquad ...(2)$

Subtract 1 from both sides of (1)

$\therefore \quad \dfrac{a}{b} - 1 = \dfrac{c}{d} - 1$

i.e., $\quad \dfrac{a-b}{b} = \dfrac{c-d}{d} \qquad \qquad ...(3)$

Dividing (2) from (3), we get

$$\frac{\dfrac{a+b}{b}}{\dfrac{a-b}{b}} = \frac{\dfrac{c+d}{d}}{\dfrac{c-d}{d}}$$

i.e., $\quad \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$

i.e., $\quad (a+b):(a-b) = (c+d):(c-d)$

Example: Find the ratio between two numbers such that their sum is 60 and difference is 8.

Solution:

Let the two numbers be x and y.

$\therefore \quad x + y = 60 \qquad \qquad ...(1)$

$\qquad x - y = 8 \qquad \qquad ...(2)$

Let us solve these questions.

Adding (1) and (2), we get

$$2x = 68$$

$$\therefore \qquad x = 34$$

Substituting $x = 34$ in (1s), we get

$$34 + y = 60$$

$$\therefore \qquad y = 60 - 34$$

i.e., $\qquad y = 26$

$\therefore$ The ratio between x and y is $\dfrac{x}{y} = \dfrac{34}{26} = \dfrac{17}{13}$

$$\therefore \qquad x : y = 17 : 13$$

Example: A ratio in its lowest terms is $\dfrac{4}{9}$. If the difference between the quantities is 25, find the quantities.

Solution:

Let the two quantities be x and y.

$$\therefore \qquad \frac{x}{y} = \frac{4}{9} = \frac{4a}{9a} \text{ (where } a \text{ is a constant and } y > x)$$

$$\therefore \qquad x = 4a \text{ and } y = 9a$$

$$y - x = 9a - 4a$$

i.e., $\qquad y - x = 5a$

$$25 = 5a$$

$$a = 5$$

$\therefore$ The quantities are 4×5 and 9×5, i.e., 20 and 45.

Example: Two numbers are in the ratio 3:5. If 7 is added to each, they are in the ratio 8:9. Find the numbers.

Solution:

Let the numbers be x and y.

x and y are in the ratio of $3 : 5$

$\therefore$ The numbers x and y can be taken as $x = 3a$ and $y = 5a$ where a is a constant and $y > x$.

If 7 is added to each, we get $3a + 7$ and $5a + 7$.

These are in the ratio $8 : 9$

$$\therefore \qquad \frac{3a + 7}{5a + 7} = \frac{8}{9}$$

Cross multiplying, we get

$$9(3a + 7) = 8(5a + 7)$$

$$27a + 63 = 40a + 56$$

i.e., $\quad 40a - 27a = 63 - 56$

$\qquad 13a = 7$

∴ $\qquad a = \dfrac{7}{13}$

∴ The numbers are $x = 3 \times \dfrac{7}{13} = \dfrac{21}{13}$ and $y = 5 \times \dfrac{7}{13} = \dfrac{35}{13}$

∴ The numbers are $\dfrac{21}{13}$ and $\dfrac{35}{13}$.

Example: What number should be added to each term in the ratio 5 : 6 so that it becomes 8 : 11?

Solution:

Let the number x be added to each term in the ratio $\dfrac{5}{6}$.

Then we get $\dfrac{5+x}{6+x} = \dfrac{8}{11}$

On cross multiplication, we get

$\qquad 11(5 + x) = 8(6 + x)$

i.e., $\quad 55 + 11x = 48 + 8x$

i.e., $\quad 11x - 8x = 48 - 55$

$\qquad 3x = -7$

$\qquad x = \dfrac{-7}{3}$

Example: The incomes of A and B are in the ratio of 5 : 6 and their expenditures are in the ratio of 4 : 5. If they both save ₹ 400 at the end of year, find the annual incomes of A and B.

Solution:

The ratio of the incomes of A and B is 5 : 6.

Therefore, their incomes may be taken as $5a$ and $6a$. Since they save ₹ 400 their expenditure is, $5a - 400$ and $6a - 400$ each.

∴ $\qquad \dfrac{5a - 400}{6a - 400} = \dfrac{4}{5}$

i.e., $\quad 5(5a - 400) = 4(6a - 400)$

$\qquad 25a - 2000 = 24a - 1600$

$\qquad 25a - 24a = 2000 - 1600$

∴ $\qquad a = 400$

∴ $\qquad A$'s income is $5 \times 400 = ₹ 2000$

$\qquad B$'s income is $6 \times 400 = ₹ 2400$

Example: If $a:b = 2:3$, $b:c = 5:7$ and $c:d = 9:3$, find $a:d$.

Solution:

$\qquad \dfrac{a}{b} = \dfrac{2}{3}, \dfrac{b}{c} = \dfrac{5}{7}, \dfrac{c}{d} = \dfrac{9}{3}$

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{2}{3} \times \frac{5}{7} \times \frac{9}{3}$$

i.e., $\quad \dfrac{a}{d} = \dfrac{10}{7}$

$$a:d = 10:7$$

Example: An article is sold at a gain of 25% on cost price. Find the ratio of selling price to cost price.

Solution:

Let the cost price of the article be x.

25% of x is $\dfrac{25x}{100} = \dfrac{x}{4}$

Therefore, the Selling price = Cost price + 25% on cost price.

$$= x + \frac{x}{4} = \frac{5x}{4}$$

$$\frac{\text{Selling price}}{\text{Cost price}} = \frac{\dfrac{5x}{4}}{x} = \frac{5}{4}$$

Therefore, selling price : cost price 5:4

Example: If $\dfrac{x}{y} = \dfrac{2}{3}$, find $\dfrac{2x^2 + 3y^2}{x^2 + y^2}$

Solution:

$$\frac{x}{y} = \frac{2}{3}$$

$$\therefore \quad y = \frac{3x}{2}$$

$$\frac{2x^2 + 3y^2}{x^2 + y^2} = \frac{2x^2 + 3\left(\dfrac{3x}{2}\right)^2}{x^2 + \left(\dfrac{3x}{2}\right)^2}$$

$$= \frac{2x^2 + 3\left(\dfrac{9x^2}{4}\right)}{x^2 + \dfrac{9x^2}{4}}$$

$$= \frac{\dfrac{8x^2 + 27x^2}{4}}{\dfrac{4x^2 + 9x^2}{4}}$$

$$= \frac{35x^2}{13x^2} = \frac{35}{13}$$

Example: Divide ₹ 5880 in the ratio 3 : 6 : 5.

Solution:

The amount of ₹ 5880 has to be divided in the ratio of 3 : 6 : 5.

Let these amounts be $3x, 6x, 5x$.

$\therefore$

$$3x + 6x + 5x = 5880$$
$$14x = 5880$$
$$x = \frac{5880}{14} = 420$$

Therefore, the three parts are $3x, 6x, 5x$.

i.e., $3 \times 420, 6 \times 420, 5 \times 420$.

i.e., ₹ 1260, ₹ 2520, ₹ 2100

Example: If $2x^2 - 3y^2 : x^2 + y^2$ is equal to $2:41$, find $x:y$.

Solution:

$$\frac{2x^2 - 3y^2}{x^2 + y^2} = \frac{2}{41}$$

On cross multiplication, we get

$$41(2x^2 - 3y^2) = 2(x^2 + y^2)$$
$$82x^2 - 123y^2 = 2x^2 + 2y^2$$

i.e.,

$$82x^2 - 2x^2 = 123y^2 + 2y^2$$
$$80x^2 = 125y^2$$
$$\frac{x^2}{y^2} = \frac{125}{80} = \frac{25}{16} = \left(\frac{5}{4}\right)^2$$
$$\frac{x}{y} = \frac{5}{4}$$

i.e., $x:y = 5:4$.

1.3 PROPORTIONS

If two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are such that $\frac{a}{b} = \frac{c}{d}$ then a, b, c, d are said to be in proportion.

The numbers a and d are called extremes and the numbers b and c are called the means. The product of the extremes is equal to the product of the means.

i.e., $\qquad a:b = c:d \Leftrightarrow ad = bc$

Example: $3:4 = 6:8 \Leftrightarrow 3 \times 8 = 24 = 4 \times 6$

1.3.1 Differences between Ratio and Proportion

The differences between ratio and proportion are distinguished as follows:

1. There are two terms in a ratio. Whereas there are four terms in a proportion.

2. Ratio is a comparison of two quantities of the same kind whereas proportion is a comparison of two ratios.

3. In the ratio, the two quantities must be of the same kind and in proportion, the first two are of one kind and the last two may be of another kind.

1.3.2 Types of Proportion

Continued Proportional

If $a:b = b:c = c:d = \dots$

i.e., If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = \dots$

Then $a, b, c, d,\dots$ are said to be in continued proportion.

Example: $\dfrac{3}{6} = \dfrac{6}{12} = \dfrac{12}{24} = \dots$ are said to be in continued proportion.

Mean Proportional

If $a:b = b:c$ then b is called the mean proportional between a and c.

$$\frac{a}{b} = \frac{b}{c}$$

$\Rightarrow \qquad b^2 = ac\,(\text{on cross multiplication})$

$\Rightarrow \qquad b = \sqrt{ac}$

Therefore, the mean proportional between a and c is given by $b = \sqrt{ac}$.

Example: If $\dfrac{3}{6} = \dfrac{6}{12}$, then the mean proportional between 3 and 12 is $\sqrt{3 \times 12} = \sqrt{36}$ $= 6$, which is the consequent of the first ratio (or the antecedent of the second ratio).

Third Proportional

If $a:b = b:c$ then c is called the third proportional between a and b.

$\Rightarrow \qquad \dfrac{a}{b} = \dfrac{b}{c} \Rightarrow b^2 = ac\,(\text{on cross multiplication})$

$\therefore \qquad c = \dfrac{b^2}{a}$ is third proportional between a and b

Example: If $\dfrac{3}{6} = \dfrac{6}{12}$, then the third proportional between 3 and 6 is $\dfrac{6^2}{3} = \dfrac{36}{3} = 12$, which is the consequent of the second ratio.

Fourth Proportional

If $a:b = c:d$ then d is called the fourth proportional of a, b, c.

i.e., $\qquad$ if $\dfrac{a}{b} = \dfrac{c}{d}$

then $\qquad ad = bc$ (on cross multiplication)

$$d = \frac{bc}{a}$$

$\therefore$ The fourth proportional of a, b, c is given by $d = \dfrac{bc}{a}$.

Example: If $2:3 = 4:6$, then 6 is the fourth proportional of 2, 3, 4.

The fourth proportional of 2, 3, 4

$= \dfrac{3 \times 4}{2} = 6$ which is the consequent of the second ratio.

Inversely Proportional

Inversely Proportional takes place when one value decreases at the same rate that the other increases.

Speed and travel time are Inversely Proportional because the faster we go the shorter the time.

- As speed goes up, travel time goes down
- As speed goes down, travel time goes up

The Rule of Proportions

If a quantity Q is divided into three parts in the ratio $p:q:r$, then

$$\frac{I \text{ part}}{p} = \frac{II \text{ part}}{q} = \frac{III \text{ part}}{r} = \frac{Q}{p+q+r}$$

$$III\,part = \left(\frac{r}{p+q+r}\right)Q$$

$$I\,part = \left(\frac{p}{p+q+r}\right)Q$$

$$II\,part = \left(\frac{q}{p+q+r}\right)Q$$

Example: Divide ₹ 2600 into three parts in the ratio $\dfrac{1}{2}:\dfrac{1}{3}:\dfrac{1}{6}$.

Solution:

By applying the rule of proportions, we can find the three parts of 2600.

$$I\,part = \left(\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}\right)2600 \qquad II\,part = \left(\frac{\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}\right)2600$$

$$= \left(\frac{\frac{1}{2}}{\frac{18+12+6}{2\cdot3\cdot6}}\right)2600 \qquad = \left(\frac{\frac{1}{3}}{\frac{18+12+6}{2\cdot3\cdot6}}\right)2600$$

$$= \frac{18}{36} \times 2600 \qquad\qquad = \frac{12}{36} \times 2600$$

$$= \frac{1}{2} \times 2600 = 1300 \qquad = \frac{1}{3} \times 2600 = 866.666 = 866.67$$

$$III\ part = \left(\dfrac{\dfrac{1}{6}}{\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{6}}\right)2600$$

$$= \left(\dfrac{\dfrac{1}{6}}{\dfrac{18+12+6}{2\cdot3\cdot6}}\right)2600$$

$$= \dfrac{6}{36}\times2600$$

$$= \dfrac{1}{6}\times2600 = 433.33$$

Therefore, the three parts are ₹ 1300, ₹ 866.67 and ₹ 433.33

Example: Find the mean proportional between 2 and 8.

Solution:

The mean proportional between 2 and 8 is $\sqrt{2\times8}=\sqrt{16}=4$.

Therefore, 4 is the mean proportional between 2 and 8.

Example: Find the third proportional of 0.5 and 1.5.

Solution:

The third proportional of a and b is $\dfrac{b^2}{a}$.

The third proportional of 0.5 and 1.5 is $\dfrac{(1.5)^2}{0.5}$

$$= \dfrac{1.5\times1.5}{0.5}=3\times1.5=4.5$$

Example: Find the fourth proportional of 8, 12, 16.

Solution:

The fourth proportional d of a, b, c is given by $d=\dfrac{bc}{a}$.

Therefore, fourth proportional of 8, 12, 16 is

$$\dfrac{12\times16}{8}=24$$

Example: The marks in Mathematics of three students Rama, Anthony and Jabbar are proportional to 5 : 4 : 3. If Anthony obtained 70 marks, what are the marks of Rama and Jabbar?

Solution:

The proportion of marks is 5 : 4 : 3

Total $= 5+4+3 = 12$

Since Anthony obtained 70 marks,

$$\dfrac{4x}{12}=70$$

$$\therefore\qquad x=\dfrac{70\times12}{4}=210$$

Therefore, the marks obtained by Rama is $\frac{5}{12} \times 210 = 87.5$ and marks obtained by Jabbar is, $\frac{3}{12} \times 210 = 52.5$.

Example: If the monthly incomes of A and B are in the ratio of $3 : 2$ and those of B and C are in the ratio of $9 : 7$, what is the ratio of the monthly incomes of A and C?

Solution:

Monthly incomes of A and $B = 3 : 2$. On multiplying both antecedent and consequent by 9, we get $A : B = 27 : 18$.

Monthly incomes of B and $C = 9 : 7$

On multiplying both antecedent and consequent by 2, we get $B : C = 18 : 14$.

$$\therefore \qquad \frac{A}{B} \times \frac{B}{C} = \frac{27}{18} \times \frac{18}{14} = \frac{27}{14}$$

Therefore, the ratio of the monthly incomes of A and C is $27 : 14$.

Example: Solve the equation $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = \dfrac{4x-1}{2}$.

Solution:

By componendo et dividendo

$$\frac{\sqrt{x+1}+\sqrt{x-1}+\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}-\sqrt{x+1}+\sqrt{x-1}} = \frac{4x-1+2}{4x-1-2}$$

$$\Rightarrow \qquad \frac{2\sqrt{x+1}}{2\sqrt{x-1}} = \frac{4x+1}{4x-1}$$

$$\Rightarrow \qquad \frac{\sqrt{x+1}}{\sqrt{x-1}} = \frac{4x+1}{4x-1}$$

Squaring $\Rightarrow \dfrac{x+1}{x-1} = \dfrac{16x^2+8x+1}{16x^2-24x+9}$

Using componendo and dividendo, we get

$$\Rightarrow \qquad \frac{2x}{2} = \frac{32x^2-16x+10}{32x-8}$$

$$\Rightarrow \qquad x = \frac{16x^2-8x+5}{16x-4}$$

$$\Rightarrow \qquad 16x^2-4x = 16x^2-8x+5$$

$$\Rightarrow \qquad 4x = 5$$

$$\Rightarrow \qquad x = \frac{5}{4}$$

1.4 GAINING RATIO

Gaining ratio is calculated at the time of retirement of a partner. Gaining ratio is the opposite of sacrificing ratio. This is the ratio gain to the existing partners of a firm when they revise the profit sharing ratio, or when the profit share of the deceased or

retired partner is shared by the other partners. This ratio is calculated by deducting the old ratio from the new ratio. The new share will be higher than the old when there is a gain.

$$\text{Gaining Ratio} = \text{New Ratio} - \text{Old Ratio}$$

Difference of old ratio and sacrifice ratio will be new ratio but total of old ratio and gaining ratio will be new ratio. Total of gaining share of each partner will be equal to the old ratio of retired partner. Goodwill will be distributed at the time of retirement in gaining ratio.

Example: Kamal, Kishore and Kunal are partners in a firm sharing profits equally. Kishore retires from the firm. Kamal and Kunal decide to share the profits in future in the ratio 4:3. Calculate the gaining ratio.

Solution:

Gaining Ratio = New ratio – Old ratio

Kamal's Gain = $4/7 - 1/3 = 5/21$

Kunal's Gain = $3/7 - 1/3 = 2/21$

Gaining Ratio = 5:2

Example: P, Q and R are partners sharing profits in the ratio of 7:2:1. P retires and the new profit sharing ratio between Q and R is 2:1. State the gaining ratio.

Solution:

Old ratio = P : Q : R

　　　　 7 : 2 : 1

New ratio = Q : R

　　　　 2 : 1

Gaining Ratio = New ratio – Old ratio

Q's gain = $2/3 - 2/10 = 14/30$

R's gain = $1/3 - 1/10 = 7/30$

Gaining Ratio = 14:7 or 2:1

1.5 SACRIFICING RATIO

When a new partner comes into the business, old partners have to give him his profit share from their portion. Thus change in profit sharing ratio is an important aspect to be considered on reconstitution by admission. In accounting, change in profit sharing ratio can be presented in various ways. The existing partners may decide to change their profit sharing ratio for various reasons. When the profit sharing ratio is revised among existing partners, there ought to be a partial sacrifice of profit share by some partners in favour of others. The sacrifice of one or a group of partners becomes the gain of the remaining partners. Total of each old partner's Sacrifice will be equal to new ratio of new admitted partner. Goodwill will be distributed at the time of admission in sacrifice ratio. Following is the formula for calculating sacrificing ratio:

$$\text{Sacrificing Ratio} = \text{Old Ratio} - \text{New Ratio}$$

When the profit sharing ratio is revised it is important to calculate the sacrificing ratio and gaining ratio. These ratios are required to adjust the value of goodwill of a firm without raising goodwill account in the books.

Case 1: The old partners give part of their share to the new partner (focus on the old partner)

Example: Calculate profit sharing ratio and sacrificing ratios, if A & B who are equal partners admit C for which A surrenders 1/2 of his share and B surrenders 1/4th of his share in favour of C.

Solution:

(a) A's Old share = 1/2

(b) Portion surrendered for C 1/2 of 1/2

i.e., $1/2 \times 1/2 = 1/4$

(c) Balance available for A = 1/4 (a – b)

(d) B's old share = 1/2

(e) Portion surrendered for C = 1/4th of 1/2

i.e., $1/2 \times 1/4 = 1/8$

(f) Balance available for B = 3/8 (1/2 – 1/8)

(g) C's share = A's contribution + B's contribution

i.e. 1/4 + 1/8 = 3/8

(h) New profit sharing ratio = 1/4 : 3/8 : 3/8

i.e., 2 : 3 : 3

Case 2: The new partner acquires his share from old partners (focus on the new partner's share).

Example: Calculate sacrificing ratio and new profit sharing ratio if A & B sharing profits and losses equally admit C into partnership for 1/3rd share in future profits, of which 2/3rd is acquired from A and 1/3rd is acquired from B.

Solution:

(a) C's share = 1/3rd of future profits

(b) A's contribution (sacrifice) = 2/3rd of 1/3rd

i.e., $2/3 \times 1/3 = 2/9$

(c) Balance available for A = A's old share – A's sacrifice

i.e., 1/2 – 2/9 = 5/18

(d) B's contribution =1/3rd of 1/3rd

i.e., $1/3 \times 1/3 = 1/9$

(e) Balance available for B = B's old share – B's contribution

i.e., 1/2 – 1/9 = 7/18

C's share = 2/9 +1/9 = 3/9

(f) New profit sharing ratio = 5/18 : 7/18 : 6/18

i.e., 5:7:6

(g) Sacrificing Ratio = 2/9 : 1/9 i.e., 2:1

Case 3: An entirely new profit sharing ratio is given

Example: Calculate sacrificing ratio if A & B sharing profits and losses equally, admit C into partnership and decide the share of future profits and losses in the ratio 3:2:2.

Solution:

A's sacrifice = 1/2 – 3/7 =7/14 – 6/14 = 1/14

B's sacrifice = 1/2 –2/7 = 7/14 – 4/14 = 3/14

Sacrificing ratio = 1:3

Check Your Progress

Choose the correct option:

1. Two numbers are respectively 20% and 50% more than a third number. The ratio of the two numbers is:

   (a) 2 : 5   (b) 3 : 5

   (c) 4 : 5   (d) 6 : 7

Fill in the blank:

2. The mean proportional between 2 and 8 is _____.

State whether the following statements are True or False:

3. In business problems related to cost-volume profit analysis or break even analysis and others, the techniques of ratio and proportions are extensively used.

4. In the ratio a : b, we call a as the first term or antecedent and b, the second term or consequent.

1.6 LET US SUM UP

- Any two quantities of the same kind compared by dividing one by the other are said to be in a ratio.

- If $\frac{p}{q}$ is a ratio then p is called the antecedent.

- Two ratios, $\frac{a}{b}$ and $\frac{c}{d}$, where b ≠ 0 and, d ≠ 0, are said to be equal if $ad = bc$.

- If two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are such that $\frac{a}{b} = \frac{c}{d}$ then a, b, c, d are said to be in proportion. The numbers a and d are called extremes and the numbers b and c are called the means.

- Proportion is a comparison of two ratios.

- If $a : b = c : d$, then $(a + b) : b = (c + d) : d$.

- If $a : b = c : d$, then $(a – b) : b = (c – d) : d$.

- $a : b = c:d$, then $(a + b) : (a – b) = (c + d) : (c – d)$

- Gaining ratio is calculated at the time of retirement of a partner. Gaining ratio is the opposite of sacrificing ratio.

- The formula for calculating gaining ratio: Gaining Ratio = New Ratio – Old Ratio.

- Goodwill will be distributed at the time of retirement in gaining ratio.

- Goodwill will be distributed at the time of admission in sacrifice ratio.

- The formula for calculating sacrificing ratio is – Sacrificing Ratio = Old Ratio – New Ratio.

1.7 LESSON END ACTIVITY

A sum of money is to be distributed among A, B, C, D in the proportion of 5: 2 : 4 : 3. If C gets ₹ 1000 more than D, what is B's share?

1.8 KEYWORDS

Gaining Ratio: It is calculated at the time of retirement of a partner.

Mean Proportion: If $a : b = b : c$, then b is called mean proportional.

Proportion: It is a comparison of two ratios.

Ratio: Any two quantities of the same kind compared by dividing one by the other are said to be in a ratio.

Sacrificing Ratio: It is ratio in which they agree to sacrifice their share of profits in favour of incoming partner.

1.9 QUESTIONS FOR DISCUSSION

1. What are the differences between ratio and proportion?

2. What do you understand by the term equality of ratios? Give their examples. Discuss the various properties of ratios.

3. Two numbers are in the ratio of 3 : 5. If 7 is added to each, they are in the ratio of 8:9. Find the numbers.

4. Marks obtained by three students are in the proportion of 2:3:5. If first student obtained 20 marks, what are the marks of second and third students?

5. Find two numbers in the ratio of 7:12 so that the greater exceeds the lesser by 275.

6. Find the mean proportional between (i) 6 and 24 (ii) $360a^4$ and $250a^2b^2$.

7. If $x : y = 3 : 4$, find the value of $x^2y + xy^2 : x^3 + y^3$.

8. If $\dfrac{a}{3} = \dfrac{b}{4} = \dfrac{c}{7}$, prove that $\dfrac{a+b+c}{c} = 2$.

9. If $(a - b - c + d):(a + b + c + d) = (a + b - c - d):(a - b + c - d)$.

   Prove that $a : d = c : b$.

10. Two numbers are respectively 20% and 50% more than a third number. Find the ratio of the two numbers.

Check Your Progress: Model Answer

1. (c)

2. 4

3. True

4. True

1.10 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

LESSON

2

PERCENTAGE AND COMMISSION

2.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Learn the concept of percentage
- Explain how to convert fractions and decimals into percentages
- Analyse how to convert percentages into fractions and decimals
- Learn about the concept of commission
- Explain results on population
- Discuss results on depreciation
- Discuss how to prepare the invoice

2.1 INTRODUCTION

The word 'percentage' is very familiar to us as it is used regularly in the media to describe anything from changes in the interest rate, to the number of people taking holidays abroad, to the success rate of the latest medical procedures or exam results. Percentages are a useful way of making comparisons, apart from being used to calculate many taxes that we pay such as VAT, income tax, domestic fuel tax and insurance tax, to name but a few. So percentages are very much part of our lives. But what does percentage actually mean?

2.2 MEANING OF PER CENT

The word per cent comes from a Latin word *per centum* meaning 'per hundred', so per cent means hundredth or the ratio of a number to 100. The symbol % is called the per cent sign.

For example, $\dfrac{50}{100} = 50\%$; $\dfrac{25}{100} = 25\%$

Per cent is commonly used in connection with taxes, discount, commissions, profit and loss.

Example: An item is on sale at 30% discount. What is the discount if the original amount was 6750 shs? What is the sale price?

Solution:

The discount is 30% of 6750TSh = $\dfrac{30}{100} \times 6750\text{TSh} = 825\text{TSh}$

The sale price will be, 6750 – 825 = 5925TSh

2.3 CONVERTING FRACTIONS AND DECIMALS INTO PERCENTAGES

All fractions and decimals can be converted into percentages. We can do this either by,

● writing the fraction or decimal as a fraction with denominator 100, or

● multiplying by 100%, which is really just $\dfrac{100}{100}$ or 1

To convert to a percentage, we obtain a fraction with denominator.

2.4 CONVERTING PERCENTAGES INTO FRACTIONS AND DECIMALS

A percentage may be converted into a fraction by first writing the percentage with a denominator of 100, then expressing it in its lowest terms.

Express as fractions in lowest terms 115% such that,

$$\frac{115 \div 5}{100 \div 5} = \frac{23}{20}$$

Percentages may be converted into decimals by shifting the decimal point two places to the left. This is equivalent to dividing by 100%.

Remember that $\dfrac{x}{100} = x\%$

To change decimals to per cent and fractions and vice versa;

Examples:

- $0.67 = \dfrac{67}{100} = 67\%$

- $0.342 = \dfrac{342}{100} = 34.2\%$

- $0.005 = 0.5\%$

- $1.65 = 165\%$

- $0.0045 = 0.45\%$

- $\dfrac{3}{4} = 75\%$

- $\dfrac{5}{8} = 0.625 = 62.5\%$

Move two decimal places to the right and put %.

2.5 ONE QUANTITY AS A PERCENTAGE OF ANOTHER

We can compare quantities using percentages. To find one quantity as a percentage of another, we write the first as a fraction of the second, then multiply by 100%.

Example: Express as a percentage: Mike ran 10 km out of 50 km

$$10 \text{ km out of } 50 \text{ km} = \dfrac{10}{50} \times 100 = 20\%$$

If the price of a commodity increases by R%, then the reduction in consumption so as not to increase the expenditure is

$$\left[\left(\dfrac{R}{100+R}\right) \times 100\right]\%$$

Example: In the new budget, the price of kerosene oil rose by 25%. By how much per cent must a person reduce his consumption so that his expenditure on it does not increase?

Solution:

Reduction in consumption $= \left[\left(\dfrac{R}{100+R}\right) \times 100\right]\%$

$$= \left[\left(\dfrac{25}{125}\right) \times 100\right]\% = 20\%$$

If the price of the commodity decreases by R%, then the increase in consumption so as to decrease the expenditure is:

$$\left[\left(\dfrac{R}{100-R}\right) \times 100\right]\%$$

2.6 THE UNITARY METHOD IN PERCENTAGE

Sometimes we know a certain percentage of the whole amount. For example, Reeta knows that 16% of her wage is deducted for tax. Her payslip shows that ₹120 is taken out for tax. She wants to know her total income before tax. The unitary method can be used to solve such problems.

Example: Find 100% of a sum of money if 16% is ₹120.

Solution:

16% of the amount is ₹120

∴   1% of the amount is $= \dfrac{₹120}{16} = ₹7.50$

∴   100% of the amount is ₹7.50×100

∴   The amount is ₹750.

2.6.1 Finding a Percentage of a Quantity

To find a percentage of a quantity, convert the percentage to a fraction or a decimal and then multiply.

Example: Find 15% of 400 kg

Solution:

15% of 400 kg

$\quad = 0.15 \times 400$ kg

$\quad = 60$ kg

2.6.2 Results on Population

Let the population of the town be P now and suppose it increases at the rate of R% per annum, then:

1. Population after n years $= P\left[1+\left(\dfrac{R}{100}\right)\right]^n$

2. Population n years ago $= \dfrac{P}{\left[1+\left(\dfrac{R}{100}\right)\right]^n}$

Example: The population of a town is 1,76,400. If it increases at the rate of 5% per annum, what will be its population 2 years hence? What was it 2 years ago?

Solution:

Population after 2 years $= 1,76,400 \times \left[1+\left(\dfrac{5}{100}\right)\right]^2$

$\qquad\qquad = 1,76,400 \times \left(\dfrac{21}{20}\right) \times \left(\dfrac{21}{20}\right)$

$\qquad\qquad = 1,94,481$

Population 2 years ago $= \dfrac{1,76,400}{\left[1+\left(\dfrac{5}{100}\right)\right]^2}$

$= \left[1,76,400 \times \left(\dfrac{20}{21}\right) \times \dfrac{20}{21}\right] = 1,60,000$

2.6.3 Results on Depreciation

Let the present value of a machine be P. Suppose it depreciates at the rate $R\%$ per annum. Then,

Value of the machine after n years $= P\left[1-\left(\dfrac{R}{100}\right)\right]^n$

Example: The value of a machine depreciates at the rate of 10% per annum. If its present price is ₹1,62,000 what will be its worth after 2 years ? What was the value of the machine 2 years ago?

Solution:

Value of the machine after 2 years

$= ₹ \left[1,62,000 \times \left(1-\dfrac{10}{100}\right)^2\right]$

$= ₹ \left[1,62,000 \times \left(\dfrac{9}{10}\right) \times \left(\dfrac{9}{10}\right)\right]$

$= ₹ 1,31,220$

Value of the machine 2 years ago

$= ₹ \left[\dfrac{1,62,000}{\left(1-\dfrac{10}{100}\right)^2}\right] = ₹ \left(1,62,000 \times \dfrac{10}{9} \times \dfrac{10}{9}\right)$

$= 2,00,000$

2.7 INTRODUCTION TO COMMISSION

There can be several possible definitions for the term commission. It can be a specially organised group that may have been put together in order to solve some issues or look into a particular matter. In finance, the term usually means something quite different. It is a payment, often expressed in percentage, for some type of service that is frequently sales oriented.

Sales tax is the tax on the sale of an item or service. It is a per cent of the purchase price and is collected by the seller.

A number of workers receive part or all of their pay by earning a commission on their work. Stockbrokers, insurance brokers, salespeople, real estate agents, bankers, and many others may be entitled to a percentage of what they sell. This is sometimes the only salary a person receives or it may be a percentage of sales/deals, on top of a straight salary.

Frequently, percentage has a set amount, though this can vary by business. People who work only on commission tend to have higher per cent reimbursements than those who work with some form of underlying salary. A retail salesperson might earn a small per cent of sales, perhaps one to two per cent, while the real estate agent or insurance broker could earn a higher per cent. In certain instances commission percentages are evaluated by quality or quantity of merchandise. The auto salesperson might earn more for selling a new car than he would for selling a used one, or could have a higher commission reimbursement rate for selling a particular model of car during special sales events.

One common payment strategy is to assign people to work on a draw versus commission basis. When sales are low, the person receives a minimum salary, but as sales increase, they may receive part percentage of sales and part salary. Alternately they may greatly exceed draw and be paid only in percentage of sales. Occasionally people have the choice between draw with small percentages and sole commission with much larger percentages. Of course, being paid by percentage of sales only can be risky if demand or price sharply declines, but gifted salespeople or other agents may find this reimbursement strategy well worth it because of the extra money they make when they sell something.

A certain set of considerations applies for those employees who make commissions. Sales work typically means lean times and anyone in sales, investment or related businesses has to know how to handle salary when less salary is earned. Those who regularly live by percentage of sales are advised to create budgets that include putting part of their money away for low sales months. Most salespeople and agents know the inherent risk of making commissions instead of a salary, and celebrate their big sales by banking or investing some of their money for those periods where no commissions are earned.

For the buyer, it's sometimes useful to know what amount of money paid goes directly to a seller or agent's salary. Some people, for example, do not want to have real estate agent sell their homes because they know that agent gets to keep a certain amount of the proceeds. Companies usually need to be direct about the percentage of a sale employees are paid, and some businesses make this a selling point. A number of online stock companies now allow people to trade at a per trade fee that is clearly advertised or other businesses brag their employees are not on commission and thus will not subject customers to high pressure sales tactics.

2.7.1 Per Cent of Sales Structure

A commission sales structure utilizing the per cent of sales structure uses a per cent to calculate commission based on sales.

$$\text{Commission Rate} \times \text{Sales} = \text{Commission}$$

Example: If Jan's salesperson sold a ₹50,000 car and Jan agreed to pay him one per cent of the sales, the salesperson would make ₹500.

2.8 HOW TO CALCULATE COMMISSION?

While many employees are paid on an hourly or salaried basis, commissioned employees are paid based on the value of goods and services that they sell. Commission-based payment is common in certain positions – sales in particular where bringing in money is an important part of the job. To calculate commission, you need to understand what system your business uses and any additional factors that may affect your total commission earnings.

2.8.1 Understanding Commission Plan

Determine what your commission is based on: Typically, commissions are paid based on the purchase price of goods and services that you sold. However, some companies use different commission bases, like the item's net profit or the item's cost to the company.

- Ask if there are any products or services excluded from the commission plan.

- A company may pay you a commission for selling certain products and services but not others.

Determine the commission rate that your company pays: The commission payment rate could be, for example, 5 per cent of the selling price of all goods sold.

The company may set different commission rates for different types of products. For example, it may pay 6 per cent commission on a product that's hard to sell and only 4 per cent commission on a product that's easy to sell.

Understand any other nuances in your commission plan: For example, in some plans, your commission rate changes after you sell a certain amount of product.

In a tier-based commission system, for example, your commission rate might increase to 7 per cent after you sell ₹50,000 worth of product.

2.8.2 Calculating the Commission

Determine the commission period: Commission payments are usually made on a monthly or biweekly basis. For example, if you are paid every two weeks, your commission period might be January 1 to January 15. That means that you are only paid for sales made during January 1 to January 15.

Calculate your total commission base you made during the period: For example, if your paid based on the purchase price of products sold and you sold ₹30,000 from January 1 to January 15, your total commission base is ₹30,000.

- If you are paid a different commission rate for different products, calculate total commission base by product. For example, if sold an equal amount of two products but they have different commission rates, note that, you sold ₹15,000 of Product A and ₹15,000 of Product B.

- If you are paid a different commission rate after you sell a certain amount of product, calculate your commission base by tier. For example, if your commission rate increases after ₹25,000 of sales, your commission base is ₹25,000 for the first tier and ₹5,000 for the second tier.

Multiply your commission rate by your commission base for the period to calculate your commission payment: For example, if you made ₹30,000 worth of sales from January 1 to January 15 and your commission rate is 5 per cent, your commission payment is ₹1,500.

- If you are paid different rates for different products, multiply each commission base by the corresponding commission rate and add the resulting figures. For example, say you sold ₹15,000 of Product A at a 3 per cent commission rate and ₹15,000 of Product B at a 6 per cent commission rate. Your commission payment for Product A is ₹450, your commission payment for Product B is ₹900, and your total commission payment is ₹1,350.

- If commission rates vary, based on amount of product sold, multiply each commission base by the commission rate for that tier and add the resulting figures. For example, say you sold ₹30,000 and you are paid a rate of 4 per cent on the

first ₹25,000 and 6 per cent on the remainder. Your commission payment is ₹1,200 for the first tier, ₹300 for the second tier, and your total commission payment is ₹1,500.

2.8.3 Making Adjustments as Necessary

Take split commissions into account: Split commissions are when more than one salesperson is involved in a sale and they share the commission. Alternatively, a regional sales manager may receive part of the commission of the salespeople in their region.

Assess any additional bonus structures or related incentives: In addition to a straight percentage, a commission structure also can include any number of more complicated incentives for a sales person or other commission-earning individual.

Look for "top of the pack" bonuses: If you know that your commissions were the top-scoring numbers for a department or team, you may be able to calculate in eligibility for top-performance bonuses.

Example: Your family is selling your house for ₹160000. The realtor charges 6%. How much will the realtor's commission be?

Solution:

$$6\% \text{ of } ₹\ 160000 = \frac{6}{100} \times 160000$$
$$= 6 \times 1600$$

Realtor's commission = ₹9,600

Example: A real-estate agent is paid a monthly salary of ₹ 900 plus commission. Last month he sold one condominium for ₹ 65,000 earning a 4% commission on the sale. How much was his commission? What was his total pay last month?

Solution:

First find his commission:

$$4\% \times ₹65,000 = c$$
$$0.04 \times 65,000 = c \quad \text{Change the percent to a decimal}$$
$$2600 = c \quad \text{Solve for c.}$$

He earned a commission of ₹2600 on the sale.

Now find his total pay for last month.

₹2600 + ₹900 = ₹3500

His total pay for last month was ₹3500.

2.9 PREPARATION OF INVOICE

An invoice is a notice you send to customers or clients to notify them that payment is due for services you have performed. An invoice should outline what services you performed, how much the client owes and where they should send your payment. In this section, we will learn how to create an invoice template you can use each time you perform a service.

Method 1: Create a Header

Include your business name: This basic information about your company should appear in professional lettering at the top of the page. If you are using word processing

software, create a new document and select "header" from the drop down menu to create a header; otherwise, simply type your company's name and center it at the top of the page.

- If you do not have a business name, type your first and last name. You may also include your middle initial if desired.

- Use Arial, Times New Roman or another professional font for the heading. The font size should be bigger than the size you will use for the body of the invoice.

Provide your contact information: Type the address where you want to receive payment, your business phone number, and your business email address just under your business name. Your contact information should be centered and written in a smaller font than your business name.

If you use your personal phone number and email address as your business phone number and email address, include those.

To make your contact information appear more readable, consider typing it on several lines. For example,

2005 Rodeo Drive

Los Angeles, CA 90010

555-666-1234

junedesign@email.com

Consider including your business logo: If you have a logo, include it in the header to the left or right of your business name. If your business name is part of your business logo, you can use the logo in place of your name.

It is not necessary to include a company logo, though it can help lend your invoice a professional air.

Method 2: Include Recipient and Invoice Information

Write the recipient's contact information: The name, address and phone number of the business you are invoicing should appear on the left side of the invoice, below the header.

If you are invoicing a person, rather than a business, include that person's name and the address and phone number you have on file for that person.

If you do not have an address and phone number for the business or person you are invoicing, include an email address instead.

Write the invoice number and other invoice information: On the right side of the page, across from the recipient's contact information, include the details of your invoice. Write the following information on separate lines on the right side of the page:

- *Invoice number:* Type INVOICE: followed by the number. Since you are the one sending the invoice, you may choose the numbering system that works best for you. You can number your invoices numerically (1, 2, 3, 4) or by date (for example, 2010-10-16, for the year, month and day of the invoice). If you choose the latter, you can add "-01" to the end to differentiate between invoices sent on the same day. In any case, make sure your number and your invoices in sequential order so you do not confuse yourself.

- *Invoice date:* Even if you choose to number your invoice by date, you should clearly write the date on a separate line.

- *Invoice due date:* If you have an agreement with the business you are invoicing about when you can expect to get paid, write that date as the due date. If you do not have an agreement, figure out the standard pay-by period in your industry; in most cases, it's 30 to 45 days.

Specify your payment terms: Under the invoice information, note whether you will take cash, cheques, credit cards, PayPal or some other form of payment.

If you charge a late fee, quickly outline this as well, if you have not outlined it somewhere else already, like in a contract.

Include your tax identification number, if applicable. Laws vary by municipality, so be sure you know your local tax code if you are to collect sales tax or some other usage tax, like the hotel tax.

Method 3: Itemize the Services Rendered

Make a chart to itemize the services. If you are using word processing software, create a table in which you will list the different services you performed for your client, or the items that were purchased from you for which you are requesting payment. Your chart should contain the following components:

Six columns with the following headings:

1. *Services:* This is where you list the task you performed, such as copyediting, or the item that was purchased from you, such as handmade jewellery.

2. *Date:* Write the date the service was performed or the item was purchased.

3. *Quantity:* Write how many pages you copyedited, how many pieces of jewellery were purchased, etc.

4. *Rate:* Write the rate you are charging for the services or items. If you charge by the hour, write the rate as ₹00.00/hr.

5. *Hours:* If you charge by the hour, write how many hours were spent performing the service.

6. *Subtotal:* Write the total amount you are charging for the service performed.

One row for every separate service performed for the client, followed by a final row reserved for the total amount you are charging the client for all services.

Calculate the total: Subtotal the amount due and add any sales tax, delivery fees or other fees and to calculate the grand total. Use bolding or highlighting to make the total stand out from the rest of the numbers.

Provide additional information: If you are invoicing a customer who has purchased items for you, describe your return policy. You may also want to thank the customer for their business and offer other products or services at the bottom of the invoice. This is the last thing customers see, so in the interest of good relations, try to be kind and appreciative.

Check Your Progress

Fill in the blanks:

1. A percentage may be converted into a fraction by first writing the percentage with a _____ of 100, then expressing it in its lowest terms.

2. _____ is a payment, often expressed in percentage, for some type of service that is frequently sales oriented.

3. An _____ is a notice you send to customers or clients to notify them that payment is due for services you have performed.

4. An invoice should outline what services you performed, how much the client owes and where they should send your _____.

2.10 LET US SUM UP

- The per cent value is computed by multiplying the numeric value of the ratio by 100.

- Per cent is commonly used in connection with taxes, discount, commissions, profit and loss.

- All fractions and decimals can be converted into percentages. To convert to a percentage, we obtain a fraction with denominator.

- A percentage may be converted into a fraction by first writing the percentage with a denominator of 100, then expressing it in its lowest terms.

- To find one quantity as a percentage of another, we write the first as a fraction of the second, then multiply by 100%.

- To find a percentage of a quantity, convert the percentage to a fraction or a decimal and then multiply.

- Let the population of the town be P now and suppose it increases at the rate of R% per annum, then: Population after n years = $P[1+(R/100)]^n$. Population n years ago = $P/[1+(R/100)]^n$.

- Commission is a payment, often expressed in percentage, for some type of service that is frequently sales oriented.

- A commission sales structure utilizing the percent of sales structure uses a percent to calculate commission based on sales.

- To calculate commission, you need to understand what system your business uses and any additional factors that may affect your total commission earnings.

- An invoice is a notice you send to customers or clients to notify them that payment is due for services you have performed.

2.11 LESSON END ACTIVITY

In a decade, the population of a town increased from 1,75,000 to 2,62,500 so the average per cent increase of population per year is.............

2.12 KEYWORDS

Commission Rate: It is usually a per cent of the selling price.

Commission: It is a fee paid to a person who makes a sale.

Depreciation: It is a method of reallocating the cost of a tangible asset over its useful life span of it being in motion.

Invoice: It is a commercial document issued by a seller to a buyer, relating to a sale transaction and indicating the products, quantities, and agreed prices for products or services the seller had provided the buyer.

Percentage: It is a number or ratio expressed as a fraction of 100. It is often denoted using the per cent sign, "%".

2.13 QUESTIONS FOR DISCUSSION

1. How to convert fractions and decimals into percentages? Convert each of the following to a percentage: (a) $\frac{9}{20}$ (b) 3

2. Explain how to convert percentages into fractions and decimals? Express as fractions in lowest terms: (a) $12\frac{1}{2}\%$ (b) 125%

3. Describe one quantity as a percentage of another. Express as a percentage: Nikita spent 5 months of the last two years overseas.

4. How to calculate commission? What is the commission rate?

5. A manufacturer sells three products, i.e. A, B and C. Product A costs 100 and sells for 150, Product B costs 150 and sells for 180, Product C costs 100 and sells for 110. On which product, he has maximum percentage of profit?

6. A manufacturer wants to earn 20% profit on a table that cost ₹1000, what amount he should charge us?

7. By selling goods for ₹ 92, a merchant loses 8% on his outlay. What is percentage profit on his outlay when he sells the same goods for ₹ 110.

8. Rahul went to a shop and bought things worth ₹ 25, out of which 30 Paise went on sales tax on taxable purchases. If the tax rate was 6%, then the cost of the tax free items was?

9. The price of a car is ₹ 3,25,000. It was insured to 85% of its price. The car was damaged completely in an accident and the insurance company paid 90% of the insurance. The difference between the price of the car and the amount received was?

10. A furniture store earns 40% profit on all sales. If the store earns ₹5,680, how much are the total sales?

Check Your Progress: Model Answer

1. Denominator

2. Commission

3. Invoice

4. Payment

2.14 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

UNIT II

LESSON

3

SIMULTANEOUS EQUATIONS

3.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Define equation
- Explain linear equations
- Know about the meaning of a quadratic equation
- Discuss solution method of quadratic equation
- Solve quadratic equations by using different methods
- Explain the solution of a system of linear equations in two variables
- Discuss the characteristics of simultaneous equations
- List out the methods of solving simultaneous linear equation

3.1 INTRODUCTION

In the previous lesson, we have studied percentage and commission in which we learnt about the meaning of percentages and carry out calculations involving percentages.

Simultaneous equations are frequently used in solving the problems of business. A statement of equality containing one or more variable is known as an equation. Two or more equations are said to be equivalent if they have the same solution. In this lesson, we study the methods of solving linear and quadratic equations. In this lesson, we will also discuss four methods of solving quadratic equations that are factoring, completing the square, using the quadratic formula and graphical method.

3.2 INTRODUCTION TO EQUATIONS

In mathematics, an equation is a statement of an equality containing one or more variables. Solving the equation consists of determining which values of the variables make the equality true.

3.2.1 Meaning of Equation

A mathematical statement used to evaluate a value. An equation can use any combination of mathematical operations, including addition, subtraction, division, or multiplication. An equation can be already established due to the properties of numbers ($2 + 2 = 4$), or can be filled solely with variables which can be replaced with numerical values to get a resulting value.

For example, the equation to calculate return on sales is:

Net income ÷ Sales revenue = Return on Sales.

When the values for net income and sales revenue are plugged into the equation, you are able to calculate the value of return on sales.

An algebraic expression involving '=' sign and holding good for all values of the unknowns involved, is called an identity.

The "=" symbol, which appears in every equation, was invented in 1557 by Robert Recorde, who considered that nothing could be more equal than parallel straight lines with the same length.

Example: $x^2 + y^2 + 2xy = (x + y)^2$

$$x^2 - y^2 = (x - y)(x + y)$$

If the equality holds for a particular value or values of the unknowns involved then it is called equation.

Example: $2x = 8$ is true for only $x = 4$.

$x^2 + 8 = 9$ is true for $x = +1$ and $x = -1$

A value of the unknown x that satisfies the equation is called a root or solution of the equation.

An equation involving only one unknown quantity is called ordinary equation.

Example: $x + 9 = 10$

$x^2 - 4x = 8$ are ordinary equations. x is the only unknown.

3.2.2 Types of Equations

Equations can be classified according to the types of operations and quantities involved. Important types include:

- An algebraic equation or polynomial equation is an equation in which both sides are polynomials.

  These are further classified by degree:

  - ❖ Linear equation for degree one
  - ❖ Quadratic equation for degree two
  - ❖ Cubic equation for degree three
  - ❖ Quartic equation for degree four
  - ❖ Quintic equation for degree five
  - ❖ Sextic equation for degree six
  - ❖ Septic equation for degree seven

- A diophantine equation is an equation where the unknowns are required to be integers.

- A transcendental equation is an equation involving a transcendental function of its unknowns.

- A parametric equation is an equation for which the solutions are sought as functions of some other variables, called parameters appearing in the equations.

- A functional equation is an equation in which the unknowns are functions rather than simple quantities.

- A differential equation is a functional equation involving derivatives of the unknown functions.

- An integral equation is a functional equation involving the anti-derivatives of the unknown functions.

- An integro-differential equation is a functional equation involving both the derivatives and the anti-derivatives of the unknown functions.

- A difference equation is an equation where the unknown is a function 'f' which occurs in the equation through $f(x)$, $f(x-1)$, ..., $f(x-k)$, for some whole integer 'k' called the order of the equation. If x is restricted to be an integer, a difference equation is the same as a recurrence relation.

3.3 LINEAR EQUATIONS

An equation of the form $ax + b = 0$ where $a \neq 0$, b are constants, is called a simple or linear equation in the variable x.

3.3.1 Solutions of Linear Equations

A value of x which satisfies the given equation is called a solution or a root.

Fundamental theorem: An equation of degree n has exactly n roots.

Hence a linear equation has exactly one root and a quadratic equation has exactly two roots.

The linear equation $ax + b = 0$ has exactly one root $x = \dfrac{-b}{a}$ (where $a \neq 0$)

Keep in mind by adding or subtracting the same quantity to both sides of an equation. This demonstrates that every equation is equivalent to an equation in which the right-hand side is zero.

Solve the following equations:

Example: $2x - 1 = 0$

Solution:

$$2x - 1 = 0$$

Here $a = 2,\ b = -1$

$\therefore$ Solution is $x = \dfrac{-b}{a} = \dfrac{-(-1)}{2}$

$\Rightarrow$ $x = \dfrac{1}{2}$ is the solution

Example: $5x - 3 = 0$

Solution:

$$5x - 3 = 0$$

Here $a = 5,\ b = -3$

$\therefore$ Solution is $x = \dfrac{-b}{a} = \dfrac{-(-3)}{5}$

$\Rightarrow$ $x = \dfrac{3}{5}$ is the solution

Example: $3x + 11 = 0$

Solution:

$$3x + 11 = 0$$
Here $a = 3, b = 11$

$\therefore$ Solution is $x = \dfrac{-b}{a} = \dfrac{-11}{3}$

$\Rightarrow$ $x = \dfrac{-11}{3}$ is the solution

Example: $7x = 0$

Solution:

$$7x = 0$$

$\Rightarrow$ $7x + 0 = 0$

Here $a = 7, b = 0$

$\therefore$ $x = \dfrac{-b}{a} = -\dfrac{0}{7} = 0$

$\therefore$ $x = 0$ is the solution

3.4 QUADRATIC EQUATION

An ordinary equation having highest power of unknown = 2 is called quadratic equation or we can say an equation of the form $ax^2 + bx + c = 0$ where $a \neq 0, b, c$ are constants is called a quadratic equation in the variable x.

Example:

(i) $3x^2 + 4x + 1 = 0$

(ii) $4x^2 - 2 = 0$

(iii) $x^2 + 8x - 1 = 0$

(iv) $7x^2 + 8 = 0$

In the above example, (ii) and (iv) are free from x term. Such an equation is called pure quadratic equation.

Because the quadratic equation involves only one unknown, it is called "univariate". The quadratic equation only contains powers of x that are non-negative integers, and therefore it is a polynomial equation, and in specific it is a second degree polynomial equation since the greatest power is two.

There are four methods of solving quadratic equations:

● The first method is Factoring.

  ❖ This is a simple method, but does not always work, because we cannot always factor a given equation.

● The second method is Completing the Square.

  ❖ This method always works, but it may involve more steps than the other methods.

● The third method is using the Quadratic Formula.

  ❖ This method always works, is simple to use, and has the advantage that it tells us easily by means of discriminant, what type of and how many solutions does the equation have.

● The fourth method is using graph i.e., Graphical Method.

We will discuss these four methods to calculate quadratic equation in detail.

3.4.1 Solutions of Quadratic Equations

The quadratic equation $ax^2 + bx + c = 0$ has exactly two roots. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

i.e., $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

These roots can also be found by factorisation.

Note: The expression $b^2 - 4ac$ is called the discriminant and is denoted by Δ.

If, $b^2 - 4ac = 0$ then the roots are rational and equal i.e., $-\dfrac{b}{2a}$.

If, $b^2 - 4ac \neq 0$ and positive, then the roots are real and distinct.

If, $b^2 - 4ac \neq 0$ and negative, then the roots are imaginary and distinct.

3.4.2 Solving Quadratic Equations by Factoring

This method relies on the fact that if the product of two real numbers is 0, then one or both of them must be 0.

Solve the following equations by the method of factoring:

Example 1: $6x^2 - 3x - 3 = 0$

Solution:

The first step is to factor the left hand side of the equation as follows:

$$6x^2 - 3x - 3 = 3(2x + 1)(x - 1) = 0$$

$$3(2x + 1)(x - 1) = 0$$

Dividing both sides by 3, we get

$$(2x + 1)(x - 1) = 0$$

For a real number to satisfy this equation, it must make either,

$$2x - 1 = 0 \text{ or } (x - 1) = 0$$

$$2x - 1 = 0 \text{ or } x - 1 = 0$$

$$2x = 1 \text{ or } x = 1$$

$$x = \frac{1}{2}, x = 1$$

So the solution of the original equation is $x = \dfrac{1}{2}$ or $x = 1$

Example 2: $(x + 6)(x + 2) = -4$

Solution:

In order to use the method of factoring to solve a quadratic equation, one side of the equation must be 0. So the first step in this case will be to make the equation in the form $ax^2 + bx + c = 0$

$$(x + 6)(x + 2) = -4$$

$$x^2 + 2x + 6x + 12 = -4$$

$$x^2 + 8x + 12 = -4$$

$$x^2 + 8x + 12 + 4 = 0$$

$$x^2 + 8x + 16 = 0$$

After expressing the equation in the quadratic form, we move ahead to factorize it:

$$x^2 + 8x + 16 = 0$$

$$x^2 + 4x + 4x + 16 = 0$$

$$(x + 4)(x + 4) = 0$$

$$x + 4 = 0 \quad \text{or } x + 4 = 0$$

So the solution of the original equation is $x = -4$

(Note that in this case the solution is one real number)

3.4.3 Solving Quadratic Equations by Completing the Square

This method involves the concept of a perfect square. A quadratic equation is said to be a perfect square if it is a square of a linear equation.

"Completing the Square" is where we...

...take a Quadratic Equation like this:

$$ax^2 + bx + c = 0$$

$\longrightarrow$

and turn it into this:

$$a(x + d)^2 + e = 0$$

For those of you in a hurry, we can tell you that:

D=b/2a

And

$e = c - b^2/4a$

But if you have time, let me show you how to "Complete the Square" yourself.

Say we have a simple expression like $x^2 + bx$. Having x twice in the same expression can make life hard. What can we do?

Well, with a little inspiration from Geometry we can convert it, like this:
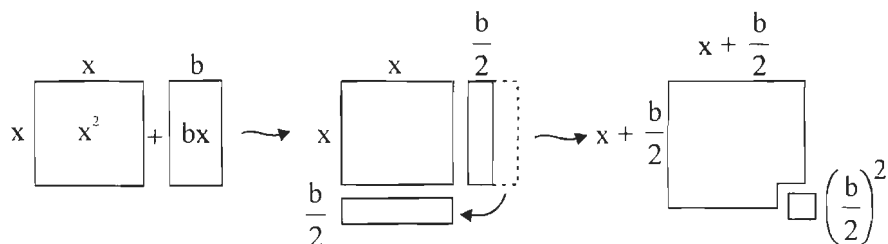
As you can see $x^2 + bx$ can be rearranged nearly into a square and we can complete the square with $(b/2)^2$

In Algebra it looks like this:

$x^2 + bx + (b/2)^2 = (x+b/2)^2$ ("Complete the Square")

So, by adding $(b/2)^2$ we can complete the square.

And $(x+b/2)^2$ has x only once, which is easier to use.

Keeping the Balance

Now ... we can't just add $(b/2)^2$ without also subtracting it too. Otherwise the whole value changes.

So let's see how to do it properly with an example:

Start with: $x^2 + 6x + 7$

("b" is 6 in this case)

Complete the Square:

$$+ \left(\frac{6}{2} \right)^2 \qquad - \left(\frac{6}{2} \right)^2$$

Also subtract the new term

Simplify it and we are done.

$$x^2 + 6x + \underbrace{\left(\frac{6}{2}\right)^2} + \underbrace{7 - \left(\frac{6}{2}\right)^2}$$

$$\left(x + \frac{6}{2}\right)^2 \quad + \quad 7 - 9 \quad = \left(x + 3\right)^2 - 2$$

The result:

$$x^2 + 6x + 7 = (x+3)^2 - 2$$

And now x only appears once.

A Shortcut Approach

Here is a quick way to get an answer you may like.

First think about the result we want: $(x+d)^2 + e$

After expanding $(x+d)^2$ that becomes: $x^2 + 2dx + d^2 + e$

Now see if we can turn our example into $x^2 + 2dx + d^2 + e$ form to discover d and e.

Solve the following equations by completing the square:

Example 1: $x^2 - 6x + 2 = 0$

Solution:

We first write the equation in the form $x^2 + bx$, it becomes

$$x^2 - 6x = -2$$

To make the left side of the equation a perfect square, we add the square of half the coefficient of x to both sides.

In this case, the coefficient of x, i.e., $b = -6$, so we add $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$

Adding 9 to both sides, we have

$$x^2 - 6x + 9 = -2 + 9 \quad [\text{but } x^2 - 6x + 9 \text{ is the same as } (x-3)^2]$$

$$(x-3)^2 = 7$$

The equation is now in the form $(x + d) = k$. Solving by using square root property, we get

$$x - 3 = \pm \sqrt{7}$$

$$x = 3 + \sqrt{7} \text{ or } x = 3 - \sqrt{7}$$

Example 2: $x^2 + 5x + 6 = 0$

Solution:

Writing the equation in the form $x^2 + bx$, we have

$$x^2 + 5x = -6$$

In this case, $b = 5$, so we add $\left(\frac{5}{2}\right)^2 = \left(\frac{25}{4}\right)$ to both sides of the equation

$$x^2 + 5x + \left(\frac{25}{4}\right) = -6 + \left(\frac{25}{4}\right) \quad \left[\text{but } x^2 + 5x + \left(\frac{25}{4}\right) = \left(x + \frac{5}{2}\right)^2\right]$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{-24 + 25}{4}$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{1}{4}$$

$$x + \frac{5}{2} = \sqrt{\frac{1}{4}}$$

$$x = \frac{-5}{2} + \frac{1}{2} \text{ or } x = \frac{-5}{2} - \frac{1}{2}$$

$$x = -2 \text{ or } x = -3$$

Example 3: $x^2 - 6x - 16 = 0$

Solution:

$x^2 - 6x = 16$

In this case, $b = -6$ so we add $\left(\frac{-6}{2}\right)^2 = 9$ to both sides of the equation:

$$x^2 - 6x + 9 = 16 + 9$$

$$x^2 - 6x + 9 = 25 \text{ (but } x^2 - 6x + 9 = (x - 3)^2)$$

$$(x - 3)^2 = 25$$

$$x - 3 = \pm\sqrt{25}$$

$$x - 3 = \pm 5$$

$$x = 3 + 5 \text{ or } x = 3 - 5$$

$$x = 8 \text{ or } x = -2$$

3.4.4 Solving Quadratic Equations by using the Quadratic Formula

This is the general formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ known as the **quadratic formula**.

This formula gives solutions to the general quadratic equation $ax^2 + bx + c = 0$, expressed in the coefficients a, b and c. We can find the solutions to any specific quadratic equation simply by substituting the appropriate values of a, b and c.

Note: The solutions to a quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by a quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If, $b^2 - 4ac > 0$, then the equation has two distinct solutions.

If, $b^2 - 4ac = 0$, then the equation has exactly one solution.

If, $b^2 - 4ac < 0$, then the equation has no real roots i.e., has complex roots.

Solve the following quadratic equations by using the formula:

Example 1: $3x^2 - 4x + 1 = 0$

Solution:

In this case, $a = 3$, $b = -4$ and $c = 1$

Substituting these values into the formula, we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{4 \pm \sqrt{16 - 12}}{6}$$

$$x = \frac{4 \pm \sqrt{4}}{6}$$

$$x = \frac{4 \pm 2}{6}$$

$$x = \frac{6}{6} \text{ or } x = \frac{2}{6}$$

$$x = 1 \text{ or } x = \frac{1}{3}$$

Example 2: $x(x + 6) = 2x - 6$

Solution:

The first step in this problem is to put the equation in the quadratic form.

$$x(x + 6) = 2x - 6$$
$$x^2 + 6x = 2x - 6$$
$$x^2 + 6x - 2x + 6 = 0$$
$$x^2 + 4x + 6 = 0$$

From this quadratic form, $a = 1$, $b = 4$ and $c = 6$

We substitute these values into the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 24}}{2}$$

$$x = \frac{-4 \pm \sqrt{-8}}{2}$$

At this point, we observe that the number under the square root sign is negative. Since we cannot take the square root of a negative number, we conclude that this quadratic equation has no real solution.

3.4.5 Graphing Quadratic Equations

If we plotted the graph of the linear equation, $y = 2x + 4$. The same process is applied with graphic quadratic equations. The difference here is that instead of obtaining a straight line, with quadratic equations we get parabola shapes.

In solving quadratic equations graphically, we will first draw the graph of the equation and then use it to obtain the solution to the equation.

Example: Solve the quadratic equation $x^2 - 5x + 4 = 0$ by using graphical method.

Solution:

Select some values of x and calculate the corresponding values of y

For $x = -2$,

Substituting -2 in the equation $y = x^2 - 5x + 4$, we get

$$(-2)^2 - 5(-2) + 4$$
$$= 4 + 10 + 4$$
$$= 18$$

Thereby obtaining a point $(-2, 18)$

Likewise, substituting -1 in the equation $y = x^2 - 5x + 4$, we get

$$(-1)^2 - 5(-1) + 4$$
$$= 1 + 5 + 4$$
$$= 10$$

Thereby obtaining a point $(-1, 10)$

And also substituting 0 in the equation, $y = x^2 - 5x + 4$, we get

$$(0)^2 - 5(0) + 4$$
$$= 0 + 0 + 4$$
$$= 4$$

Thereby obtaining a point $(0, 4)$

We do the same procedure for the rest of the x values and get our table of values as;

x	-2	-1	0	1	2	3	4	5	6
y	18	10	4	0	-2	-2	0	4	10

The next step is to plot the graph using the obtained table of values:
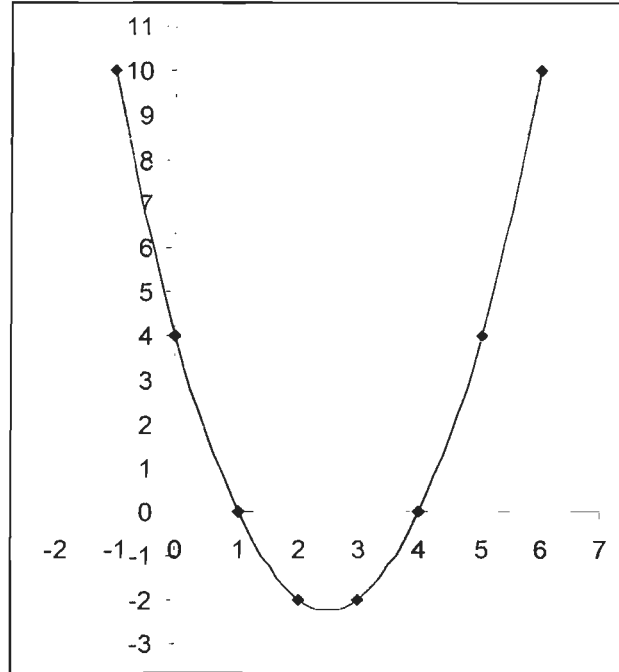
The solution of the quadratic equation. $x^2 - 5x + 4 = 0$ can be obtained from the graph, $y = x^2 - 5x + 4$ at the point where $y = 0$. In this case, $y = 0$ when $x = 1$ and when $x = 0$.

Therefore the solution of our equation is $x = 1$ or $x = 4$.

Solve the quadratic equation in the above example by factoring, and compare the result.

Example: Without drawing, find the points where the line $y = -2x - 4$ touches the graph.

$$y = x^2 + 3x - 10$$

Solution:

Let
$$y = x^2 + 3x - 10 \qquad \qquad ...(i)$$

$$y = -2x - 4 \qquad \qquad ...(ii)$$

To solve the two equations simultaneously, substitute equation (i) into equation (ii), this means we replace y by $-2x - 4$

We get,

$$-2x - 4 = x^2 + 3x - 10$$

$$x^2 + 3x - 10 + 2x + 4 = 0 \qquad \text{(combine like terms)}$$

$$x^2 + 5x - 6 = 0$$

We have ended up with a quadratic equation, which we can solve by using any method:

By factoring:

$$x^2 + 5x - 6 = 0$$

$$x^2 - x + 6x - 6 = 0$$

$$(x - 1)(x + 6) = 0$$

$x = 1$ or $x = -6$ then for the two values of x find the corresponding values of y

Find y when $x = 1$

$y = -2x - 4$

$y = -2(1) - 4$

$y = -2 - 4$

$y = -6$

Find y when $x = -6$

$y = -2x - 4$

$y = -2(-6) - 4$

$y = 12 - 4$

$y = 8$

So there are two points of intersection between the graph $y = x^2 + 3x - 10$ and $y = -2x - 4$ which are $(1, -6)$ and $(-6, 8)$.

Solve the quadratic equation:

Example: $x^2 - 5x + 6 = 0$

Solution:

$x^2 - 5x + 6 = 0$

(a) By using the formula:

$$a = 1, b = -5, c = 6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow$ $$x = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$\Rightarrow$ $$x = \frac{5 \pm \sqrt{1}}{2}$$

$\Rightarrow$ $$x = \frac{5+1}{2} \text{ and } \frac{5-1}{2}$$

i.e., $x = 3, x = 2$ are the solutions.

(b) By factorisation

$$x^2 - 5x + 6 = 0$$

$\Rightarrow$ $x^2 - 2x - 3x + 6 = 0$

$\Rightarrow$ $x(x - 2) - 3(x - 2) = 0$

$\Rightarrow$ $(x - 2)(x - 3) = 0$

$\Rightarrow$ $x - 2 = 0, x - 3 = 0$

$\Rightarrow$ $x = 2, x = 3$ are the solutions.

Example: $4x^2 - 1 = 0$

Solution:

$4x^2 - 1 = 0$

(a) By using the formula

$$a = 4, b = 0, c = -1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow$
$$x = \frac{-0 \pm \sqrt{0 + 16}}{2 \times 4}$$

$\Rightarrow$
$$x = \pm \frac{4}{2 \times 4}$$

$\Rightarrow$
$$x = \frac{1}{2}, \frac{-1}{2} \text{ are the solutions.}$$

(b) By factorisation

$$4x^2 - 1 = 0$$

$\Rightarrow$
$$(2x)^2 - 1^2 = 0$$

$\Rightarrow$
$$(2x + 1)(2x - 1) = 0 \text{ [using } a^2 - b^2 = (a + b)(a - b)]$$

$\Rightarrow$
$$2x + 1 = 0, 2x - 1 = 0$$

$\Rightarrow$
$$x = \frac{-1}{2}, x = \frac{1}{2} \text{ are the solutions.}$$

Example: $6x^2 - 5x + 1 = 0$

Solution:

(a) By using the formula

$$a = 6, b = -5, c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow$
$$x = \frac{5 \pm \sqrt{25 - 24}}{12}$$

$\Rightarrow$
$$x = \frac{5 \pm 1}{12}$$

$\Rightarrow$
$$x = \frac{5 + 1}{12}, \frac{5 - 1}{12}$$

$\Rightarrow$
$$x = \frac{6}{12}, \frac{4}{12}$$

$\Rightarrow$
$$x = \frac{1}{2}, \frac{1}{3}$$

(b) By factorisation

$$6x^2 - 5x + 1 = 0$$

$\Rightarrow$
$$6x^2 - 2x - 3x + 1 = 0$$

$\Rightarrow$
$$2x(3x - 1) - 1(3x - 1) = 0$$

$\Rightarrow$
$$(3x - 1)(2x - 1) = 0$$

$\Rightarrow$
$$3x - 1 = 0, 2x - 1 = 0$$

$$\Rightarrow \qquad 3x = 1, 2x = 1$$

$$\Rightarrow \qquad x = \frac{1}{3}, x = \frac{1}{2}$$

Example: $x^2 + 6x + 8 = 0$

Solution:

(a) By using the formula

$$a = 1, b = 6, c = 8$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-6 \pm \sqrt{36 - 32}}{2}$$

$$\Rightarrow \qquad x = \frac{-6 \pm \sqrt{4}}{2}$$

$$\Rightarrow \qquad x = \frac{-6 \pm 2}{2}$$

$$\Rightarrow \qquad x = \frac{-6 + 2}{2}, \frac{-6 - 2}{2}$$

$$\Rightarrow \qquad x = -2, -4 \text{ are the solutions.}$$

(b) By factorisation

$$x^2 + 6x + 8 = 0$$

$$\Rightarrow \qquad x^2 + 2x + 4x + 8 = 0$$

$$\Rightarrow \qquad x(x + 2) + 4(x + 2) = 0$$

$$\Rightarrow \qquad (x + 2)(x + 4) = 0$$

$$\Rightarrow \qquad x + 2 = 0, x + 4 = 0$$

$$\Rightarrow \qquad x = -2, x = -4$$

Example: $x^2 + x + 1 = 0$

Solution:

(a) By using the formula

$$a = 1, b = 1, c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore \qquad x = \frac{-1+\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}$$

Example: $2x^2 + x - 1 = 0$

Solution:

(a) By using the formula

$$2x^2 + x - 1 = 0$$

$$a = 2, b = 1, c = -1$$

$$\therefore \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{1 + 8}}{2 \times 2}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{9}}{4}$$

$$\Rightarrow \qquad x = \frac{-1 \pm 3}{4}$$

$$\Rightarrow \qquad x = \frac{-1+3}{4}, \frac{-1-3}{4}$$

$$\Rightarrow \qquad x = \frac{1}{2}, -1$$

(b) By factorisation

$$2x^2 + x - 1 = 0$$

$$\Rightarrow \qquad 2x^2 + 2x - x - 1 = 0$$

$$\Rightarrow \qquad 2x(x+1) - 1(x+1) = 0$$

$$\Rightarrow \qquad (x+1)(2x-1) = 0$$

$$\Rightarrow \qquad x + 1 = 0, 2x - 1 = 0$$

$$\Rightarrow \qquad x = -1, x = \frac{1}{2}.$$

Example: $x^2 + x - 1 = 0$

Solution:

$$x^2 + x - 1 = 0$$

$$a = 1, b = 1, c = -1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{1 + 4}}{2 \times 1}$$

$$\Rightarrow \qquad x = \frac{-1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \qquad x = \frac{-1 + \sqrt{5}}{2}, \frac{-1 - \sqrt{5}}{2}$$

Example: Solve the equation by formula method $12x^2 + 23x = 24$.

Solution:

$$12x^2 + 23x = 24$$

$$\Rightarrow \qquad 12x^2 + 23x - 24 = 0$$

Here $\qquad a = 12, b = 23, c = -24$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-23 \pm \sqrt{23^2 - 4(12)(-24)}}{2 \times 12}$$

$$= \frac{-23 \pm \sqrt{529 + 1152}}{24}$$

$$= \frac{-23 \pm \sqrt{1681}}{24}$$

$$= \frac{-23 \pm 41}{24}$$

i.e., $\qquad x = \dfrac{-23 + 41}{24}$ and $x = \dfrac{-23 - 41}{24}$

i.e., $\qquad x = \dfrac{18}{24}$ and $x = \dfrac{-64}{24}$

i.e., $\qquad x = \dfrac{3}{4}$ and $x = \dfrac{-8}{3}$.

Example: Solve the equation

$$\frac{2}{x-4} + \frac{5}{x-3} = \frac{7}{x-1}$$

Solution:

Multiply the given equation by the LCM

$\qquad (x-4)(x-3)(x-1)$, we get

$\qquad 2(x-3)(x-1) + 5(x-4)(x-1) = 7(x-4)(x-3)$

$\Rightarrow \qquad 2x^2 - 8x + 6 + 5x^2 - 25x + 20 = 7x^2 - 49x + 84$

$\Rightarrow \qquad 7x^2 - 33x + 26 = 7x^2 - 49x + 84$

$\Rightarrow \qquad 7x^2 - 33x + 26 - 7x^2 + 49x - 84 = 0$

$\Rightarrow \qquad 16x - 58 = 0$

$$\Rightarrow \qquad x = \frac{58}{16}$$

$$\Rightarrow \qquad x = \frac{29}{8}$$

Example: Solve the equation

$$\frac{a}{x} - \frac{a}{b} = \frac{b}{x} - \frac{b}{a} \quad \text{where } a \neq b$$

Solution:

Multiply the given equation by the LCM:

abx, we get

$$a^2 b - a^2 x = ab^2 - b^2 x$$

$$\Rightarrow \qquad a^2 x - b^2 x = a^2 b - ab^2$$

$$\Rightarrow \qquad (a^2 - b^2) x = ab(a - b)$$

$$\Rightarrow \qquad (a+b)(a-b)x = ab(a-b)$$

$$\Rightarrow \qquad x = \frac{ab(a-b)}{(a+b)(a-b)}$$

$$\Rightarrow \qquad x = \frac{ab}{a+b}$$

Example: Solve the equation

$$\frac{x}{5} - \frac{2x}{7} = \frac{1}{3}$$

Solution:

Multiply the given equation by the LCM $5 \times 7 \times 3$

We get $\qquad 21x - 30x = 35$

$$\Rightarrow \qquad -9x = 35$$

$$\Rightarrow \qquad x = -\frac{35}{9}$$

Example: $0.8x - 0.7 + 1.7x = 2.75 - 4.7x$

Solution:

Rearranging the equation, we get

$$0.8x + 1.7x + 4.7x = 2.75 + 0.7$$

$$\Rightarrow \qquad 7.2x = 3.45$$

$$\Rightarrow \qquad x = \frac{3.45}{7.2}$$

$$\Rightarrow \qquad x = \frac{345}{720}$$

$$x = \frac{69}{144} = \frac{23}{48} = 0.479$$

∴ $\qquad\qquad x = 0.479$

Example: Find two numbers whose sum is 58 and difference is 10.

Solution:

Let x be one of the numbers.

∴ The other number has to be $58 - x$ as the sum of two numbers has to be 58.

The difference is given to be 10.

∴ $\qquad\qquad x - (58 - x) = 10$

⇒ $\qquad\qquad x - 58 + x = 10$

⇒ $\qquad\qquad 2x = 58 + 10$

⇒ $\qquad\qquad 2x = 68$

⇒ $\qquad\qquad x = 34$

and the other number is $58 - x = 58 - 34 = 24$.

∴ The numbers are 34 and 24.

3.5 SIMULTANEOUS LINEAR EQUATIONS

If two linear equations in two variables are satisfied simultaneously by the same values of the variables, then they are called simultaneous linear equations in two variables.

A system of equations

$$a_1 x + b_1 y = c_1$$

$a_2 x + b_2 y = c_2$ where a_1, b_1, a_2, b_2, c_1, c_2 are all constants is called general form or Standard form of simultaneous equations in 2 variables x and y. We get a unique value for x and y as the solution to such a system.

3.5.1 Characteristics of Simultaneous Equations

● Simultaneous Equation involves two variables, usually in x and y.

● These equations involve two equations: one linear and the other non-linear.

3.5.2 Types of Methods to Solve Simultaneous Linear Equation

In this section, we will discuss certain types of methods which help to solve simultaneous equation.

Comparison Method: In this method, one of the variables is expressed in terms of the other from both the equations. Then by comparing we get a linear equation in one variable which can be solved. Then by substituting this value of one variable in any of the given equations, we get the value of the other variable.

Example: To solve:

$$x + 2y = 7$$

$$2x - y = 4$$

$$x + 2y = 7$$

$$x = 7 - 2y \quad\quad \text{...(1)}$$

$$2x - y = 4$$

$$2x = 4 + y$$

$$x = 4 + y/2 \quad\quad \text{...(2)}$$

From (1) and (2)

$$7 - 2y = 4 + y/2$$

$$2(7 - 2y) = 4 + y$$

$$14 - 4y = 4 + y$$

$$14 - 4 = 4y + y$$

$$10 = 5y$$

$$y = 2$$

Substituting $y = 2$ in (1)

$$x = 7 - 2(2)$$

$$x = 7 - 4$$

$$x = 3$$

Substitution Method: In this method, one of the variables is expressed in terms of the other from one of the given equations. By substituting this in the remaining equation we get a linear equation which can be solved. Again putting this value in any one of the equations, we get the value of the remaining variable.

Example: To solve:

$$x + 2y = 7$$

Consider

$$2x - y = 4$$

$$2x = 4 + y$$

$$2x - 4 = y.$$

Substituting $y = 2x - 4$ in $x + 2y = 7$

$$x + 2(2x - 4) = 7$$

$$x + 4x - 8 = 7$$

$$5x - 8 = 7$$

$$5x = 7 + 8$$

$$5x = 15$$

$$x = 3$$

Substituting $x = 3$ in $y = 2x - 4$

$$y = 2(3) - 4$$

$$y = 6 - 4$$

$$y = 2$$

Elimination Method: In this method, the given equations are multiplied by suitable constants, so that the co-efficient of one of the variables in both the equations become numerically equal. Then by adding/subtracting the transformed equations we get a linear equation which can be solved. By substituting, we get the value of another variable.

Example: To solve:

$$x + 2y = 7 \qquad \text{...(1)}$$

$$2x - y = 4 \qquad \text{...(2)}$$

Equation 1: $x + 2y = 7$

Equation 2 × 2 = $\underline{4x - 2y = 8}$

Adding $5x = 15$

$$x = 3$$

Substituting $x = 3$ in (1), we get

$$x + 2y = 7$$

$$3 + 2y = 7$$

$$2y = 7 - 3$$

$$2y = 4$$

$$y = 2$$

Rule of Cross Multiplication (RCM)

If $a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$, is the system. Then,
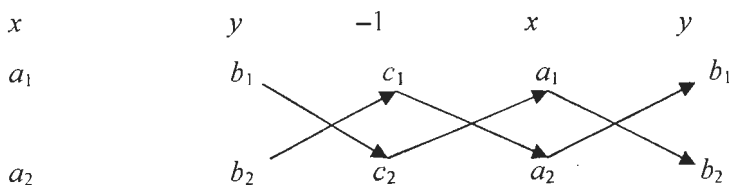
$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{-1}{a_1b_2 - b_1a_2}$$

Hence,

$$x = -\frac{\left(b_1c_2 - b_1c_2\right)}{a_1b_2 - b_1a_2}$$

$$y = -\frac{(c_1a_2 - c_2a_1)}{a_1b_2 - b_1a_2}$$

Example: To solve,

$$x + 2y = 7$$

$$2x - y = 4$$

$$\begin{array}{ccccc} x & y & -1 & x & y \\ 1 & 2 & 7 & 1 & 2 \\ 2 & -1 & 4 & 2 & -1 \end{array}$$
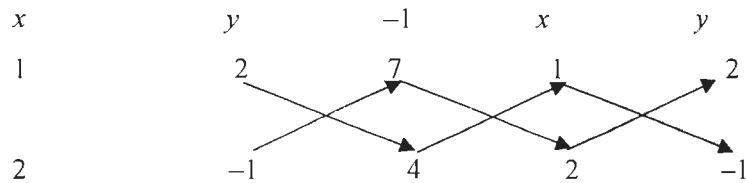
$$\frac{x}{2(4)-7(-1)}=\frac{y}{7(2)-4(1)}=\frac{-1}{1(-1)-2(2)}$$

$$\frac{x}{8+7}=\frac{y}{14-4}=\frac{1}{1+4}$$

$$\frac{x}{15}=\frac{y}{10}=\frac{1}{5}$$

$$x=\frac{15}{5} \text{ and } y=\frac{10}{5}$$

$$x=3 \text{ and } y=2$$

Solving Graphically

Simultaneous equations can also be solved graphically. If the graphs of each linear equation are drawn, then the solution to the system of simultaneous equations is the coordinate of the point at which the two graphs intersect.

For example:

$$xy = 2y \qquad \qquad ...(1)$$

$$2x - 3 \qquad \qquad ...(2)$$

The graphs of the two equations are shown below.
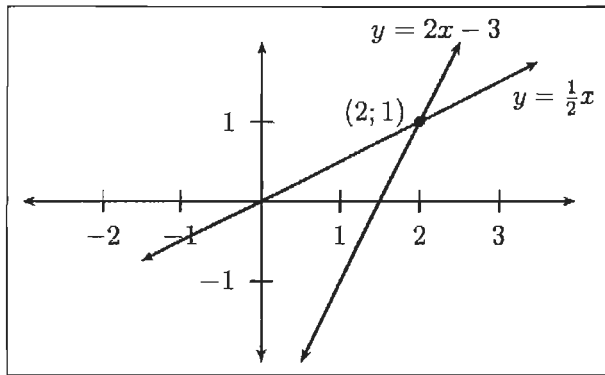
Figure 3.1: Graph of given Equations

The intersection of the two graphs is (2;1). So the solution to the system of simultaneous equations is.......... $x = 2$ and $y = 1$.

We can also check the solution using algebraic methods.

Substitute equation (1) into equation (2):

$$x = 2y$$

$$\therefore \qquad y - 2(2y) - 3$$

Then solve for y:

$$y - 4y = -3$$

$$-3y = -3$$

$$\therefore \quad\quad y = 1$$

Substitute the value of y back into equation (1):

$$x = 2(1)$$

$$\therefore \quad\quad x = 2$$

Notice that both methods give the same solution.

3.6 PROBLEMS ON COMMERCIAL APPLICATION

Example: A number decreased by 13 is same as 8 decreased by 2 times the number. Find the number.

Solution:

Let the number be x.

A number decreased by $13 = x - 13$. ...(1)

8 decreased by 2 times the number $= 8 - 2x$. ...(2)

Given (1) = (2)

$$x - 13 = 8 - 2x$$

$$x + 2x = 8 + 13$$

$$3x = 21$$

$$x = 7$$

So the required number is 7.

Example: Raju's Age is 3 times Abhijith's age. If their total age is 92 years. Find their ages.

Solution:

Let Abhijith's Age be x.

Then Raju's age $= 3x$

Given total age $= 92$

$$x + 3x = 92$$

$$4x = 92$$

$$x = 92/4 = 23$$

Abhijith's Age = 23 years and Raju's Age = 23 × 3 = 69 years.

Example: A man is paid for each day ₹ 22 if he works and forfeits for each day ₹ 5 if he is idle. If at the end of 45 days, he gets ₹ 747. How many days was he idle?

Solution:

Let 'x' be the number of days idle. Then $(45 - x)$ days a man works.

For one working day he is paid ₹ 22.

For $(45 - x)$ working days he gets $22 (45 - x)$ ...(1)

For one day idle, he loses ₹ 5

For x days he loses ₹ $5x$ ...(2)

Given, he gets ₹ 747

i.e.,
$$22(45 - x) - 5x = 747$$
$$990 - 22x - 5x = 747$$
$$990 - 27x = 747$$
$$-27x = 747 - 990$$
$$-27x = -243$$
$$x = (-243)/(-27) = 9$$
$$x = 9$$

Hence, he is idle for 9 days.

Example: The sum of a number and its reciprocal is 4.25. Find the number.

Solution:

Let the number be x.

Its reciprocal $= 1/x$

Given their sum $= 4.25$

$$x + \frac{1}{x} = 4.25$$
$$\frac{x^2 + 1}{x} = \frac{17}{4}$$
$$4x^2 + 4 = 17x$$
$$4x^2 - 17x + 4 = 0$$
$$4x^2 - 16x - x + 4 = 0$$
$$4x(x - 4) - 1(x - 4) = 0$$
$$(4x - 1)(x - 4) = 0$$
$$4x - 1 = 0 \text{ or } x - 4 = 0$$
$$4x = 1$$
$$x = \frac{1}{4} \text{ or } x = 4$$

So the required number is 4.

Example: There are two examinations rooms A and B. If 10 students are sent from A to B, then the number of students in each room is the same. If 20 candidates are sent from B to A, then the number of students in A is double the number of students in B. The number of students in room A is:

Solution:

Let the number of students in rooms A and B be x and y respectively.

Then, $x - 10 = y + 10 \implies x - y = 20$ .... (1)

and $x + 20 = 2(y - 20) \implies x - 2y = -60$ .... (2)

Solving (1) and (2), we get:

$$x = 100, y = 80.$$

∴   The required answer A = 100.

Example: The price of 2 sarees and 4 shirts is ₹ 1600. With the same money one can buy 1 saree and 6 shirts. If one wants to buy 12 shirts, how much shall be have to pay?

Solution:

Let the price of a saree and a shirt be ₹ x and ₹ y respectively.

Then, $2x + 4y = 1600$ ....(i)

and $x + 6y = 1600$ ....(ii)

Divide equation (i) by 2, we get the below equation.

$\Rightarrow$ $x + 2y = 800$ ...(iii)

Now subtract (iii) from (ii)

$$x + 6y = 1600$$
$$x + 2y = 800$$
$$\overline{4y = 800}$$

Therefore, $y = 200$.

Now apply value of y in (iii)

$\Rightarrow$ $x + 2 \times 200 = 800$

$\Rightarrow$ $x + 400 = 800$

Therefore $x = 400$

Solving (i) and (ii), we get

$x = 400, y = 200$.

$\therefore$ Cost of 12 shirts = ₹ (12×200) = ₹ 2400.

Check Your Progress

Fill in the blanks:

1. Theory of equations are frequently used in solving the _____ of business.

2. A statement of equality containing one or more variable is known as _____

3. A value of x which satisfies the given equation is called a _____ or a _____

4. A _____ has exactly one root and a quadratic equation has exactly two roots.

5. The expression _____ is called the discriminant and is denoted by Δ.

3.7 LET US SUM UP

- A linear expression equated to zero is called a linear equation. Thus the general form of a linear equation is $ax + b = 0$ where $a \neq 0$.

- An equation of the type: $ax + by + c = 0$ is called a linear equation in two variables. A linear equation with more than one variable has infinite number of solutions.

- A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$.

- a is called the coefficient of x^2; b is called the coefficient of x and c is called the constant term. A quadratic equations can be solved by factoring, completing the square, using the quadratic formula or graphically.

- Solutions of a system of linear equations can be found by: (i) substitution method, (ii) elimination method and (iii) cross multiplication method.

- If a, b and c are real constants with $a \neq 0$, an equation that can be written as $ax^2 + bx + c = 0$ is called a quadratic equation.

- The formula for solving quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- An equation of degree n has exactly n roots. Hence a linear equation has exactly one root and a quadratic equation has exactly two roots.

- Sometimes you will be asked to find 2 unknown values by solving 2 equations at the same time. These types of equations are called simultaneous equations.

3.8 LESSON END ACTIVITY

Discuss in group, the nature of roots of quadratic equation $ax^2 + bx + c = 0$. Assuming different values of a, b and c find roots and comment upon their nature.

3.9 KEYWORDS

Equation: A statement that two expressions (connected by the sign=) are equal.

Linear Equation: A Linear expression equated to zero is called a linear equation.

Quadratic Equation: An equation of the form, $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are constant, is called a quadratic equation.

Root: A value of the variable which satisfies the given equation is called a solution or root.

Simultaneous Equations: These are also known as a system of equations or an equation system. These are a finite set of equations for which common solutions are sought.

3.10 QUESTIONS FOR DISCUSSION

1. When the solutions of quadratic equation are termed as rational and equal, real and distinct and imaginary and distinct.

2. Solve the following equation:

   (i) $4x + 8 = 2x - 5$

   (ii) $\dfrac{3}{x-1} + \dfrac{4}{x-2} = \dfrac{7}{x-3}; x \neq 1, 2 \text{ or } 3$

3. The incomes of 3 brothers Ajith, Bharath and Chandan are in the ratio 6:5:4 and their expenditure are in the ratio 3:2:1. If Ajith saves ₹ 120 out of his income ₹ 1500. Find the savings of Chandan and Bharath.

4. The number consists of 2 digits. The sum of the digits is 6. If the right digit is increased by 4, the new number formed will he equal to $\dfrac{1}{7}$ the original number. Find the numbers.

5. A commodity was produced by using 3 units of labour and 2 unit of capital. The total cost comes to ₹ 62. If another commodity is produced by using 4 units of labour and one unit of capital, the cost comes to ₹ 56. What is the cost per unit of labour and of capital?

6. Solve the following equations:

   (i) $2x - 3 = 0$ (ii) $4x + 5 = 0$ (iii) $3x - 12 = 0$

   (iv) $7x = 0$ (v) $8x + 16 = 0$ (vi) $9x - \dfrac{3}{2} = 0$

   (vii) $2x + \dfrac{1}{2} = 0$ (viii) $17x - 1 = 0$ (ix) $10x + 50 = 0$

   (x) $7x - 14 = 0$ (xi) $\dfrac{\sqrt{x+1}}{\sqrt{x}} + \dfrac{\sqrt{x}}{\sqrt{x+1}} = 0$ (xii) $2 \cdot 7x + 8 \cdot 1 = 0$

7. If the length of a room is increased by 6 mts and the breadth decreased by 4 mts., its area diminishes by 12 sq. mts. Again if the length is diminished by 3 mts. and breadth increased by 5 mis., the area increases by 60 sq mts. Find the length and breadth of the room.

8. John is 2 years older than Ravi. If the sum of their ages is 22 years, find out their separate ages.

9. Divide 120 into 2 parts so that 6 times one part is 4 more than twice the second part.

10. Solve the following equation: $5(x - 2)^2 - 6 + 13(x - 2) = 0$

11. Solve the following quadratic equations by factorization.

    (i) $18x^2 - 9x - 2 = 0$

    (ii) $8x^2 + 14x + 3 = 0$

12. Draw the graph of the equation $y = x^2 - x - 6$ and use it to find the solution of the equation $x^2 - x - 6 = 2$.

13. Solve the following equations:

    (i) $x^2 + x - 6 = 0$

    (ii) $\sqrt{2x - 1} + \sqrt{x - 1} = 5$

    (iii) $\sqrt{x - 2} + \sqrt{2x - 4} = 6$

    (iv) $x^4 - 16 = 0.$

14. Determine whether the solutions of $5x^2 + 2x - 4 = 0$ are real or not real.

15. Solve the following equations by completing the square:

    (i) $x^2 - 6x + 34 = 0$

    (ii) $x^2 - 4x - 7 = 0$

    (iii) $3x^2 - 9x + 12 = 0$

Check Your Progress: Model Answer

1. Problems

2. An equation

3. Solution, root

4. Linear equation

5. $b^2 - 4ac$

3.11 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

LESSON

4

INVOICE PREPARATION AND AVERAGE

CONTENTS

4.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Explain the concept and uses of invoice
- Describe the preparation of invoice
- Classify the different types of invoices
- Explain the meaning of average and its types
- Discuss the arithmetic mean and its calculation
- Explain median and its calculation
- Describe the usage and calculation of mode
- Explain the concept of geometric mean and harmonic mean

4.1 INTRODUCTION

An invoice is a commercial document issued by businesses to indicate a transaction and request payment for the product or service provided.

In today's business world, there are certain expectations around how we talk about money. An invoice is a powerful communication tool that ensures payment in a timely manner; it also makes a good impression on clients and sets the stage for long-term relationships.

Summarization of the data is a necessary function of any statistical analysis. As a first step in this direction, the huge mass of unwieldy data is summarized in the form of tables and frequency distributions. In order to bring the characteristics of the data into sharp focus, these tables and frequency distributions need to be summarized further. A measure of central tendency or an average is very essential and an important summary measure in any statistical analysis.

In statistics, an average is defined as the number that measures the central tendency of a given set of numbers.

When most people think average, they are thinking of a specific type of average called the arithmetic mean. But there are actually three different types of averages: the mean, median and mode.

4.2 MEANING OF INVOICE AND ITS USES

An invoice represents a receipt for the sale of goods and services. How an invoice is prepared also applies to the preparation of a sales order, a purchase order, a delivery receipt and a price quotation.

To get the total price for a certain item, we simply multiply the quantity of the item by the unit price. We have to make sure that the units in the quantity agree with the units in the unit price, i.e., if quantity is expressed in pieces, the unit price should be per piece; if quantity is expressed in dozens, the unit price should be per dozen. Then you add the individual total prices to get the grand total.

After the dispatch of goods, the seller prepares an invoice of the goods sold in which the quantity and quality of goods and their price are mentioned. Discount, if any, is deducted from the total amount, to this are added the seller's other expenses. Railway receipt number is also mentioned in the invoice if the goods have been sent by train.

4.2.1 Uses of Invoice

The following are the uses of invoice:

- It informs the buyer about the price of the goods and other expenses he has to pay.

- If the invoice reaches buyer before the goods, he can make arrangement for their resale.

- The buyer can compare the invoice with his order.

- After taking delivery of the goods he can compare the contents of the packages with the invoice and point discrepancy, if any to the seller.

- Pay octroi etc. on the basis of the invoice.

- Necessary entries can be made in the books of accounts on the basis of invoice.

4.3 PREPARATION OF INVOICE

Invoice is prepared in duplicate. The original copy is sent to the buyer and the duplicate is kept for future reference. The usual contents of the invoice are:

- Name and address of the seller
- No. and date of the invoice
- No. of the purchase order
- Name and address of the buyer
- Place where it has been made
- Terms of trade
- Details about quantity of goods like weight or length etc.
- Date
- Separate price of each item and total price.
- Trade discount, if any.
- Expenses incurred on sending the goods.
- Advance payment received (if paid by the buyer)
- Net amount payable.
- Details about mode of sending the goods
- Errors and omissions excepted
- Special information, if any
- Signature of the seller.

Basically you can divide the Invoice into three main parts: The Header, Body and Footer.

1. *Invoice Header (Top Part)*

   ❖ Company/Business letterhead: Mention company/business name, address, telephone & email and web address if available. In some countries, you must also include your business registration number.

   ❖ The words "Invoice" or "Bill" clearly written towards the top of the page. If you are registered to bill and collect tax, in some countries you may instead have to use the words "Tax Invoice". You must confirm with your local tax authority if you need to use the words "Tax Invoice."

❖ If your registered to bill and collect tax, you may also need to include your Tax Registration number. This can usually be indicated just below the words "Tax Invoice."

❖ An invoice number. This is a running serial number that you maintain. You should only have one number per invoice issued. Hence, no 2 invoices should have the same number.

❖ An invoice date

❖ Your payment terms or how soon you expect to be paid, e.g. "COD", "Cash", "30 days", etc.

❖ Your customer or client name and address.

❖ "Our Ref#" or "Our Reference". Here you can enter your quotation number if you issued a quotation prior to the invoice.

❖ "Your Ref#" or "Customer Ref". If you were given a Purchase order or Work Order by your customer, here you should enter the PO or WO number so the customer can match your invoice with their own paper work.

2. *Invoice Body (Middle Part)*

❖ A description of the goods your supplying, quantity, unit of measure, price per unit and total amount for individual items. In the case of services, your brief scope of work and amount for individual items.

3. *Invoice Footer (Bottom)*

❖ Total Amount of all individual items.

❖ If applicable, a tax amount and total after tax.

❖ Payment instructions if necessary. Tell recipients how to make out their check (cheque) payment. If you expect payments by wire transfer, you should provide your bank account number and details here.

Other comments you may have like delivery instructions, goods return policy, overdue payment policy, etc.

4.3.1 Format of Invoice

If your customer accepts an electronic format of invoice, you should prepare your invoice in PDF format.

PDF format is good for electronic delivery as:

1. PDF cannot be altered,

2. It can be opened freely in different platforms.

3. The layout of the invoice can be kept and printed correctly in different platforms.

	N-Tex	TIN No : 123456789012
	4/1, Ramasamy Street	CST No. 369852 Dt: 12/10/87
MK COMPUTER MARK	Erode - 638001, Tamil Nadu	
	Phone : 0424 2217571 Mobile :	
	Email :	

Buyer	Invoice No : 500 Invoice Date : 02 / 10 / 2007
Rajasekar Tex	PO Nos :
	PO Date :
Erode	DC Nos :
	DC Date :

S.No.	Description	Qty/Unit	Rate	Per	Amount
1	Sun Lungies	150 Nos	35.00	Nos	5,250,00
2	Supreme Print Lungies 2.00 Mtrs	50 Nos	48.50	Nos	2,425,00
	Output Vat 12.50%				303.13
	Output Vat 4.00				210.00
	Round Off LESS				−0.13
	Total Amount				**8,188.00**

Amount in words : RUPEES EIGHT THOUSAND ONE HUNDRED EIGHTY-EIGHT ONLY

Through   : Cheran Roadways

Vehicle/Receipt No :

Packing Details :

* Subject to Erode Jurisdiction

* We are not responsible for any loss or damage in transit

E. & O.E.

For N-Tex

Authorized Signatory

Computer Generated Invoice

4.4 TYPES OF INVOICE

The various types of invoices used in business are discussed below:

● *Local invoices:* In these invoices only the cost of the goods less any trade discount is borne by the seller. All expenses of packing, cartage, loading and freight for carrying goods to the place of buyer are shown extra and charged to the buyer. It means that the cost includes cost of the goods only and all other expenses are extra and are recovered from the buyer.

- *At station invoice:* This implies that all costs upto the stage of putting the goods at railway station will be borne by the seller and are included in the cost of goods but expenses beyond that, i.e., the railway fare, insurance, etc., will be borne by the buyer.

- *Free on Rails (FOR) invoices:* Under this all the costs, i.e., cost of goods, cost of packing, carrying the goods to railway station, loading them in wagons, are borne by the seller and further expenses are borne by the buyer, i.e., they are charged over and above the cost of goods.

- *Cost and Freight (C and F) invoice:* In such type of invoices it is presumed that cost which the seller is charging includes cost of goods, cost of packing, freight for carrying goods to the buyer and all other incidental expenses. All expenses other than above, i.e.. insurance will be charged extra.

- *Cost, Insurance and freight, (CI & F) invoice:* In such invoices cost charged by the seller includes cost of goods, cost of packing, freight and insurance. Other expenses are charged extra.

- *Franco invoice:* In such invoices all costs upto putting the goods at the door of the buyer are borne by the seller, i.e., the price which he has quoted includes all expenses incurred in carrying the goods to the buyer's place. For example, Franco cost invoice will mean that cost charged by the seller includes cost of the goods, cost of packing, freight, insurance and local transportation charges.

4.5 EFFICIENT INVOICING

These are the six steps recommended to be considered while preparing for efficient billing and invoicing:

1. *Keep accurate records of work done:* In order to invoice, one need to know what work has been carried out. It's hard to remember at the end of the month what work has been done at the beginning of the month, so one needs to track work as it goes along. If billing is done at per hour, make sure time-tracking software is used so one knows exactly how much work has been done. Use time-management platforms to record tasks done and don't forget to include materials if relevant.

2. *Decide how often invoicing is done:* One may want to invoice more regularly for products as opposed to services, and may negotiate a different deal with each client. However, it is important to pay attention to cash flow and not allow too much time between invoices. Once or twice a month is the recommended frequency. Don't forget to send out invoices at the time one have set; sending a bill long after the work was done disrupts the cash flow and may cause accounting issues for customers.

3. *Define the accepted payment methods:* If one wants his/her customers to pay quickly, make it as easy as possible for them to pay. One can accept a range of payment options, from credit card and check to PayPal and other online methods. Although some payment methods will also cost in transaction fees, it is likely to be worth as one is get paid more speedily.

4. *Set up your invoices to include a "payment due by" date:* In addition to let customers know expect payment within, say, 14 days. it is helpful to include an actual date on the invoice. This way the customer doesn't have to calculate or conveniently become confused about when the payment is expected. To ensure timely payment, one can either charge a late fee, or even more effective, offer a discount for early payment. (When a price is quoted to the client, add discount percentage on to the total so that one don't lose out by offering the deal.) Remember that payment will arrive only after an invoice is send out, so don't put

it off and don't be passive. If payment is not forthcoming, chase it down and politely and firmly make it clear that you expect immediate payment.

5. *Create professional-looking invoices:* If one wants his/her business to be taken seriously, he/she need serious-looking invoices. Make sure they include all the pertinent information and are printed nicely. Using online software which creates the invoices for and sends them via email is the easiest way to do this.

6. *Don't forget the human touch:* If invoices are send to large companies, it's a good idea to pick up the phone and introduce to someone in the accounting department of that particular company. Since the person now "knows" invoice sender, the invoice will get personal and speedy attention when it arrives. If there are any issues to be ironed out, the process will go more smoothly once a phone contact has been made. For invoices sent to smaller companies or individuals, consider sending a short and friendly note along with request for payment works well.

4.6 CONCEPT OF AVERAGE

"A single value which can represent the whole set of data is called an average". If the average tends to lie or indicate the center of the distribution it is called the measure of central tendency, or sometimes they locate the general position of the data so they are also called the measure of location.

The average of a distribution has been defined in various ways. Some of the important definitions are mentioned below:

"An average is an attempt to find one single figure to describe the whole of figures".

— Clark and Sekkade

"Average is a value which is typical or representative of a set of data".

— Murray R. Spiegal

"An average is a single value within the range of the data that is used to represent all the values in the series. Since an average is somewhere within the range of data it is sometimes called a measure of central value".

— Croxton and Cowden

"A measure of central tendency is a typical value around which other figures congregate".

—Sipson and Kafka

4.6.1 Functions of an Average

● *To present huge mass of data in a summarised form:* It is very difficult for human mind to grasp a large body of numerical figures. A measure of average is used to summarise such data into a single figure which makes it easier to understand and remember.

● *To facilitate comparison:* Different sets of data can be compared by comparing their averages. For example, the level of wages of workers in two factories can be compared by mean (or average) wages of workers in each of them.

● *To help in decision making:* Most of the decisions to be taken in research, planning, etc., are based on the average value of certain variables. For example, if the average monthly sales of a company are falling, the sales manager may have to take certain decisions to improve it.

4.6.2 Desirable Qualities of a Good Average

An average which possesses all or most of the following qualities (characteristics) is considered a good average:

1. It should be easy to calculate and simple to understand.

2. It should be clearly defined by a mathematical formula.

3. It should not be affected by extreme values.

4. It should be based on all the observations.

5. It should be capable of further mathematical treatment.

6. It should have sample stability.

4.6.3 Types of Averages

Mathematical averages are:

1. Arithmetic Mean

2. Median

3. Mode

4. Geometric Mean

5. Harmonic Mean

All types of averages are further discussed in detail.

4.7 ARITHMETIC MEAN

Arithmetic mean is the most commonly used average or measure of the central tendency applicable only in case of quantitative data; it is also simply called the "mean". Arithmetic mean is defined as:

"Arithmetic mean is a quotient of sum of the given values and number of the given values".

Before the further discussion of arithmetic mean, we shall introduce certain notations. It will be assumed that there are n observations whose values are denoted by $X_1, X_2, \ldots$ X_n respectively. The sum of these observations $X_1 + X_2 + \ldots + X_n$ will be denoted in abbreviated form as $\sum_{i=1}^{n} X_i$, where Σ (called sigma) denotes summation sign. The subscript of X, i.e., 'i' is a positive integer, which indicates the serial number of the observation. Since there are n observations, variation in i will be from 1 to n. This is indicated by writing it below and above Σ, as written earlier. When there is no ambiguity in range of summation, this indication can be skipped and we may simply write $X_1 + X_2 + \ldots + X_n = \Sigma X_i$.

Arithmetic mean is defined as the sum of observations divided by the number of observations. It can be computed in two ways: (i) Simple arithmetic mean and (ii) weighted arithmetic mean. In case of simple arithmetic mean, equal importance is given to all the observations while in weighted arithmetic mean, the importance given to various observations is not same.

4.7.1 Calculation of Simple Arithmetic Mean

The formula and method used to calculate arithmetic mean depends upon the way observations are given. Here different ways of calculating arithmetic mean are discussed below:

When Individual Observations are Given

Let there be n observations $X_1, X_2, ... X_n$. Their arithmetic mean can be calculated either by direct method or by short cut method. The arithmetic mean of these observations will be denoted by $\overline{X}$.

Direct Method

Under this method, $\overline{X}$ is obtained by dividing sum of observations by number of observations, i.e.,

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Short-cut Method

This method is used when the magnitude of individual observations is large. The use of short-cut method is helpful in the simplification of calculation work.

Let A be any assumed mean. We subtract A from every observation. The difference between an observation and A, i.e., $X_i - A$ is called the deviation of i th observation from A and is denoted by d_i. Thus, we can write; $d_1 = X_1 - A, d_2 = X_2 - A, ... d_n = X_n - A$. On adding these deviations and dividing by n, we get

$$\frac{\Sigma d_i}{n} = \frac{\Sigma(X_i - A)}{n} = \frac{\Sigma X_i - nA}{n} = \frac{\Sigma X_i}{n} - A$$

or

$$\overline{d} = \overline{X} - A \text{ (where } \overline{d} = \frac{\Sigma d_i}{n})$$

On rearranging, we get $\overline{X} = A + \overline{d} = A + \frac{\Sigma d_i}{n}$

This result can be used for the calculation of $\overline{X}$.

Remarks: Theoretically we can select any value as assumed mean. However, for the purpose of simplification of calculation work, the selected value should be as nearer to the value of as possible.

Example: The following figures relate to monthly output of cloth of a factory in a given year:

Months	:	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Output (in '000 metres)	:	80	88	92	84	96	92	96	100	92	94	98	86

Calculate the average monthly output.

Solution:

Using Direct Method

$$\bar{X} = \frac{80 + 88 + 92 + 84 + 96 + 92 + 96 + 100 + 92 + 94 + 98 + 96}{12}$$

= 91.5 ('000 mtrs)

Using Short-cut Method

Let A = 90.

X_i	80	88	92	84	96	92	96	100	92	94	98	86	Total
$d_i = X_i - A$	−10	−2	2	−6	6	2	6	10	2	4	8	−4	$\Sigma d_i = 18$

$$\therefore \ \bar{X} = 90 + \frac{18}{12} = 90 + 1.5 = 91.5 \text{ thousand mtrs}$$

When Data are in the Form of an Ungrouped Frequency Distribution

Let there be n values X_1, X_2, ... X_n out of which X_1 has occurred f_1 times, X_2 has occurred f_2 times, ... X_n has occurred f_n times. Let N be the total frequency. *i.e.,*

$N = \sum_{i=1}^{n} f_i$. Alternatively, this can be written as follows:

Values	X_1	X_2	- - -	X_n	Total Frequency
Frequency	f_1	f_2	- - -	f_n	N

Direct Method

The arithmetic mean of these observations using direct method is given by

$$\frac{\underbrace{X_1 + X_1 + ... + X_1}_{f_1 \ times} + \underbrace{X_2 + ... + ... + X_2}_{f_2 \ times} + \underbrace{... + ... + X_n + ... + X_n}_{f_n \ times}}{f_1 + f_2 + ... + f_n}$$

Since $X_1 + X_1 + ... + X_1$ added f_1 times can also be written $f_1 X_1$. Similarly, by writing other observation in same manner, we have

$$\bar{X} = \frac{f_1 X_1 + f_2 X_2 ...... + f_n x_n}{f_1 + f_2 + ... + f_n} = \frac{\sum_{i=1}^{n} f_i X_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i X_i}{N} \qquad ... (1)$$

Short-cut Method

As before, we take the deviations of observations from an arbitrary value A. The deviation of ith observation from A is $d_i = X_i - A$.

Multiplying both sides by f_i we have $f_i d_i = f_i (X_i - A)$

Taking sum over all the observations

$$\sum f_i d_i = \sum f_i (X_i - A) = \sum f_i X_i - A \sum f_i = \sum f_i X_i - A.N$$

Dividing both sides by N, we have

$$\frac{\sum f_i d_i}{N} = \frac{\sum f_i X_i}{N} - A = \bar{X} - A \quad \text{or} \quad \bar{X} = A + \frac{\sum f_i d_i}{N} = A + \bar{d}$$

Example: The following is the frequency distribution of age of 670 students of a school. Compute the arithmetic mean of the data.

X (in years)	5	6	7	8	9	10	11	12	13	14
Frequency	25	45	90	165	112	96	81	26	18	12

Solution:

Direct Method

The computations are shown in the following table:

X	5	6	7	8	9	10	11	12	13	14	Total
f	25	45	90	165	112	96	81	26	18	12	$\Sigma f = 670$
fX	125	270	630	1320	1008	960	891	312	234	168	$\Sigma fX = 5918$

$$\overline{X} = \frac{\Sigma fX}{\Sigma f} = \frac{5918}{670} = 8.83 \text{ years.}$$

Short-cut Method

The method of computations are shown in the following table:

X	5	6	7	8	9	10	11	12	13	14	Total
f	25	45	90	165	112	96	81	26	18	12	670
d = X − 8	−3	−2	−1	0	1	2	3	4	5	6	
fd	−75	−90	−90	0	112	192	243	104	90	72	558

∴ $$\overline{X} = A + \frac{\Sigma fd}{N} = 8 + \frac{558}{670} = 8 + 0.83 = 8.83 \text{ years.}$$

When Data are in the Form of a Grouped Frequency Distribution

In a grouped frequency distribution, there are classes along with their respective frequencies. Let l_i be the lower limit and u_i be the upper limit of ith class. Further, let the number of classes be n, so that $i = 1, 2,\ldots n$. Also let f_i be the frequency of ith class. This distribution can written in tabular form, as shown.

Class Intervals	Frequency (f)
$l_1 - u_1$	f_1
$l_2 - u_2$	f_2
⋮	⋮
$l_n - u_n$	f_n
Total Frequency	$= \Sigma f_i = N$

Note: Here u_1 may or may not be equal to l_2, *i.e.*, the upper limit of a class may or may not be equal to the lower limit of its following class.

It may be recalled here that, in a grouped frequency distribution, we only know the number of observations in a particular class interval and not their individual magnitudes. Therefore, to calculate mean, we have to make a fundamental assumption

that the observations in a class are uniformly distributed. Under this assumption, the mid-value of a class will be equal to the mean of observations in that class and hence can be taken as their representative. Therefore, if X_i is the mid-value of ith class with frequency f_i, the above assumption implies that there are f_i observations each with magnitude X_i ($i = 1$ to n).

Remarks: The accuracy of arithmetic mean calculated for a grouped frequency distribution depends upon the validity of the fundamental assumption. This assumption is rarely met in practice. Therefore, we can only get an approximate value of the arithmetic mean of a grouped frequency distribution.

Example: Calculate arithmetic mean of the following distribution:

Class Intervals	:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	:	3	8	12	15	18	16	11	5

Solution:

Here only short-cut method will be used to calculate arithmetic mean but it can also be calculated by the use of direct-method.

Class Intervals	Mid Values (X)	Frequency	d = X – 35	fd
0–10	5	3	–30	–90
10–20	15	8	–20	–160
20–30	25	12	–10	–120
30–40	35	15	0	0
40–50	45	18	10	180
50–60	55	16	20	320
60–70	65	11	30	330
70–80	75	5	40	200
Total		**88**		**660**

$\therefore$
$$\overline{X} = A + \frac{\Sigma fd}{N} = 35 + \frac{660}{88} = 42.5$$

Step Deviation Method or Coding Method

In a grouped frequency distribution, if all the classes are of equal width, say 'h', the successive mid-values of various classes will differ from each other by this width. This fact can be utilised for reducing the work of computations.

Let us define $u_i = \dfrac{X_i - A}{h}$. Multiplying both sides by f_i and taking sum over all the observations we have, $\displaystyle\sum_{i=1}^{n} f_i u_i = \frac{1}{h} \sum_{i=1}^{n} f_i (X_i - A)$

or
$$h \sum_{i=1}^{n} f_i u_i = \sum_{i=1}^{n} f_i X_i - A \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} f_i X_i - A.N$$

Dividing both sides by N, we have

$$h.\frac{\sum_{i=1}^{n} f_i u_i}{N} = \frac{\sum_{i=1}^{n} f_i X_i}{N} - A = \bar{X} - A$$

$$\therefore \quad \bar{X} = A + h.\frac{\sum_{i=1}^{n} f_i u_i}{N} \qquad \qquad \ldots (2)$$

4.7.2 Weighted Arithmetic Mean

In the computation of simple arithmetic mean, equal importance is given to all the items. But this may not be so in all situations. If all the items are not of equal importance, then simple arithmetic mean will not be a good representative of the given data. Hence, weighing of different items becomes necessary. The weights are assigned to different items depending upon their importance, *i.e.*, more important items are assigned more weight. For example, to calculate mean wage of the workers of a factory, it would be wrong to compute simple arithmetic mean if there are a few workers (say managers) with very high wages while majority of the workers are at low level of wages. The simple arithmetic mean, in such a situation, will give a higher value that cannot be regarded as representative wage for the group. In order that the mean wage gives a realistic picture of the distribution, the wages of managers should be given less importance in its computation. The mean calculated in this manner is called weighted arithmetic mean. The computation of weighted arithmetic is useful in many situations where different items are of unequal importance, *e.g.*, the construction index numbers, computation of standardised death and birth rates, *etc.*

Formulae for Weighted Arithmetic Mean

Let X_1, X_2 ....., X_n be n values with their respective weights w_1, w_2 ....., w_n. Their weighted arithmetic mean denoted as $\bar{X}_w$ is given by,

1. $\bar{X}_w = \dfrac{\Sigma w_i X_i}{\Sigma w_i}$ (Using direct method)

2. $\bar{X}_w = A + \dfrac{\Sigma w_i d_i}{\Sigma w_i}$ (where $d_i = X_i - A$) (Using short-cut method),

3. $\bar{X}_w = A + \dfrac{\Sigma w_i u_i}{\Sigma w_i} \times h$ (where $u_i = \dfrac{X_i - A}{h}$) (Using step-deviation method)

4.7.3 Properties of Arithmetic Mean

Arithmetic mean of a given data possesses the following properties:

1. The sum of deviations of the observations from their arithmetic mean is always zero.

    According to this property, the arithmetic mean serves as a point of balance or a centre of gravity of the distribution; since sum of positive deviations (*i.e.*, deviations of observations which are greater than $\bar{X}$) is equal to the sum of negative deviations (*i.e.*, deviations of observations which are less than $\bar{X}$).

    Proof:

    Let X_1, X_2 ....., X_n be n observations with respective frequencies $f_1, f_2 ....., f_n$. Let Σf_i = N, be the total frequency. Thus, $\bar{X} = \dfrac{\Sigma f_i X_i}{N}$.

Let $d_i = X_i - \bar{X}$, where $i = 1, 2 \ldots n$. Multiplying both sides by f_i and taking sum over all the observations, we have

$$\Sigma f_i d_i = \Sigma f_i (X_i - \bar{X}) = \Sigma f_i X_i - \bar{X} \Sigma f_i$$

$$= \Sigma f_i X_i - \bar{X}.N = 0 \quad \text{(Since } \Sigma f_i X_i = N\bar{X}) \text{. Hence proved.}$$

2. The sum of squares of deviations of observations is minimum when taken from their arithmetic mean. Because of this, the mean is sometimes termed as 'least square' measure of central tendency.

Proof:

The sum of squares of deviations of observations from arithmetic mean

$$= \Sigma f_i \left(X_i - \bar{X} \right)^2$$

Similarly, we can define sum of squares of deviations of observations from any arbitrary value A as $S = \Sigma f_i (X_i - A)^2$ .... (1)

We have to show that S will be minimum when $A = \bar{X}$.

To prove this, we try to find that value of A for which S is minimum. The necessary and sufficient conditions for minimum of S are:

$$\frac{dS}{dA} = 0 \quad and \quad \frac{d^2S}{dA^2} > 0$$

Differentiating (1) *w.r.t.* A, we have

$$\frac{dS}{dA} = -2\Sigma f_i (X_i - A) = 0 \text{, for minima.} \quad\quad .... (2)$$

On dividing both sides by -2, we have

$$\Sigma f_i (X_i - A) = 0 \quad \text{or} \quad \Sigma f_i X_i - NA = 0$$

Or $\dfrac{\Sigma f_i X_i}{N} - A = 0$ (on dividing both sides by N)

Thus, $\bar{X} - A = 0$ or $A = \bar{X}$.

Further, to show that S is minimum, it will be shown that $\dfrac{d^2S}{dA^2} > 0$ at $A = \bar{X}$.

Differentiating (2) further *w.r.t.* A, we have

$$\frac{d^2S}{dA^2} = -2\Sigma f_i (0-1) = 2\Sigma f_i = 2N \text{, which is always positive.}$$

Hence S is minimum when $A = \bar{X}$.

3. Arithmetic mean is capable of being treated algebrically.

This property of arithmetic mean highlights the relationship between $\bar{X}$, $\Sigma f_i X_i$ and N. According to this property, if any two of the three values are known, the third can be easily computed. This property is obvious and requires no proof.

4. If $\overline{X}_1$ and N_1 are the mean and number of observations of a series and $\overline{X}_2$ and N_2
are the corresponding magnitudes of another series, then the mean of the combined series of $N_1 + N_2$ observations is given by $\overline{X} = \dfrac{N_1\overline{X}_1 + N_2\overline{X}_2}{N_1 + N_2}$.

Proof:

To find mean of the combined series, we have to find sum of its observations.

Now, the sum of observations of the first series, *i.e.*, $\Sigma f_1 X_1 = N_1 \overline{X}_1$ and the sum of observations of the second series, *i.e.*, $\Sigma f_2 X_2 = N_2 \overline{X}_2$.

$\therefore$ The sum of observations of the combined series, *i.e.*, $N_1 \overline{X}_1 + N_2 \overline{X}_2$.

Thus, the combined mean $\overline{X} = \dfrac{N_1\overline{X}_1 + N_2\overline{X}_2}{N_1 + N_2}$.

This result can be generalised: If there are k series each with mean $\overline{X}_i$ and number of observations equal to N_i, where $i = 1, 2 \dots k$, the mean of the combined series of $N_1 + N_2 + \dots + N_k$ observations is given by

$$\overline{X} = \frac{N_1\overline{X}_1 + N_2\overline{X}_2 + \dots + N_k\overline{X}_k}{N_1 + N_2 + \dots + N_k} = \frac{\displaystyle\sum_{i=1}^{k} N_i X_i}{\displaystyle\sum_{i=1}^{k} N_i}$$

5. If a constant B is added (subtracted) from every observation, the mean of these observations also gets added (subtracted) by it.

Proof:

Let $\overline{X}$ be the mean of the observations $X_1, X_2, \dots X_n$ with respective frequencies as $f_1, f_2 \dots f_n$. When B is added to every observations, let $u_i = X_i + B$.

Multiply both sides by f and take sum over all the observations, we get

$$\Sigma f_i u_i = \Sigma f_i(X_i + B) = \Sigma f_i X_i + NB$$

Dividing both sides by N, we get

$$\frac{\Sigma f_i u_i}{N} = \frac{f_i X_i}{N} + B \quad \text{or} \quad \overline{u} = \overline{X} + B$$

i.e., The mean of $u_i = X_i + B$ is obtained by adding B to the mean of X_i values.

Similarly, it can be shown that if $v_i = X_i - B$, then $\overline{V} = \overline{X} - B$

6. If every observation is multiplied (divided) by a constant b, the mean of these observations also gets multiplied (divided) by it.

Proof:

Let us define $w_i = \beta X_i$. Multiplying both sides by f_i and taking sum over all the observations, we get $\Sigma f_i w_i = \beta \Sigma f_i X_i$.

Dividing both sides by N, we get $\dfrac{\sum f_i w_i}{N} = \beta \dfrac{f_i X_i}{N}$ or $\overline{w} = \beta \overline{X}$

Similarly, it can be shown that if $D_i = \dfrac{X_i}{\beta}$, then $\bar{D} = \dfrac{\bar{X}}{\beta}$

Using properties 5 and 6, we can derive the following results:

If $Y_i = a + bX_i$, then $\Sigma f_i Y_i = \Sigma f_i(a + bX_i)$ or $\Sigma f_i Y_i = a\Sigma f_i + b\Sigma f_i X_i$.

Dividing both sides by $N (= \Sigma f_i)$, we have

$$\frac{\sum f_i Y_i}{N} = a + b\,\frac{f_i X_i}{N} \quad \text{or} \quad \bar{Y} = a + b\bar{X}$$

This shows that relationship between the means of two variables is same as the relationship between the variables themselves.

7. If some observations of a series are replaced by some other observations, then the mean of original observations will change by the average change in magnitude of the changed observations.

Proof:

Let mean of n observations be $\bar{X} = \dfrac{X_1 + X_2 + \dots + X_n}{n}$. Further, Let X_1, X_2, X_3 are replaced by the respective observations Y_1, Y_2, Y_3. Therefore, the change in magnitude of the changed observations $= (Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3)$.

Hence average change in magnitude $= \dfrac{(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3)}{n}$.

Thus, New $\bar{X}$ = Old $\bar{X}$ + Average change in magnitude.

4.8 MEDIAN

Median of distribution is that value of the variant which divides it into two equal parts. In terms of frequency curve, the ordinate drawn at median divides the area under the curve into two equal parts. Median is a positional average because its value depends upon the position of an item and not on its magnitude.

4.8.1 Determination of Median

The various ways of determining median are discussed below:

When Individual Observations are Given

The following steps are involved in the determination of median:

● The given observations are arranged in either ascending or descending order of magnitude.

● Given that there are n observations, the median is given by:

❖ The size of $\left(\dfrac{n+1}{2}\right)^{\text{th}}$ observations, when n is odd.

❖ The mean of the sizes of $\dfrac{n}{2}^{\text{th}}$ and $\left(\dfrac{n}{2}+1\right)^{\text{th}}$ observations, when n is even.

Example: Find median of the following observations:

20, 15, 25, 28, 18, 16, 30.

Solution:

Writing the observations in ascending order, we get 15, 16, 18, 20, 25, 28, 30.

Since $n = 7$, i.e., odd, the median is the size of $\left(\dfrac{7+1}{2}\right)^{th}$ i.e., 4th observation.

Hence, median, denoted by $M_d = 20$.

Note: The same value of M_d will be obtained by arranging the observations in descending order of magnitude.

Example: Find median of the data: 245, 230, 265, 236, 220, 250.

Solution:

Arranging these observations in ascending order of magnitude, we get

220, 230, 236, 245, 250, 265. Here n = 6, i.e., even.

$\therefore$ Median will be arithmetic mean of the size of $\left(\dfrac{6}{2}\right)^{th}$, i.e., 3rd and $\left(\dfrac{6}{2}+1\right)^{th}$, i.e., 4th

observations. Hence $M_d = \dfrac{236 + 245}{2} = 240.5$

When Ungrouped Frequency Distribution is Given

In this case, the data are already arranged in the order of magnitude. Here, cumulative frequency is computed and the median is determined in a manner similar to that of individual observations.

Example: Locate median of the following frequency distribution:

Variable (X) :	10	11	12	13	14	15	16
Frequency (f) :	8	15	25	20	12	10	5

Solution:

X	:	10	11	12	13	14	15	16
f	:	8	15	25	20	12	10	5
c.f.	:	8	23	48	68	80	90	95

Here $N = 95$, which is odd. Thus, median is size of $\left[\dfrac{95+1}{2}\right]^{th}$ i.e., 48th observation.

From the table 48th observation is 12, $\therefore M_d = 12$.

Alternative Method

$\dfrac{N}{2} = \dfrac{95}{2} = 47.5$. Looking at the frequency distribution we note that there are 48 observations which are less than or equal to 12 and there are 72 (i.e., 95 – 23) observations which are greater than or equal to 12. Hence, median is 12.

When Grouped Frequency Distribution is Given (Interpolation Formula)

The determination of median, in this case, will be explained with the help of the following example.

Example: The following table shows the daily sales of 230 footpath sellers of Chandni Chowk:

Sales (in ₹) :	0-500	500-1000	1000-1500	1500-2000
No. of Sellers:	12	18	35	42

Sales (in ₹) :	2000-2500	2500-3000	3000-3500	3500-4000
No. of Sellers:	50	45	20	8

Locate the median of the above data using

1. only the less than type ogive, and

2. both, the less than and the greater than type ogives.

Solution:

To draw ogives, we need to have a cumulative frequency distribution.

Class Intervals	Frequency	Less than c.f.	More than c.f.
0-500	12	12	230
500-1000	18	30	218
1000-1500	35	65	200
1500-2000	42	107	165
2000-2500	50	157	123
2500-3000	45	202	73
3000-3500	20	222	28
3500-4000	8	230	8

1. Using the less than type ogive
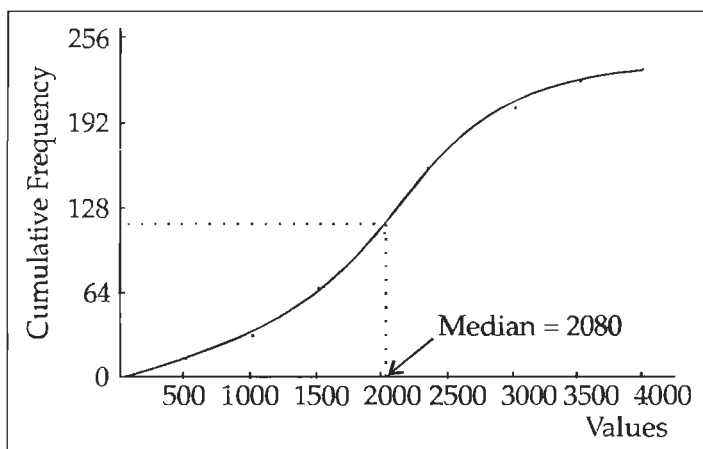
The value $\dfrac{N}{2} = 115$ is marked on the vertical axis and a horizontal line is drawn from this point to meet the ogive at point S. Drop a perpendicular from S. The point at which this meets X- axis is the median.

2. Using both types of ogives
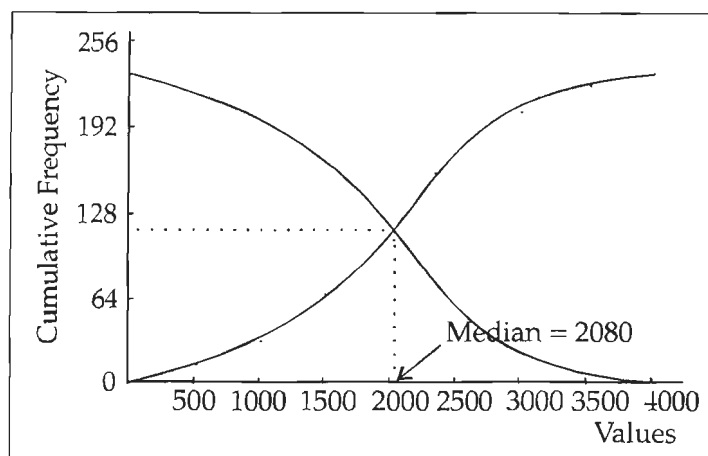
A perpendicular is dropped from the point of intersection of the two ogives. The point at which it intersects the X-axis gives median. It is obvious from above figures that median = 2080.

4.8.2 Other Partition or Positional Measures

Median of a distribution divides it into two equal parts. It is also possible to divide it into more than two equal parts. The values that divide a distribution into more than two equal parts are commonly known as partition values or *fractiles*. Some important partition values are discussed in the following sections.

Quartiles

The values of a variable that divide a distribution into four equal parts are called quartiles. Since three values are needed to divide a distribution into four parts, there are three quartiles, *viz.* Q_1, Q_2 and Q_3, known as the first, second and the third quartile respectively.

For a discrete distribution, the first quartile (Q_1) is defined as that value of the variate such that at least 25% of the observations are less than or equal to it and at least 75% of the observations are greater than or equal to it.

For a continuous or grouped frequency distribution, Q_1 is that value of the variate such that the area under the histogram to the left of the ordinate at Q_1 is 25% and the area to its right is 75%. The formula for the computation of Q_1 can be written by making suitable changes in the formula of median.

After locating the first quartile class, the formula for Q_1 can be written as follows:

$$Q_1 = L_{Q_1} + \frac{\left(\dfrac{N}{4} - C\right)}{f_{Q_1}} \times h$$

Here, L_{Q_1} is lower limit of the first quartile class, h is its width, f_{Q_1} is its frequency and C is cumulative frequency of classes preceding the first quartile class.

By definition, the second quartile is median of the distribution. The third quartile (Q_3) of a distribution can also be defined in a similar manner.

For a discrete distribution, Q_3 is that value of the variate such that at least 75% of the observations are less than or equal to it and at least 25% of the observations are greater than or equal to it.

For a grouped frequency distribution, Q_3 is that value of the variate such that area under the histogram to the left of the ordinate at Q_3 is 75% and the area to its right is 25%. The formula for computation of Q_3 can be written as

$$Q_3 = L_{Q_3} + \frac{\left(\dfrac{3N}{4} - C\right)}{f_{Q_3}} \times h, \text{ where the symbols have their usual meaning.}$$

Deciles

Deciles divide a distribution into 10 equal parts and there are, in all, 9 deciles denoted as $D_1, D_2, \ldots\ldots D_9$ respectively.

For a discrete distribution, the ith decile D_i is that value of the variate such that at least $(10i)\%$ of the observation are less than or equal to it and at least $(100 - 10i)\%$ of the observations are greater than or equal to it ($i = 1, 2, \ldots\ldots 9$).

For a continuous or grouped frequency distribution, D_i is that value of the variate such that the area under the histogram to the left of the ordinate at D_i is $(10i)\%$ and the area to its right is $(100 - 10i)\%$. The formula for the ith decile can be written as

$$D_i = L_{D_i} + \frac{\left(\dfrac{iN}{10} - C\right)}{f_{D_i}} \times h \quad (i = 1, 2, \ldots\ldots 9)$$

Percentiles

Percentiles divide a distribution into 100 equal parts and there are, in all, 99 percentiles denoted as $P_1, P_2, \ldots\ldots P_{25}, \ldots\ldots P_{40}, \ldots\ldots P_{60}, \ldots\ldots P_{99}$ respectively.

For a discrete distribution, the kth percentile P_k is that value of the variate such that at least $k\%$ of the observations are less than or equal to it and at least $(100 - k)\%$ of the observations are greater than or equal to it.

For a grouped frequency distribution, P_k is that value of the variate such that the area under the histogram to the left of the ordinate at P_k is $k\%$ and the area to its right is $(100 - k)\%$. The formula for the kth percentile can be written as

$$P_k = L_{P_k} + \frac{\left(\dfrac{kN}{100} - C\right)}{f_{P_k}} \times h, \quad (k = 1, 2, \ldots\ldots 99)$$

Remarks:

- We may note here that $P_{25} = Q_1, P_{50} = D_5 = Q_2 = M_d, P_{75} = Q_3, P_{10} = D_1, P_{20} = D_2$, etc.

- In continuation of the above, the partition values are known as Quintiles (Octiles) if a distribution is divided in to 5 (8) equal parts.

- The formulae for various partition values of a grouped frequency distribution, given so far, are based on 'less than' type cumulative frequencies. The corresponding formulae based on 'greater than' type cumulative frequencies can be written in a similar manner, as given below:

$$Q_1 = U_{Q_1} - \frac{\left(\dfrac{3N}{4} - C\right)}{f_{Q_1}} \times h \qquad\qquad Q_3 = U_{Q_3} - \frac{\left(\dfrac{N}{4} - C\right)}{f_{Q_3}} \times h$$

$$D_i = U_{D_i} - \frac{\left[\left(N - \frac{iN}{10}\right) - C\right]}{f_{D_i}} \times h \qquad P_k = U_{P_k} - \frac{\left[\left(N - \frac{kN}{100}\right) - C\right]}{f_{P_k}} \times h$$

Here $U_{Q_1}, U_{Q_3}, U_{D_i}, U_{P_k}$ are the upper limits of the corresponding classes and C denotes the greater than type cumulative frequencies.

Example: Locate Median, $Q_1, Q_3, D_4, D_7, P_{15}, P_{60}$ and P_{90} from the following data:

Daily Profit (in ₹) :	75	76	77	78	79	80	81	82	83	84	85
No. of Shops :	15	20	32	35	33	22	20	10	8	3	2

Solution:

First we calculate the cumulative frequencies, as in the following table:

Daily Profit (in ₹)	75	76	77	78	79	80	81	82	83	84	85
No. of Shops (f)	15	20	32	35	33	22	20	10	8	3	2
Less than c.f.	15	35	67	102	135	157	177	187	195	198	200

1. **Determination of Median:** Here $\frac{N}{2} = 100$. From the cumulative frequency column, we note that there are 102 (greater than 50% of the total) observations that are less than or equal to 78 and there are 133 observations that are greater than or equal to 78. Therefore, $M_d = ₹ 78$.

2. **Determination of Q_1 and Q_3:** First we determine $\frac{N}{4}$ which is equal to 50. From the cumulative frequency column, we note that there are 67 (which is greater than 25% of the total) observations that are less than or equal to 77 and there are 165 (which is greater than 75% of the total) observations that are greater than or equal to 77. Therefore, $Q_1 = ₹ 77$. Similarly, $Q_3 = ₹ 80$.

3. **Determination of D_4 and D_7:** From the cumulative frequency column, we note that there are 102 (greater than 40% of the total) observations that are less than or equal to 78 and there are 133 (greater than 60% of the total) observations that are greater than or equal to 78. Therefore, $D_4 = ₹ 78$. Similarly, $D_7 = ₹ 80$.

4. **Determination of P_{15}, P_{60} and P_{90}:** From the cumulative frequency column, we note that there are 35 (greater than 15% of the total) observations that are less than or equal to 76 and there are 185 (greater than 85% of the total) observations that are greater than or equal to 76. Therefore, $P_{15} = ₹ 76$. Similarly, $P_{60} = ₹ 79$ and $P_{90} = ₹ 82$.

4.9 MODE

Mode is that value of the variate which occurs maximum number of times in a distribution and around which other items are densely distributed. In the words of Croxton and Cowden, "The mode of a distribution is the value at the point around which the items tend to be most heavily concentrated. It may be regarded the most typical of a series of values." Further, according to A.M. Tuttle, "Mode is the value which has the greatest frequency density in its immediate neighbourhood."

If the frequency distribution is regular, then mode is determined by the value corresponding to maximum frequency. There may be a situation where concentration of observations around a value having maximum frequency is less than the

concentration of observations around some other value. In such a situation, mode cannot be determined by the use of maximum frequency criterion. Further, there may be concentration of observations around more than one value of the variable and, accordingly, the distribution is said to be bi-model or multi-model depending upon whether it is around two or more than two values.

The concept of mode, as a measure of central tendency, is preferable to mean and median when it is desired to know the most typical value, *e.g.,* the most common size of shoes, the most common size of a ready-made garment, the most common size of income, the most common size of pocket expenditure of a college student, the most common size of a family in a locality, the most common duration of cure of viral-fever, the most popular candidate in an election, etc.

4.9.1 Determination of Mode

When data are either in the form of individual observations or in the form of ungrouped frequency distribution

Given individual observations, these are first transformed into an ungrouped frequency distribution. The mode of an ungrouped frequency distribution can be determined in two ways, as given below:

1. By inspection or

2. By method of grouping

By Inspection

When a frequency distribution is fairly regular, then mode is often determined by inspection. It is that value of the variate for which frequency is maximum. By a fairly regular frequency distribution, we mean that as the values of the variable increase the corresponding frequencies of these values first increase in a gradual manner and reach a peak at certain value and, finally, start declining gradually in, approximately, the same manner as in case of increase.

Example: Compute mode of the following data:

$$3, 4, 5, 10, 15, 3, 6, 7, 9, 12, 10, 16, 18,$$
$$20, 10, 9, 8, 19, 11, 14, 10, 13, 17, 9, 11$$

Solution:

Writing this in the form of a frequency distribution, we get

Values	:	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Frequency	:	2	1	1	1	1	1	3	4	2	1	1	1	1	1	1	1	1	1

$\therefore$ Mode = 10

Remarks:

● If the frequency of each possible value of the variable is same, there is no mode.

● If there are two values having maximum frequency, the distribution is said to be bi-modal.

By Method of Grouping

This method is used when the frequency distribution is not regular. Let us consider the following example to illustrate this method.

Example: Determine the mode of the following distribution:

X	:	10	11	12	13	14	15	16	17	18	19
f	:	8	15	20	100	98	95	90	75	50	30

Solution:

This distribution is not regular because there is sudden increase in frequency from 20 to 100. Therefore, mode cannot be located by inspection and hence the method of grouping is used. Various steps involved in this method are as follows:

1. Prepare a table consisting of 6 columns in addition to a column for various values of X.

2. In the first column, write the frequencies against various values of X as given in the question.

3. In second column, the sum of frequencies, starting from the top and grouped in twos, are written.

4. In third column, the sum of frequencies, starting from the second and grouped in twos, are written.

5. In fourth column, the sum of frequencies, starting from the top and grouped in threes are written.

6. In fifth column, the sum of frequencies, starting from the second and grouped in threes are written.

7. In the sixth column, the sum of frequencies, starting from the third and grouped in threes are written.

The highest frequency total in each of the six columns is identified and analysed to determine mode. We apply this method for determining mode of the above example.

Analysis Table

Columns	V	A	R	I	A	B	L	E		
	10	11	12	13	14	15	16	17	18	19
1				1						
2					1	1				
3				1	1					
4				1	1	1				
5					1	1	1			
6						1	1	1		
Total	0	0	0	3	4	4	2	1	0	0

Since the value 14 and 15 are both repeated maximum number of times in the analysis table, therefore, mode is ill defined. Mode in this case can be approximately located by the use of the following formula.

Mode = 3 Median − 2 mean

Calculation of Median and Mean

X	10	11	12	13	14	15	16	17	18	19	Total
f	8	15	20	100	98	95	90	75	50	30	581
c.f.	8	23	43	143	241	336	426	501	551	581	
fX	80	165	240	1300	1372	1425	1440	1275	900	570	8767

$$\text{Median} = \text{Size of} \left(\frac{581+1}{2} \right)^{th}, \text{ i.e., } 291^{st} \text{ observation} = 15. \text{ Mean } \frac{8767}{581} = 15.09$$

Mode = 3 × 15 − 2 × 15.09 = 45 − 30.18 = 14.82

Remarks:

If the most repeated values, in the above analysis table, were not adjacent, the distribution would have been bi-modal, i.e., having two modes.

When Data are in the Form of a Grouped Frequency Distribution

The following steps are involved in the computation of mode from a grouped frequency distribution:

(i) *Determination of modal class:* It is the class in which mode of the distribution lies. If the distribution is regular, the modal class can be determined by inspection, otherwise, by method of grouping.

(ii) *Exact location of mode in a modal class (interpolation formula):* The exact location of mode, in a modal class, will depend upon the frequencies of the classes immediately preceding and following it. If these frequencies are equal, the mode would lie at the middle of the modal class interval. However, the position of mode would be to the left or to the right of the middle point depending upon whether the frequency of preceding class is greater or less than the frequency of the class following it. The exact location of mode can be done by the use of interpolation formula, developed below:
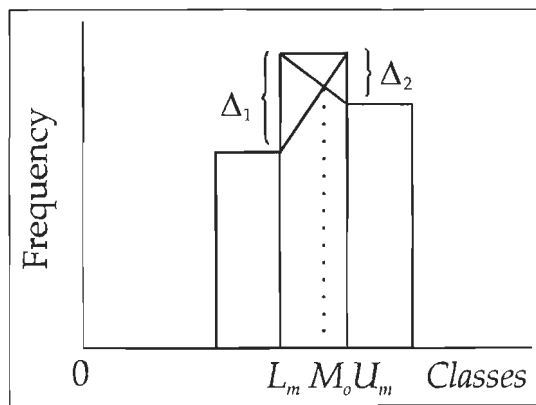
Figure 4.1: Location of Mode

Let the modal class be denoted by $L_m - U_m$, where L_m and U_m denote its lower and the upper limits respectively. Further, let f_m be its frequency and h its width. Also let f_1 and f_2 be the respective frequencies of the immediately preceding and following classes.

We assume that the width of all the class intervals of the distribution are equal. If these are not equal, make them so by regrouping under the assumption that frequencies in a class are uniformly distributed.

Make a histogram of the frequency distribution with height of each rectangle equal to the frequency of the corresponding class. Only three rectangles, out of the complete histogram, that are necessary for the purpose are shown in the above figure.

Let $\Delta_1 = f_m - f_1$ and $\Delta_2 = f_m - f_2$. Then the mode, denoted by M_o, will divide the modal class interval in the ratio $\dfrac{\Delta_1}{\Delta_2}$.

To derive a formula for mode, the point M_o in the figure, should be such that

$$\frac{M_0 - L_m}{U_m - M_o} = \frac{\Delta_1}{\Delta_2} \text{ or } M_o\Delta_2 - L_m\Delta_2 = U_m\Delta_1 - M_o\Delta_1$$

$$\Rightarrow \quad (\Delta_1 + \Delta_2)M_o = L_m\Delta_2 + U_m\Delta_1 = L_m\Delta_2 + (L_m + h)\Delta_1 \text{ (where } U_m = L_m + h)$$

$$= (\Delta_1 + \Delta_2) L_m + \Delta_1 h$$

Dividing both sides by $\Delta_1 + \Delta_2$, we have

$$M_o = L_m + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times h \qquad \qquad ....(1)$$

By slight adjustment, the above formula can also be written in terms of the upper limit (U_m) of the modal class.

$$M_o = U_m - h + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times h = U_m - \left[1 - \frac{\Delta_1}{\Delta_1 + \Delta_2} \right] \times h$$

$$= U_m - \left[\frac{\Delta_2}{\Delta_1 + \Delta_2} \times h \right] \qquad \qquad ....(2)$$

Replacing Δ_1 by $f_m - f_1$ and Δ_2 by $f_m - f_2$, the above equations can be written as

$$M_o = L_m + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h \qquad \qquad ....(3)$$

and $\quad M_o = U_m - \dfrac{f_m - f_2}{2f_m - f_1 - f_2} \times h \qquad \qquad ....(4)$

Note: The above formulae are applicable only to a unimodal frequency distribution.

Example: The monthly profits (in ₹) of 100 shops are distributed as follows:

Profit per Shop :	0-100	100-200	200-300	300-400	400-500	500-600
No. of Shops :	12	18	27	20	17	6

Determine the 'modal value' of the distribution graphically and verify the result by calculation.

Solution:

Since the distribution is regular, the modal class would be a class having the highest frequency. The modal class, of the given distribution, is 200-300.

Graphical Location of Mode

To locate mode we draw a histogram of the given frequency distribution. The mode is located as shown in the figure. From the figure, mode = ₹ 256.
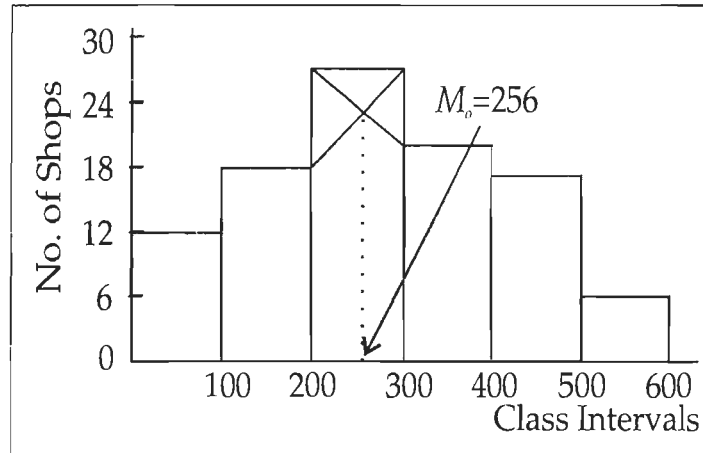
Determination of Mode by Interpolation Formula

Since the modal class is 200-300, $L_m = 200$, $\Delta_1 = 27 - 18 = 9$, $\Delta_2 = 27 - 20 = 7$ and $h = 100$.

$$\therefore \quad M_o = 200 + \frac{9}{9+7} \times 100 = ₹\ 256.25$$

Calculation of Mode when either Δ_1 or Δ_2 is negative

The interpolation formula, for the calculation of mode, is applicable only if both Δ_1 and Δ_2 are positive. If either Δ_1 or Δ_2 is negative, we use an alternative formula that gives only an approximate value of the mode.

We recall that the position of mode, in a modal class, depends upon the frequencies of its preceding and following classes, denoted by f_1 and f_2 respectively. If $f_1 = f_2$, the mode will be at the middle point which can be obtained by adding $\dfrac{f_2}{f_1 + f_2} \times h$ to the lower limit of the modal class or, equivalently, it can be obtained by subtracting $\dfrac{f_1}{f_1 + f_2} \times h$ from its upper limit. We may note that $\dfrac{f_1}{f_1 + f_2} = \dfrac{f_2}{f_1 + f_2} = \dfrac{1}{2}$ when $f_1 = f_2$.

Further, if $f_2 > f_1$, the mode will lie to the right of the mid-value of modal class and, therefore, the ratio $\dfrac{f_2}{f_1 + f_2}$ will be greater than $\dfrac{1}{2}$. Similarly, if $f_2 < f_1$, the mode will lie to the left of the mid-value of modal class and, therefore, the ratio $\dfrac{f_2}{f_1 + f_2}$ will be less than $\dfrac{1}{2}$. Thus, we can write an alternative formula for mode as:

$$\text{Mode} = L_m + \frac{f_2}{f_1 + f_2} \times h \quad \text{or equivalently,} \quad \text{Mode} = U_m - \frac{f_1}{f_1 + f_2} \times h.$$

Remarks: The above formula gives only an approximate estimate of mode vis-a-vis the interpolation formula.

4.10 GEOMETRIC MEAN

The geometric mean of a series of n positive observations is defined as the nth root of their product.

Calculation of Geometric Mean

Individual Series

If there are n observations, $X_1, X_2, \ldots\ldots X_n$, such that $X_i > 0$ for each i, their Geometric Mean (GM) is defined as

$$GM = (X_1.X_2...X_n)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} X_i\right)^{\frac{1}{n}}, \text{ where the symbol } \prod \text{ is used to denote the product}$$

of observations.

To evaluate GM, we have to use logarithms. Taking log of both sides, we have

$$\log (GM) = \frac{1}{n}\log(X_1.X_2...X_n)$$

$$= \frac{1}{n}\left[\log X_1 + \log X_2 + ... + \log X_n\right] = \frac{\Sigma \log X_i}{n}$$

Taking antilog of both sides, we have

$$GM = \text{antilog}\left[\frac{\Sigma \log X_i}{n}\right]$$

This result shows that the GM of a set of observations is the antilog of the arithmetic mean of their logarithms.

Ungrouped Frequency Distribution

If the data consists of observations $X_1, X_2, \ldots\ldots X_n$ with respective frequencies $f_1, f_2, \ldots\ldots$ f_n, where $\sum_{i=1}^{n} f_i = N$, the geometric mean is given by

$$GM = \left[\underbrace{X_1.X_1.....X_1}_{f_1 \text{ times}}\underbrace{X_2.....X_2}_{f_2 \text{ times}}.....\underbrace{X_n.....X_n}_{f_n \text{ times}}\right]^{\frac{1}{N}} = \left[X_1^{f_1}.X_2^{f_2}...X_n^{f_n}\right]^{\frac{1}{N}}$$

Taking log of both sides, we have

$$\log (GM) = \frac{1}{N}\left[\log X_1^{f_1} + \log X_2^{f_2} + ... + \log X_n^{f_n}\right]$$

$$= \frac{1}{N}\left[f_1 \log X_1 + f_2 \log X_2 + ... + f_n \log X_n\right] = \frac{\sum_{i=1}^{n} f_i \log X_i}{N}$$

or $GM = \text{antilog}\left(\frac{1}{N}\sum_{i=1}^{n} f_i \log X_i\right)$, which is again equal to the antilog of the arithmetic mean of the logarithm of observations.

Weighted Geometric Mean

If various observations, X_1, X_2, ...X_n, are not of equal importance in the data, weighted geometric mean is calculated. Weighted *GM* of the observations X_1, X_2, ...X_n with respective weights as w_1, w_2 ......w_n is given by :

$$GM = \text{antilog} \left[\frac{\Sigma w_i \log X_i}{\Sigma w_i} \right]$$, i.e., weighted geometric mean of observations is equal to

the antilog of weighted arithmetic mean of their logarithms.

Geometric Mean of the Combined Group

If G_1, G_2, ... G_k are the geometric means of k groups having n_1, n_2, ... n_k observations respectively, the geometric mean G of the combined group consisting of $n_1 + n_2 + ... + n_k$ observations is given by

$$G = \text{antilog} \left[\frac{n_1 \log G_1 + n_2 \log G_2 + ... + n_k \log G_k}{n_1 + n_2 + ... + n_k} \right] \text{antilog} \left[\frac{\Sigma n_i \log G_i}{\Sigma n_i} \right]$$

To determine the average rate of change of price for the entire period when the rate of change of prices for different periods are given

Let P_0 be the price of a commodity in the beginning of the first year. If it increases by $k_1\%$ in the first year, the price at the end of 1st year (or beginning of second year) is given by

$$P_1 = P_0 + P_0 \frac{k_1}{100} = P_0 \left(1 + \frac{k_1}{100} \right) = P_0(1 + r_1), \text{ where } r_1 = \frac{k_1}{100} \text{ denotes the rate of}$$

increase per rupee in first year. Similarly, if the price changes by $k_2\%$ in second year, the price at the end of second year is given by

$$P_2 = P_1 + P_1 \frac{k_2}{100} = P_1 \left(1 + \frac{k_2}{100} \right) = P_1(1 + r_2)$$

Replacing the value of P_1 as $P_0(1 + r_1)$, we can write

$$P_2 = P_0(1 + r_1)(1 + r_2)$$

Proceeding in this way, if $100r_i\%$ is the rate of change of price in the ith year, the price at the end of nth period, P_n, is given by

$$P_n = P_0(1 + r_1)(1 + r_2) ...... (1 + r_n) \qquad .... (1)$$

Further, let $100r \%$ per year be the average rate of increase of price that gives the price P_n at the end of n years. Therefore, we can write

$$P_n = P_0(1 + r)(1 + r) ...... (1 + r) = P_0(1 + r)^n \qquad .... (2)$$

Equating (1) and (2), we can write

$$(1 + r)^n = (1 + r_1)(1 + r_2) ...... (1 + r_n)$$

or $$(1 + r) = \left[(1 + r_1)(1 + r_2)...(1 + r_n) \right]^{\frac{1}{n}} \qquad .... (3)$$

This shows that $(1 + r)$ is geometric mean of $(1 + r_1)$, $(1 + r_2)$, ... and $(1 + r_n)$.

From (3), we get

$$r = \left[(1 + r_1)(1 + r_2)...(1 + r_n) \right]^{\frac{1}{n}} - 1 \qquad .... (4)$$

Note: Here r denotes the per unit rate of change. This rate is termed as the rate of increase or the *rate of growth* if positive and the rate of decrease or the *rate of decay* if negative.

Average Rate of Growth of Population

The average rate of growth of price, denoted by r in the above section, can also be interpreted as the average rate of growth of population. If P_0 denotes the population in the beginning of the period and P_n the population after n years, using Equation (2), we can write the expression for the average rate of change of population per annum as

$$r = \left(\frac{P_n}{P_0}\right)^{\frac{1}{n}} - 1.$$

Similarly, equation (4), given above, can be used to find the average rate of growth of population when its rates of growth in various years are given.

Remarks: The formulae of price and population changes, considered above, can also be extended to various other situations like growth of money, capital, output, etc.

Suitability of Geometric Mean for Averaging Ratios

It will be shown here that the geometric mean is more appropriate than arithmetic mean while averaging ratios.

Let there be two values of each of the variables x and y, as given below:

x	y	Ratio$\left(\dfrac{x}{y}\right)$	Ratio$\left(\dfrac{y}{x}\right)$
40	60	2 / 3	3 / 2
20	80	1 / 4	4

Now *AM* of (x/y) ratios $= \dfrac{\frac{2}{3}+\frac{1}{4}}{2} = \dfrac{11}{24}$ and the *AM* of (y/x) ratios $= \dfrac{\frac{3}{2}+4}{2} = \dfrac{11}{4}$. We note that their product is not equal to unity.

However, the product of their respective geometric means, *i.e.,* $\dfrac{1}{\sqrt{6}}$ and $\sqrt{6}$, is equal to unity.

Since it is desirable that a method of average should be independent of the way in which a ratio is expressed, it seems reasonable to regard geometric mean as more appropriate than arithmetic mean while averaging ratios.

Properties of Geometric Mean

1. As in case of arithmetic mean, the sum of deviations of logarithms of values from the log GM is equal to zero.

   This property implies that the product of the ratios of *GM* to each observation, that is less than it, is equal to the product the ratios of each observation to *GM* that is greater than it. For example, if the observations are 5, 25, 125 and 625, their *GM* = 55.9. The above property implies that

   $$\frac{55.9}{5} \times \frac{55.9}{25} = \frac{125}{55.9} \times \frac{625}{55.9}$$

2. Similar to the arithmetic mean, where the sum of observations remains unaltered if each observation is replaced by their *AM*, the product of observations remains unaltered if each observation is replaced by their *GM*.

4.11 HARMONIC MEAN

The harmonic mean of *n* observations, none of which is zero, is defined as the reciprocal of the arithmetic mean of their reciprocals.

Calculation of Harmonic Mean

Individual Series

If there are *n* observations $X_1, X_2, \ldots\ldots X_n$, their harmonic mean is defined as

$$HM = \cfrac{n}{\cfrac{1}{X_1} + \cfrac{1}{X_2} + \ldots + \cfrac{1}{X_n}} = \cfrac{n}{\sum\limits_{i=1}^{n} \cfrac{1}{X_i}}$$

Ungrouped Frequency Distribution

For ungrouped data, *i.e.*, each $X_1, X_2, \ldots\ldots X_n$, occur with respective frequency $f_1, f_2 \ldots\ldots f_n$, where $\Sigma f_i = N$ is total frequency, the arithmetic mean of the reciprocals of observations is given by $\dfrac{1}{N} \sum\limits_{i=1}^{n} \dfrac{f_i}{X_i}$.

Thus, $HM = \cfrac{N}{\Sigma \cfrac{f_i}{X_i}}$

Weighted Harmonic Mean

If $X_1, X_2, \ldots\ldots X_n$ are *n* observations with weights $w_1, w_2, \ldots\ldots w_n$ respectively, their weighted harmonic mean is defined as follows:

$$HM = \cfrac{\Sigma w_i}{\Sigma \cfrac{w_i}{X_i}}$$

Choice between Harmonic Mean and Arithmetic Mean

The harmonic mean, like arithmetic mean, is also used in averaging of rates like price per unit, kms per hour, work done per hour, etc., under certain conditions. To explain the method of choosing an appropriate average, consider the following illustration.

Let the price of a commodity be ₹ 3, 4 and 5 per unit in three successive years. If we take A.M. of these prices, *i.e.*, $\dfrac{3+4+5}{3} = 4$, then it will denote average price when equal quantities of the commodity are purchased in each year. To verify this, let us assume that 10 units of commodity are purchased in each year.

∴ Total expenditure on the commodity in 3 years = 10 × 3 + 10 × 4 + 10 × 5.

Also, average price

$$= \frac{\text{Total expenditure}}{\text{Total quantity purchased}} = \frac{10 \times 3 + 10 \times 4 + 10 \times 5}{10 + 10 + 10} = \frac{3 + 4 + 5}{3},$$

which is arithmetic mean of the prices in three years.

Further, if we take harmonic mean of the given prices, i.e., $\dfrac{3}{\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}}$, it will denote the average price when equal amounts of money are spent on the commodity in three years. To verify this let us assume that ₹ 100 is spent in each year on the purchase of the commodity.

$\therefore$ Average price $= \dfrac{\text{Total expenditure}}{\text{Total quantity purchased}} = \dfrac{300}{\dfrac{100}{3}+\dfrac{100}{4}+\dfrac{100}{5}} = \dfrac{3}{\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}}$

Next, we consider a situation where different quantities are purchased in the three years. Let us assume that 10, 15 and 20 units of the commodity are purchased at prices of ₹ 3, 4 and 5 respectively.

$$\text{Average price} = \dfrac{\text{Total expenditure}}{\text{Total quantity purchased}} = \dfrac{3\times10+4\times15+5\times20}{10+15+20},$$

which is weighted arithmetic mean of the prices taking respective quantities as weights.

Further, if ₹ 150, 200 and 250 are spent on the purchase of the commodity at prices of ₹ 3, 4 and 5 respectively, then

$$\text{Average price} = \dfrac{150+200+250}{\dfrac{150}{3}+\dfrac{200}{4}+\dfrac{250}{5}}, \text{ where } \dfrac{150}{3}, \dfrac{200}{4} \text{ and } \dfrac{250}{5} \text{ are the quantities}$$

purchased in respective situations.

The above average price is equal to the weighted harmonic mean of prices taking money spent as weights.

Therefore, to decide about the type of average to be used in a given situation, the first step is to examine the rate to be averaged. It may be noted here that a rate represents a ratio, e.g., $\text{Price} = \dfrac{\text{Money}}{\text{Quantity}}$, $\text{Speed} = \dfrac{\text{Distance}}{\text{Time}}$, $\text{Work done per hour} = \dfrac{\text{Work done}}{\text{Time taken}}$, etc.

We have seen above that arithmetic mean is appropriate average of prices $\left(\dfrac{\text{Money}}{\text{Quantity}}\right)$ when quantities, that appear in the denominator of the rate to be averaged, purchased in different situations are given. Similarly, harmonic mean will be appropriate when sums of money, that appear in the numerator of the rate to be averaged, spent in different situations are given.

To conclude, we can say that the average of a rate, defined by the ratio p/q, is given by the arithmetic mean of its values in different situations if the conditions are given in terms of q and by the harmonic mean if the conditions are given in terms of p. Further, if the conditions are same in different situations, use simple *AM* or *HM* and otherwise use weighted *AM* or *HM*.

Check Your Progress

Fill in the blanks:

1. _____ informs the buyer about the price of the goods and other expenses he has to pay.

2. Different sets of data can be compared by comparing their _____.

3. Arithmetic mean is the most commonly used average or measure of the central tendency applicable only in case of _____ data.

4. Median is a _____ average because its value depends upon the position of an item and not on its magnitude.

5. If the frequency distribution is regular, then mode is determined by the value corresponding to _____ frequency.

4.12 LET US SUM UP

- An invoice represents a receipt for the sale of goods and services. How an invoice is prepared also applies to the preparation of a sales order, a purchase order, a delivery receipt, and a price quotation.

- Invoice is prepared in duplicate. The original copy is sent to the buyer and the duplicate is kept for future reference.

- The various types of invoices used in business are local invoices; at station invoice; free on rails invoices; cost and freight invoices; cost, insurance and freight invoice; and franco invoice.

- A single value which can represent the whole set of data is called an average.

- If the average tends to lie or indicate the centre of the distribution it is called the measure of central tendency, or sometimes they locate the general position of the data so they are also called the measure of location.

- Different sets of data can be compared by comparing their averages.

- Arithmetic mean is the most commonly used average or measure of the central tendency applicable only in case of quantitative data; it is also simply called the "mean".

- The accuracy of arithmetic mean calculated for a grouped frequency distribution depends upon the validity of the fundamental assumption.

- The computation of weighted arithmetic is useful in many situations where different items are of unequal importance, e.g., the construction index numbers, computation of standardised death and birth rates, etc.

- Median of distribution is that value of the variant which divides it into two equal parts. In terms of frequency curve, the ordinate drawn at median divides the area under the curve into two equal parts.

- The values that divide a distribution into more than two equal parts are commonly known as partition values or fractiles.

- Mode is that value of the variate which occurs maximum number of times in a distribution and around which other items are densely distributed.

- When a frequency distribution is fairly regular, then mode is often determined by inspection.

- The harmonic mean of n observations, none of which is zero, is defined as the reciprocal of the arithmetic mean of their reciprocals.

4.13 LESSON END ACTIVITY

Collect all the invoices (approximately 10 nos.) of various purchases you or your family member done in past one month and analyse the details mentioned on those invoices. Based on that analysis prepare a report and discuss in class with your classmates.

4.14 KEYWORDS

Invoice: An invoice or a bill is a commercial document issued by a seller to a buyer, relating to a sale transaction and indicating the products, quantities, and agreed prices for products or services the seller had provided the buyer.

Average: An average is a single value within the range of the data that is used to represent all the values in the series.

Arithmetic Mean: Arithmetic mean is defined as the sum of observations divided by the number of observations.

Geometric Mean: The geometric mean of a series of n positive observations is defined as the nth root of their product.

Harmonic Mean: The harmonic mean of n observations, none of which is zero, is defined as the reciprocal of the arithmetic mean of their reciprocals.

Measure of Central Tendency: A measure of central tendency is a typical value around which other figures congregate.

Median: Median of distribution is that value of the variate which divides it into two equal parts.

Mode: Mode is that value of the variate which occurs maximum number of times in a distribution and around which other items are densely distributed.

Percentiles: Percentiles divide a distribution into 100 equal parts and there are, in all, 99 percentiles denoted as $P_1, P_2, \ldots\ldots P_{25}, \ldots\ldots P_{40}, \ldots\ldots P_{60}, \ldots\ldots P_{99}$ respectively.

Quartiles: The values of a variable that divide a distribution into four equal parts are called quartiles.

Weighted Arithmetic Mean: Weights are assigned to different items depending upon their importance, i.e., more important items are assigned more weight.

Weighted Geometric Mean: Weighted geometric mean of observations is equal to the antilog of weighted arithmetic mean of their logarithms.

4.15 QUESTIONS FOR DISCUSSION

1. Explain invoice and its uses.

2. Briefly explain the preparation of an invoice; also prepare a format invoice to support your explanation.

3. What are the functions of an average? Discuss the relative merits and demerits of various types of statistical averages.

4. Give the essential requisites of a measure of 'Central Tendency'. Under what circumstances would a geometric mean or a harmonic mean be more appropriate than arithmetic mean?

5. Compute arithmetic mean of the following series:

Marks	:	0-10	10-20	20-30	30-40	40-50	50-60
No. of Students	:	12	18	27	20	17	6

6. Calculate arithmetic mean of the following data:

Mid-Values	:	10	12	14	16	18	20
Frequency	:	3	7	12	18	10	5

7. Locate median of the following data:

65, 85, 55, 75, 96, 76, 65, 60, 40, 85, 80, 125, 115, 40

8. In a class of 16 students, the following are the marks obtained by them in statistics. Find out the lower quartile, upper quartile, seventh decile and thirty fifth percentile.

S. No.	:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Marks	:	5	12	17	23	28	31	37	41	42	49	54	58	65	68	17	77

9. Find out median from the following:

No. of Workers	:	1-5	6-10	11-15	16-20	21-25
No. of Factories	:	3	8	13	11	5

10. Find median from the following distribution:

X	:	1	2	3	4	5-9	10-14	15-19	20-25
f	:	5	10	16	20	30	15	8	6

11. Determine the mode of the following data:

58, 60, 31, 62, 48, 37, 78, 43, 65, 48

12. Find geometric mean from the following daily income (in ₹) of 10 families:

85, 70, 15, 75, 500, 8, 45, 250, 40 and 36.

Check Your Progress: Model Answer

1. Invoice
2. Averages
3. Quantitative
4. Positional
5. Maximum

4.16 SUGGESTED READINGS

Navaneetham P.A., *Business Mathematics*, Jai Publications, Trichy.

Parmanand Gupta, Om P. Chug, *Comprehensive Business Mathematics*, Laxmi Publications, 2005.

Joel Lerner, *Schaum's Outline of Basic Business Mathematics*, McGraw-Hill Professional, 1999.

Soper, *Mathematics for Economics and Business*, Blackwell Publishing, 2004.

UNIT III

LESSON

5

ELEMENTARY MATRICES

5.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Define matrix
- Explain Algebra of matrices
- List out the types of matrices
- Applying matrix Algebra to solve a system of linear equations

5.1 INTRODUCTION

In the previous lesson, we studied about the simultaneous equations and how to solve simultaneous equations using various methods.

In this lesson, we will discuss the elementary matrices. Matrices were first introduced to solve systems of linear equations. In 1750, G. Cramer gave a rule called Cramer's rule to solve the simultaneous equations. Sir Arthur Cayley introduced the theory of matrices.

If all the equations of a system or model are linear, then matrix algebra provides an efficient method of their solution than the traditional method of elimination of variables.

5.2 INTRODUCTION TO MATRIX

Matrices originated as mere stores of information but, at present, have found very wide applications. They play a very vital role not only in mathematics but also in communication theory, network analysis, theory of structures, quantum mechanics, biology, sociology, economics, psychology, statistics, etc.

5.2.1 Meaning of Matrix

A matrix is an array (or arrangement) of certain numbers along rows and columns. Horizontal arrangements are called rows and vertical arrangements are called columns. The numbers are called the elements.

Example: $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 7 \\ -1 & 4 & 10 \end{bmatrix}$

$C = \begin{bmatrix} -3 & 5 \\ 0 & 6 \\ 7 & -12 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 10 & 7 \\ 11 & -2 & 8 \end{bmatrix}$

5.2.2 Order of a Matrix

If a matrix has m rows and n columns then the matrix is said to be of order $m \times n$.

In the above examples, A is of order 2×2, B is of order 2×3, C is of order 3×2, D is of order 3×3.

5.2.3 Equality of Matrices

Two matrices A and B are said to be equal if they are of the same order and the corresponding elements of A and B are equal:

Example: (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 8 & -7 & 4 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 8 & -7 & 4 \end{bmatrix}_{2 \times 3}$

The orders are same and the corresponding elements are equal.

$\therefore \quad A = B$

(2) If $A = \begin{bmatrix} x & 2 \\ 0 & y \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$ then

$A = B \longleftrightarrow x = -1$ and $y = 4$.

5.3 TYPES OF MATRICES

In this section, we will learn the different types of matrices: rectangular matrix, square matrix, diagonal matrix, scalar matrix, unit matrix (or identity matrix), null matrix (or zero matrix) and row matrix. The different types of matrices are as follows:

1. *Rectangular Matrix:* A matrix of order $m \times n$ is called a rectangular matrix.

Example: $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 7 & 0 \end{bmatrix}_{2 \times 3}$ is a rectangular matrix.

2. **Square Matrix:** A matrix in which the number of rows is equal to the number of columns (i.e., $m \times m$ matrix) is called square matrix.

   Example: $A = \begin{bmatrix} -1 & 4 \\ 7 & 0 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 2 & 6 & -11 \\ 5 & 0 & 8 \\ 7 & -4 & 1 \end{bmatrix}_{3 \times 3}$

3. **Diagonal Matrix:** A square matrix in which all the elements except the principal diagonal elements are zero, is called a diagonal matrix.

   Example: $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ are diagonal matrices.

4. **Scalar Matrix:** A diagonal matrix in which all the principal diagonal elements are equal, is called a scalar matrix.

   Example: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

5. **Unit Matrix (or Identity Matrix):** A scalar matrix in which all the principal diagonal elements are equal to 1 is called a unit matrix.

   Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$

   A unit (or identity) matrix is denoted by I.

6. **Null Matrix (or Zero Matrix):** A matrix in which all the elements are 0 is called a null matrix. A null (or zero) matrix is denoted by 0.

   Example: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$

   $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3}$

7. **Row Matrix:** A matrix having only one row and any number of columns (i.e., $1 \times n$ matrix) is called a row matrix.

   Example: $\begin{bmatrix} 1 & -3 & 0 \end{bmatrix}_{1 \times 3}$

8. **Column Matrix:** A matrix having only one column and any number of rows (i.e., $m \times 1$ matrix) is called a column matrix.

   Example: $\begin{bmatrix} 2 \\ -7 \\ 0 \end{bmatrix}_{3 \times 1}$

Example: Keerthi buys 8 dozen of pens, 10 dozen of pencils and 4 dozen of rubber. Pens cost ₹ 18 per dozen, pencils ₹ 9 per dozen and rubber ₹ 6 per dozen. Represent the quantities bought by a row matrix and prices by a column matrix and hence obtain the total cost.

Solution:

Let A be the row matrix of quantities and B be the column matrix of prices.

$$\therefore \quad A = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix} \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix} = [144 + 90 + 24] = [258]$$

Total cost is ₹ 258.

5.4 OPERATION OF MATRICES

There are various operations of matrix such as addition, subtraction, multiplication of two matrices as follows:

5.4.1 Addition of Matrices

Addition of two matrices A and B is defined if and only if they are of the same order.

If A and B are matrices of the same order then their sum $A + B$ is obtained by adding the corresponding elements of A and B.

Example: $\quad A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & 10 \end{bmatrix}_{2\times3}, B = \begin{bmatrix} 0 & -1 & 7 \\ 5 & 8 & 15 \end{bmatrix}_{2\times3}$

then $\quad A + B = \begin{bmatrix} 2+0 & -1+(-1) & 0+7 \\ 4+5 & 7+8 & 10+15 \end{bmatrix}_{2\times3}$

$$= \begin{bmatrix} 2 & -2 & 7 \\ 9 & 15 & 25 \end{bmatrix}_{2\times3}$$

$$B + A = \begin{bmatrix} 0+2 & (-1)+(-1) & 7+0 \\ 5+4 & 8+7 & 15+10 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 2 & -2 & 7 \\ 9 & 15 & 25 \end{bmatrix}_{2\times3}$$

$$\therefore \quad A + B = B + A.$$

Properties of Matrix Addition

(a) *Matrix Addition is Commutative:* If A and B are two comparable matrices, then $A + B = B + A$. As in case of scalars, this property implies that it is immaterial whether B is added to A or vice-versa.

(b) *Matrix Addition is Associative:* If A, B and C are three comparable matrices, then

$$A + (B + C) = C + (A + B) = B + (A + C)$$

This property, like the case of scalars, implies that it is immaterial whether $(B + C)$ is added to A or $(A + B)$ is added to C or $(A + C)$ is added to B. According to this property, to add three matrices, we can first add any two of them and then add this to the third matrix.

(c) **Scalar Multiplication is Distributive over Matrix Addition:** Given a scalar k and matrices A and B of the same order, we can write $k(A + B) = kA + kB$.

This property implies that k times the sum of two matrices is equal to the sum of k times of each matrix.

(d) **Existence of Additive Identity:** The null matrix O is said to be additive identity of any matrix A, of the same order because $A + O = A$, i.e., the identity of A does not change when O is added to it.

(e) **Existence of Additive Inverse:** If A and B are two comparable matrices, such that $A + B = O$, where O is a null matrix of the same order as that of A or B, then B is said to be additive inverse of A and vice-versa. Obviously, $B = -A$ i.e., negative of a matrix is its additive inverse.

(f) **Matrix Addition admits Cancellation Law:** Matrix addition, like scalars, admits cancellation law. If A, B and C are three comparable matrices such that $A + C = B + C$, then $A = B$. To show this, we add $-C$ to both sides of the above equation. Thus, $A + C - C = B + C - C$ or $A + (C - C) = B + (C - C)$ or $A + O = B + O$ or $A = B$.

5.4.2 Subtraction of Matrices

Subtraction of two matrices A and B is defined if and only if they are of the same order.

If A and B are matrices of the same order then their difference $A - B$ is obtained by subtracting the elements of B by the corresponding elements of A.

Example: If $A = \begin{bmatrix} 2 & 0 \\ 5 & -1 \\ 4 & 7 \end{bmatrix}_{3 \times 2}$, $B = \begin{bmatrix} -1 & 8 \\ 9 & 0 \\ 7 & -3 \end{bmatrix}_{3 \times 2}$

then $A - B = \begin{bmatrix} 2 - (-1) & 0 - 8 \\ 5 - 9 & -1 - 0 \\ 4 - 7 & 7 - (-3) \end{bmatrix}_{3 \times 2}$

$= \begin{bmatrix} 3 & -8 \\ -4 & -1 \\ -3 & 10 \end{bmatrix}_{3 \times 2}$

$B - A = \begin{bmatrix} -1 - 2 & 8 - 0 \\ 9 - 5 & 0 - (-1) \\ 7 - 4 & -3 - 7 \end{bmatrix}_{3 \times 2}$

$= \begin{bmatrix} -3 & 8 \\ 4 & 1 \\ 3 & -10 \end{bmatrix}_{3 \times 2}$

$\therefore \quad A - B \neq B - A.$

5.4.3 Scalar Multiplication

If A is a matrix of order $m \times n$ and k is a scalar, then the matrix kA is obtained by multiplying all the elements of A by k.

Example: If $A = \begin{bmatrix} 2 & -5 & 4 \\ -7 & 3 & 10 \end{bmatrix}_{2 \times 3}$ then

$$2A = \begin{bmatrix} 4 & -10 & 8 \\ -14 & 6 & 20 \end{bmatrix}_{2 \times 3}$$

and

$$\frac{1}{2} A = \begin{bmatrix} 1 & \dfrac{-5}{2} & 2 \\ -\dfrac{7}{2} & \dfrac{3}{2} & 5 \end{bmatrix}_{2 \times 3}$$

Example: Two businessmen are trading in shares have three banking company shares as shown in the following table:

Merchant	Vijaya Bank	Canara Bank	Corporation Bank
Mr. Jain	200	100	300
Mr. Gupta	250	150	100

The approximate prices (in ₹) of three banking company shares in three stock exchange market are given below.

	Vijaya Bank	Canara Bank	Corporation Bank
Bangalore	39	40	38
Bombay	40	50	35
New Delhi	35	45	42

In which market each of the above businessmen has to sell their shares to get maximum receipt. Solve by matrix multiplication method.

Solution:

$$\text{Let } A = \begin{bmatrix} VB & CB & Cor.B \\ 200 & 100 & 300 \\ 250 & 150 & 100 \end{bmatrix}_{2 \times 3} \begin{matrix} \\ Jain \\ Gupta \end{matrix}$$

$$B = \begin{bmatrix} Bang. & Bomb. & N.Delhi \\ 39 & 40 & 35 \\ 40 & 50 & 45 \\ 38 & 35 & 42 \end{bmatrix}_{3 \times 3} \begin{matrix} \\ VB \\ CB \\ Cor.B \end{matrix}$$

$$\therefore \quad AB = \begin{bmatrix} 200 & 100 & 300 \\ 250 & 150 & 100 \end{bmatrix} \begin{bmatrix} 39 & 40 & 35 \\ 40 & 50 & 45 \\ 38 & 35 & 42 \end{bmatrix}$$

$$= \begin{bmatrix} 200 \times 39 + 100 \times 40 + 300 \times 38 & 200 \times 40 + 100 \times 50 + 300 \times 35 \\ 250 \times 39 + 150 \times 40 + 100 \times 38 & 250 \times 40 + 150 \times 50 + 100 \times 35 \end{bmatrix}$$

$$\begin{matrix} 200 \times 35 + 100 \times 45 + 300 \times 42 \\ 250 \times 35 + 150 \times 45 + 100 \times 42 \end{matrix}$$

$$= \begin{bmatrix} 7800+4000+11400 & 8000+5000+10500 & 7000+4500+12600 \\ 9750+6000+3800 & 10000+7500+3500 & 8950+6750+4200 \end{bmatrix}$$

$$= \begin{bmatrix} \text{Bang.} & \text{Bomb.} & \text{ND} \\ 23200 & 23500 & 24100 \\ 19550 & 21000 & 19900 \end{bmatrix}_{2\times3} \begin{matrix} \\ \text{Jain} \\ \text{Gupta} \end{matrix}$$

Jain has to sell his shares in New Delhi and Gupta has to sell his shares in Bombay to get maximum receipt.

5.4.4 Multiplication of Matrices

Multiplication of matrices is defined if and only if the number of columns of the first matrix is equal to the number of rows of the second matrix. i.e., if A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$ then only AB is defined and AB will be a matrix of order $m \times p$. The mode of multiplication is always row $\times$ column.

Let
$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2\times3} \text{ and } B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}_{3\times2}$$

then
$$AB = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1x_1 + b_1x_2 + c_1x_3 & a_1y_1 + b_1y_2 + c_1y_3 \\ a_2x_1 + b_2x_2 + c_2x_3 & a_2y_1 + b_2y_2 + c_2y_3 \end{bmatrix}_{2\times2}$$

Also
$$BA = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1a_1 + y_1a_2 & x_1b_1 + y_1b_2 & x_1c_1 + y_1c_2 \\ x_2a_1 + y_2a_2 & x_2b_1 + y_2b_2 & x_2c_1 + y_2c_2 \\ x_3a_1 + y_3a_2 & x_3b_1 + y_3b_2 & x_3c_1 + y_3c_2 \end{bmatrix}_{3\times3}$$

From these, we observe that $AB \neq BA$. Hence in general $AB \neq BA$.

In case of addition of matrix $A + B = B + A$, but in case of matrix multiplication $AB \neq BA$.

Properties of Matrix Multiplication

(a) *Matrix Multiplication is not, in general, Commutative:* If A and B are two matrices conformable for the product AB and BA, then in general $AB \neq BA$.

In view of this fact, the terms, pre-multiply and post-multiply, are often used to specify the order of multiplication. For example, BA can be obtained either by pre-multiplication of A by B or by post-multiplication of B by A. Matrix multiplication is, however, commutative under the following exceptional circumstances:

(i) When one matrix is a square and the other an identity matrix.

(ii) When one matrix is inverse of the other.

(b) *Matrix Multiplication is Associative:* If A, B and C are three matrices such that conformability conditions for the product ABC are satisfied, then $ABC = (AB)C = A(BC)$.

This Property implies that, to get the product ABC, it is immaterial whether A is post-multiplied by B and then the result is post-multiplied by C or B is post-multiplied by C and then the result is pre-multiplied by A etc.

(c) *Matrix Multiplication is Distributive over Addition:* Distributive law is $A(B + C) = AB + AC$, provided that conformability conditions for multiplication as well as addition are satisfied.

(d) *Existence of Multiplicative Identity Matrix:* The identity matrix I is said to be the multiplicative identity matrix or simply the identity matrix of any matrix, say A of the same order, because $IA = AI = A$, i.e., the identity of A remains unaffected as a result of its multiplication by I.

(e) *Existence Multiplicative Inverse:* Two scalars, a and b are said to be inverse of each other if $ab = ba = 1$. In a similar way, two square matrices of the same order, say A and B, are said to be multiplicative inverse of each other if $AB = BA = I$.

(f) *Matrix Multiplication does not always admit Cancellation Law:* According to this law, if $AB = AC$, it does not necessarily imply that $B = C$. To illustrate this, consider:

$$A = \begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 8 & 7 \\ -6 & -5 \end{bmatrix}$$

Now, $$AB = \begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}$$

And $$AC = \begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix}\begin{bmatrix} 8 & 7 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}$$

We note that $AB = AC$ while $B \ne C$.

(g) *The product of two Matrices can be a Null Matrix with none of them being Null Matrix.*

To illustrate this, let $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 9 & 15 \end{bmatrix}$

Then, $$AB = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 9 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Notes:

(i) If $AB = O$, then it is not necessary that BA will also be a null matrix.

(ii) Properties (*f*) and (*g*) hold true only for singular matrices.

(h) *Positive Integral Powers of Matrices:* If A is a square matrix, then the product $A.A$ is denoted by A^2. Further, we can write $A^2A = (A.A)A = A^3$. We should note that, like real numbers, we have $A^nA^m = A^mA^n = A^{m-n}$ for matrices.

Notes:

(i) Using the above equation, it is possible to find some matrices which commute with respect to multiplication. If we let $B = A^n$ and $C = A^m$, then obviously, $BC = CB$.

(ii) A matrix A is said to be Idempotent if $A = A^2 = A^3 = \ldots\ldots = A^m$.

(iii) A matrix A is said to be Involuntary if $A^2 = I$.

(iv) A matrix A is said to be Nilpotent if $A^n = O$, where n (the index of matrix A) is any positive integer.

Example: Two oil merchants have the following stock of oil (in kg):

Merchant	Groundnut	Sunflower	Coconut
A	250	300	150
B	400	350	100

The approximate prices (in ₹ per kg) of three types of oil in 3 markets are:

Market	Groundnut	Sunflower	Coconut
X	70	50	150
Y	60	55	140
Z	55	60	132

In which market each of the above businessmen has to sell his stocks to get maximum receipt? Solve by matrix multiplication method.

Solution:

$$\text{Let } P = \begin{bmatrix} G.N. & S.F. & C.N \\ 250 & 300 & 150 \\ 400 & 350 & 100 \end{bmatrix} \begin{matrix} \\ A \\ B \end{matrix}$$

$$Q = \begin{bmatrix} X & Y & Z \\ 70 & 60 & 55 \\ 50 & 55 & 60 \\ 150 & 140 & 132 \end{bmatrix} \begin{matrix} \\ GN \\ SF \\ CN \end{matrix}$$

$$\therefore \quad PQ = \begin{bmatrix} 250 & 300 & 150 \\ 400 & 350 & 100 \end{bmatrix} \begin{bmatrix} 70 & 60 & 55 \\ 50 & 55 & 60 \\ 150 & 140 & 132 \end{bmatrix}$$

$$= \begin{bmatrix} 250 \times 70 + 300 \times 50 + 150 \times 150 & 250 \times 60 + 300 \times 55 + 150 \times 140 \\ 400 \times 70 + 350 \times 50 + 100 \times 150 & 400 \times 60 + 350 \times 55 + 100 \times 140 \end{bmatrix}$$

$$\begin{bmatrix} 250 \times 55 + 300 \times 60 + 150 \times 132 \\ 400 \times 55 + 350 \times 60 + 100 \times 132 \end{bmatrix}$$

$$= \begin{bmatrix} 17500 + 15000 + 22500 & 15000 + 16500 + 21000 \\ 28000 + 17500 + 15000 & 24000 + 19250 + 14000 \end{bmatrix}$$

$$\begin{bmatrix} 13750 + 18000 + 19800 \\ 22000 + 21000 + 13200 \end{bmatrix}$$

$$= \begin{bmatrix} X & Y & Z \\ 55000 & 52500 & 51550 \\ 60500 & 57250 & 56200 \end{bmatrix} \begin{matrix} \\ A \\ B \end{matrix}$$

A has to sell his oil stock in market X and B also has to sell his oil stock in market X to get maximum receipt.

Example: At Bangalore merchant A has 300 bags of rice, 600 bags of wheat and 800 bags of Ragi and another merchant B has 250 bags, 700 bags and 1000 bags of same food grains. The prices (in ₹) at three cities are:

Place	Rice	Wheat	Ragi
Mysore	100	90	80
Mangalore	110	80	70
Kolar	120	70	80

To which city, each merchant will send his supply in order to get maximum gross receipts? Solve by matrix multiplication method.

Solution:

$$\text{Let } P = \begin{bmatrix} 300 & 600 & 800 \\ 250 & 700 & 1000 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$$

$$Q = \begin{bmatrix} 100 & 110 & 120 \\ 90 & 80 & 70 \\ 80 & 70 & 80 \end{bmatrix}$$

$$= \begin{bmatrix} 30000 + 54000 + 64000 & 33000 + 48000 + 56000 \\ 25000 + 63000 + 80000 & 27500 + 56000 + 70000 \end{bmatrix}$$

$$\begin{matrix} 36000 + 42000 + 64000 \\ 30000 + 49000 + 80000 \end{matrix}$$

$$= \begin{bmatrix} \text{Mysore} & \text{Mangalore} & \text{Kolar} \\ 148000 & 137000 & 142000 \\ 168000 & 153500 & 179000 \end{bmatrix} \begin{matrix} \\ A \\ B \end{matrix}$$

A has to send his supply to Mysore and B has to send his supply to Kolar to get maximum gross receipts.

5.5 TRANSPOSE OF A MATRIX

If A is a matrix of order $m \times n$, then the matrix obtained by interchanging the rows and columns is called the transpose of A and is denoted by A' or A^T. A' will be a matrix of order $n \times m$.

Example: If $A = \begin{bmatrix} 1 & 3 & -5 \\ -4 & 7 & 8 \end{bmatrix}_{2 \times 3}$ then $A' = \begin{bmatrix} 1 & -4 \\ 3 & 7 \\ -5 & 8 \end{bmatrix}_{3 \times 2}$

Worked examples on addition, substraction, scalar multiplication, multiplication of matrices and transpose of matrices.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}$, find

$$A + B, A - B, 2A + 3B, 2A - 3B, 5A + B, A - 7B$$

Solution:

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+0 \\ 3+4 & 4+7 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 7 & 11 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1+1 & 2-0 \\ 3-4 & 4-7 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -3 \end{bmatrix}$$

$$2A + 3B = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 2-3 & 4-0 \\ 6+12 & 8+21 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 18 & 29 \end{bmatrix}$$

$$2A - 3B = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 3\begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 2+3 & 4-0 \\ 6-12 & 8-21 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -6 & -13 \end{bmatrix}$$

$$5A + B = 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 5-1 & 10+0 \\ 15+4 & 20+7 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 19 & 27 \end{bmatrix}$$

$$A - 7B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 7\begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1+7 & 2-0 \\ 3-28 & 4-49 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ -25 & -45 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} -1 & 5 \\ 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -8 \end{bmatrix}$

Verify $(A+B)' = A' + B'$ and $(A-B)' = A' - B'$

Solution:

$$A = \begin{bmatrix} -1 & 5 \\ 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & -8 \end{bmatrix}$$

$$\therefore \quad A' = \begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix}, B' = \begin{bmatrix} 2 & 3 \\ 1 & -8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -1 & 5 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -8 \end{bmatrix} = \begin{bmatrix} -1+2 & 5+1 \\ 0+3 & 6-8 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 3 & -2 \end{bmatrix}$$

$$\therefore \quad (A+B)' = \begin{bmatrix} 1 & 3 \\ 6 & -2 \end{bmatrix} \qquad \qquad ...(1)$$

$$A' + B' = \begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & -8 \end{bmatrix} = \begin{bmatrix} -1+2 & 0+3 \\ 5+1 & 6-8 \end{bmatrix}$$

$$\Rightarrow \quad A' + B' = \begin{bmatrix} 1 & 3 \\ 6 & -2 \end{bmatrix} \qquad \qquad ...(2)$$

From (1) and (2) ; $(A+B)' = A' + B'$

$$A - B = \begin{bmatrix} -1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -8 \end{bmatrix} = \begin{bmatrix} -1-2 & 5-1 \\ 0-3 & 6+8 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -3 & 14 \end{bmatrix}$$

$$\therefore \quad (A-B)' = \begin{bmatrix} -3 & -3 \\ 4 & 14 \end{bmatrix} \qquad \qquad ...(3)$$

$$A' - B' = \begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & -8 \end{bmatrix} = \begin{bmatrix} -1-2 & 0-3 \\ 5-1 & 6+8 \end{bmatrix}$$

$$\Rightarrow \quad A' - B' = \begin{bmatrix} -3 & -3 \\ 4 & 14 \end{bmatrix} \qquad \qquad ...(4)$$

From (3) and (4) $(A - B)' = A' - B'$.

Example: Find the matrices A and B given that

$$2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} \text{ and } 3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

Solution:

$$2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} \qquad \qquad ...(1)$$

$$3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix} \qquad \qquad ...(2)$$

Multiply (1) by 2

$$\therefore \quad (1) \times 2 \Rightarrow 4A + 2B = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

$$(2) \Rightarrow 3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

Subtracting, we get $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}$

Substituting this in (1), we get

$$2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix} + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\therefore \quad B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 10 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & 3-0 & 1-2 \\ 1-0 & 4-10 & 0+10 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ find AB and BA.

Solution:

$$AB = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1) + (-2)(1) & 2(1) + (-2)1 \\ -2(1) + 2(1) & (-2)1 + 2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 2-2 \\ -2+2 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+1(-2) & 1(-2)+1(2) \\ 1(2)+1(-2) & 1(-2)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & -2+2 \\ 2-2 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: Simplify $\begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1)+(-1)0 & 1(0)+(-1)(-8) \\ (-2)(-1)+(-3)0 & (-2)(0)+(-3)(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -1+0 & 0+8 \\ 2+0 & 0+24 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 2 & 24 \end{bmatrix}.$$

Example: Find x and y if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2(4)+3(-1) \\ 4(4)+5(-1) \end{bmatrix} = \begin{bmatrix} 8-3 \\ 16-5 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Equating the corresponding elements, we get

$x = 5, \ y = 11.$

Example: If $A = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$, prove that $A^2 - 4I = 0$ where I is the unit matrix of second order.

Solution:

$$A^2 = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0+4 & 0+0 \\ 0+0 & 4+0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$-4I = -4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\therefore \quad A^2 - 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & 0+0 \\ 0+0 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\therefore \quad A^2 - 4I = 0.$$

Example: If $A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$, prove that $A^2 - 8A + 13I = 0$.

Solution:

$$A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 9+2 & 3+5 \\ 6+10 & 2+25 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 16 & 27 \end{bmatrix}$$

$$-8A = -8\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -24 & -8 \\ -16 & -40 \end{bmatrix}$$

$$13I = 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

Adding, $A^2 - 8A + 13I = \begin{bmatrix} 11 & 8 \\ 16 & 27 \end{bmatrix} + \begin{bmatrix} -24 & -8 \\ -16 & -40 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

$$= \begin{bmatrix} 11-24+13 & 8-8+0 \\ 16-16+0 & 27-40+13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad A^2 - 8A + 13I = 0.$$

Example: If $A = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 1 & -1 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution:

$$A' = \begin{bmatrix} 5 & 0 \\ 2 & 7 \\ -1 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 7 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5(-3)+2(4)+(-1)1 & 5(1)+2(7)+(-1)(-1) \\ 0(-3)+7(4)+1(1) & 0(1)+7(7)+1(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -15+8-1 & 5+14+1 \\ 0+28+1 & 0+49-1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -8 & 20 \\ 29 & 48 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} -8 & 29 \\ 20 & 48 \end{bmatrix} \qquad \qquad ...(1)$$

$$B'A' = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 7 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 7 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-3)5 + 4(2) + 1(-1) & (-3)0 + 4(7) + 1(1) \\ 1(5) + 7(2) + (-1)(1) & 1(0) + 7(7) + (-1)1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 + 8 - 1 & 0 + 28 + 1 \\ 5 + 14 + 1 & 0 + 49 - 1 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} -8 & 29 \\ 20 & 48 \end{bmatrix} \qquad \qquad ...(2)$$

From (1) and (2), $(AB)' = B'A'$

Example: If $A = \begin{bmatrix} 2 & -4 \\ 4 & 1 \end{bmatrix}$, find $(A')^2 - A' + I$.

Solution:

$$A = \begin{bmatrix} 2 & -4 \\ 4 & 1 \end{bmatrix}, \quad \therefore A' = \begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}$$

$$(A')^2 = A'A' = \begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 16 & 8 + 4 \\ -8 - 4 & -16 + 1 \end{bmatrix}$$

$$\therefore \quad (A')^2 = \begin{bmatrix} -12 & 12 \\ -12 & -15 \end{bmatrix}$$

$$-A' = \begin{bmatrix} -2 & -4 \\ 4 & -1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad (A')^2 - A' + I = \begin{bmatrix} -12 - 2 + 1 & 12 - 4 + 0 \\ -12 + 4 + 0 & -15 - 1 + 1 \end{bmatrix} = \begin{bmatrix} -13 & 8 \\ -8 & -15 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$, find A^3.

Solution:

$$A^2 = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 12-4 \\ 3-1 & 4+1 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2.A = \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 39+8 & 52-8 \\ 6+5 & 8-5 \end{bmatrix}$$

$$\therefore \quad A^3 = \begin{bmatrix} 47 & 44 \\ 11 & 3 \end{bmatrix}$$

Example: Find the product of $A = [x \quad y \quad z], B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Solution:

A is a matrix of order 1×3, B is of 3×3 and C is of 3×1.

$\therefore \quad ABC$ will be a matrix of order 1×1.

$$AB = [x \quad y \quad z]_{1 \times 3}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3 \times 3}$$

$$= \begin{bmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \end{bmatrix}_{1 \times 3}$$

$$(AB)C = \begin{bmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[(ax+hy+gz)\,x + (hx+by+fz)y + (gx+fy+cz)z \right]$$

$$= [ax^2 + hxy + gzx + hxy + by^2 + fyz + gzx + fyz + cz^2]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx]_{1 \times 1}$$

Example: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$.

Solution:

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$-4A = \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix}$$

$$-5I = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Adding $\Rightarrow A^2 - 4A - 5I = \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore \quad A^2 - 4A - 5I = 0.$

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 7 & -5 \\ 0 & 18 & 10 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 & 8 \\ 0 & 6 & 3 \\ 5 & 7 & 11 \end{bmatrix}$, verify that $2(A+B) = 2A + 2B$.

Solution:

$$A + B = \begin{bmatrix} 1-2 & -2+4 & 3+8 \\ 4+0 & 7+6 & -5+3 \\ 0+5 & 8+7 & 10+11 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 11 \\ 4 & 13 & -2 \\ 5 & 15 & 21 \end{bmatrix}$$

$$2(A+B) = 2\begin{bmatrix} -1 & 2 & 11 \\ 4 & 13 & -2 \\ 5 & 15 & 21 \end{bmatrix}$$

$$2(A+B) = \begin{bmatrix} -2 & 4 & 22 \\ 8 & 26 & -4 \\ 10 & 30 & 42 \end{bmatrix} \qquad \ldots(1)$$

$$2A = 2\begin{bmatrix} 1 & -2 & 3 \\ 4 & 7 & -5 \\ 0 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 6 \\ 8 & 14 & -10 \\ 0 & 16 & 20 \end{bmatrix}$$

$$2B = 2\begin{bmatrix} -2 & 4 & 8 \\ 0 & 6 & 3 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 16 \\ 0 & 12 & 6 \\ 10 & 14 & 22 \end{bmatrix}$$

$$\therefore \quad 2A + 2B = \begin{bmatrix} 2 & -4 & 6 \\ 8 & 14 & -10 \\ 0 & 16 & 20 \end{bmatrix} + \begin{bmatrix} -4 & 8 & 16 \\ 0 & 12 & 6 \\ 10 & 14 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 2-4 & -4+8 & 6+16 \\ 8+0 & 14+12 & -10+6 \\ 0+10 & 16+14 & 20+22 \end{bmatrix}$$

$$2A + 2B = \begin{bmatrix} -2 & 4 & 22 \\ 8 & 26 & -4 \\ 10 & 30 & 42 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), $2(A+B) = 2A + 2B$.

Example: If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, prove that $A^2 + B^2 - 2I = 0$.

Solution:

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-2I = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Adding, $A^2 + B^2 - 2I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 1+1-2 & 0+0+0 \\ 0+0+0 & 1+1-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad A^2 + B^2 - 2I = 0.$$

Example: Solve for x, y, z given that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 8+15+24 \\ 16+25+36 \\ 28+40+54 \end{bmatrix} = \begin{bmatrix} 47 \\ 77 \\ 122 \end{bmatrix}$$

$\therefore$ $x = 47$, $y = 77$, $z = 122$.

Example: Evaluate $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 21 \\ 18 & 15 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-27 & 3+2-24 & 5+4-21 \\ 2+6-18 & 4+8-15 & 6+10-12 \end{bmatrix} = \begin{bmatrix} -26 & -19 & -12 \\ -10 & -3 & 4 \end{bmatrix}$$

Example: Find the matrix X such that $A + 2X = B$ given that

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -5 \\ 4 & 9 \end{bmatrix}.$$

Solution:

$A + 2X = B$

$\therefore$ $2X = B - A$

$\therefore$ $X = \dfrac{1}{2}(B - A)$

$$= \frac{1}{2}\left\{ \begin{bmatrix} 2 & -5 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} 5 & -1 \\ 4 & 7 \end{bmatrix} \right\} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ 0 & 2 \end{bmatrix}$$

i.e., $X = \begin{bmatrix} -\dfrac{3}{2} & -2 \\ 0 & 1 \end{bmatrix}$

5.6 LET US SUM UP

● In this lesson, we have studied the concepts of matrices and its importance in solving real world problems of business.

● While a matrix is an array of numbers arranged into certain number of rows and columns, a determinant is a scalar.

● Unlike scalars, the basic operations such as addition, subtraction and multiplication can be performed only if certain conditions are satisfied by the participating matrices. Like scalars division of one matrix by another is not defined.

5.7 LESSON END ACTIVITY

Analyze the significance of minor and cofactors calculation. The cost of 5 kg of rice, 2 kg of sugar and 5 kg of wheat is ₹ 23. The cost of 4 kg of rice, 4 kg of sugar and 2 kg of wheat is ₹ 19. The cost of 3 kg of rice, 2 kg of sugar and 4 kg of wheat is ₹ 18. Find the rate per kg of each of these commodities.

5.8 KEYWORDS

Cofactor: A cofactor of an element a_{ij}, denoted by C_{ij}, is its minor with appropriate sign.

Column Matrix: A matrix having only one column.

Matrix: An array of numbers arranged in certain numbers of rows and columns.

Minor: A minor of an element a_{ij} denoted by M_{ij}, is a sub-determinant of $|A|$ obtained by deleting its ith row and j^{th} column.

Rank: A non-negative integral value that represents the minimum number of non-dependent rows or columns in a matrix.

Rectangular Matrix: A matrix consisting of m rows and n columns, where $m \neq n$.

Row Matrix: A matrix having only one row.

Singular Matrix: A matrix whose determinant is zero.

Square Matrix: If the number of rows of a matrix is equal to its number of columns, the matrix is said to be a square matrix.

Symmetric Matrix: A square matrix A $[a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all values of i and j.

5.9 QUESTIONS FOR DISCUSSION

1. Discuss with examples, different types of matrices.

2. Given the following column vectors

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Find (i) $A + B$, (ii) $A + B + C$ and (iii) $2A - B + C$

3. Find the value of x if

(i) $\begin{vmatrix} 2 & 1 & 3 \\ 8 & 4 & x \\ -7 & 5 & 1 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ -1 & 2 & 7 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} x & 2 & 2 \\ 2 & x & 2 \\ 2 & 2 & 0 \end{vmatrix} = 0$ (iv) $\begin{vmatrix} x & -3 & -3 \\ -3 & 0 & -3 \\ -3 & -3 & x \end{vmatrix} = 0$

(v) $\begin{vmatrix} x & 2 & x+2 \\ 3 & 5 & 8 \\ 0 & 1 & -3 \end{vmatrix} = 0$ (vi) $\begin{vmatrix} 2 & x & -4 \\ 6 & 3 & -12 \\ -1 & 2 & 3 \end{vmatrix} = 0$

4. Find the adjoint and inverse of the following matrices and verify that $A\,(\text{Adj } A) = (\text{Adj } A)A = |A|\,I$.

(i) $\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ -3 & -1 & -2 \end{bmatrix}$

(x) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ (xi) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ (xii) $\begin{bmatrix} 2 & -3 & 5 \\ 5 & 2 & -7 \\ -4 & 3 & 1 \end{bmatrix}$

5. Using Cramer's rule, solve the following equations:

(i) $x + y = 1$ (ii) $2x + y = -1$ (iii) $3x + 2y = 7$
     $x - y = 3$ $x - 2y = 2$ $x - 2y = 0$

(iv) $-3x + 2y = -7$ (v) $4x + 2y = -1$ (vi) $x - 2y = 1$
      $8x - y = 10$ $-2x + y = 0$ $5x + 7y = 3$

(vii) $x + y = 7$ (viii) $2x - 3y = 4$ (ix) $x - y + 2z = 3$
       $2x + y = 8$ $3x + 2y = 5$ $2x + z = 1$
                                                         $3x + 2y + z = 4$

(x) $3x + y + 2z = 3$ (xi) $x + y + z - 7 = 0$ (xii) $x + 2y - z = 2$

      $2x - 3y - z = -3$ $x + 2y + 3z - 16 = 0$ $2x + z = 0$

      $x + 2y + z = 4$ $x + 3y + 4z - 22 = 0$ $x + 3y - 2z = 3$

(xiii) $x - 2y = 0$ (xiv) $3x - y + 2z = 13$ (xv) $x + 2y + z = -1$

      $2x - y + z = 4$ $2x + y - z = 3$ $x + y + 2z = 3$

      $3x + y - 2z = -3$ $x + 3y - 5z = -8$ $3x + 2y + 3z = 5$

(xvi) $x + 3y + 2z = 5$ (xvii) $x + y + 2z = 9$ (xviii) $2x + y - z = 2$

      $2x + y + z = 3$ $3x + 2y + z = 10$ $x - 2y + z = 5$

      $5x + 2y + 3z = 6$ $x + 2y + 3z = 14$ $x + y + 2z = 3$

(xix) $x - y - 2z = 3$ (xx) $3x + 2y + z = 17$

      $2x + y + z = 5$ $x + z = 7$

      $4x - y - 2z = 11$ $2x + y = 7$

6. 3 kg Ragi and 7 kg of Jowar cost ₹ 35.50. 7 kg of Ragi and 3 kg of Jowar cost ₹ 29.50. Find the costs per kg of Ragi and Jowar.

7. Find the matrix A if $2A + B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

8. Prove that $\begin{bmatrix} a^2 + b^2 & b^2 + c^2 \\ a^2 + c^2 & a^2 + b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix} = \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$.

9. If $A = \begin{bmatrix} 3 & 0 \\ -3 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 7 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 7 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$, find the matrix X such that $2A + 3B - X = C$.

10. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, verify that $(A + B)' = A' + B'$ and $(A - B)' = A' - B'$.

11. If $A = \begin{bmatrix} -1 & 2 \\ 4 & -3 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 & 5 \\ 4 & 2 & 6 \end{bmatrix}$ find AB and BA.

12. Find $\left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \right\} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

13. Find AB and BA given that $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $B = \begin{bmatrix} a & -b \\ b & -a \end{bmatrix}$.

14. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, show that $A^2 = 0$.

15. Find a and b if $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} a & 1 \\ 5 & b \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -3 & 5 \end{bmatrix}$.

16. If $A = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 6 & -7 \end{bmatrix}$, verify that $(AB)' = B'A'$.

17. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, prove that $A^2 - 5A + 7I = 0$ where I is the unit matrix of order 2×2.

18. If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, verify that

   (i) $A(B + C) = AB + AC$

   (ii) $A + (B + C) = (A + B) + C$

   (iii) $A - (B - C) \neq (A - B) - C$

19. Find x and y given that $\begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

20. If $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, prove that $A^2 - 4I = 0$.

21. Solve for A and B given that

   $$2A + B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 4 & 0 \end{bmatrix}, \quad 3A + 2B = \begin{bmatrix} -4 & 6 & 2 \\ 2 & -3 & 5 \end{bmatrix}.$$

22. Simplify $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$.

23. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $(A')^2 - 3A + 4I$.

24. Evaluate $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3 \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$.

25. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$

   Find (i) $A' + B'$, (ii) $A' - B'$, (iii) AB, (iv) BA, (v) $(AB)'$ and (vi) $(BA)'$.

26. Solve for x, y, z given that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$.

27. Solve for x, y, z given that $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$.

28. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$ where I is the unit matrix of order 3×3.

29. Find a, b, c if

$$4\begin{bmatrix} a & 2 & 3 \\ 4 & 5 & -6 \\ 7 & -8 & 6 \end{bmatrix} - 5\begin{bmatrix} 3 & 2 & 1 \\ 2 & b & -4 \\ 9 & -8 & 7 \end{bmatrix} = \begin{bmatrix} 13 & -2 & 7 \\ 6 & -5 & -4 \\ -17 & 2c & 1 \end{bmatrix}.$$

30. Solve for x $\begin{bmatrix} x^2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2x & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 7 \end{bmatrix}.$

Check Your Progress: Model Answer

1. True

2. True

3. False

5.10 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

UNIT IV

LESSON

6

LOGARITHMS AND ANTILOGARITHMS

6.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

● Define the term logarithms and powers

● Understand the combinations of logarithms

● Discuss particular argument values and base values

● Understand the different applications of logarithms

● Describe antilogarithms

6.1 INTRODUCTION

In the previous lesson, we studied about the matrices. In this lesson, we will learn what logarithms are and evaluate some basic logarithms. This will prepare you for future work with logarithm expressions and functions. Logarithms are another way of thinking about exponents.

6.2 LOGARITHMS

John Napier was born in 1550. He died on 4th April, 1667 in Edinburgh. A mathematician John Napier introduced the concept of logarithm for the first time in 17th century. Later, Henry Briggs, a British mathematician born in Feb. 1561 in Yorkshire, England, prepared and published logarithm tables. He died on 26th January, 1663 in Oxford, England. Logarithm tables made complicated numerical calculations both — easy and fast. Today with the advent of desk calculators and computers, the work of numerical calculations has become easier and faster, thus reducing the usefulness of logarithm tables. All the while they are useful for calculations in the study of science and mathematics.

Many natural phenomena vary as a power function a^x, so that they display exponential change over widely measurable ranges of the exponent x. Scientists are therefore often interested in the value of this exponent, which is called a logarithm.

6.2.1 Logarithms and Powers

Quite often when analyzing a power function $y = a^x$, it is y that is the known quantity, and the exponent x is the value that one wishes to find.

One then needs a function $x = f(y)$ that is the **inverse** of the power function, which is called a **logarithm**:

$$x = \log_a y$$

y is called the **argument** of the logarithm.

Another way of saying this is that x is the exponent to which a must be raised to equal y:

$$x = \log_a y \textbf{ if and only if } a^x = y.$$

When going from a logarithmic relation to a power relation, it helps to visually imagine the "log" disappearing as you bring the base a to the other side of the equality and raise it to the power x:

$$a^x = y$$

$$x = \log_a y$$

Note that we can always very generally write:

$$y = a^{\log_a y},$$

which emphasizes that **a logarithm is just an exponent of a power function**.

Examples:

$$x = \log_2 8$$
$$2^x = 8 = 2^3$$
$$x = 3$$

6.3 COMBINATIONS OF LOGARITHMS

Some simple combinations of logarithms illustrate their utility.

6.3.1 Logarithms of Products

When the argument of a logarithm is a product of two numbers, $\log_a (y \cdot z)$, it can be split apart into the *sum* of two logarithms:

$$\log_a (y \cdot z) = \log_a y + \log_a z$$

Explanation:

$$x = \log_a (y \cdot z)$$

$$a^x = y \cdot z$$

$$= a^{\log_a y} \cdot a^{\log_a z}$$

$$= a^{\log_a y + \log_a z}$$

$$x = \log_a y + \log_a z$$

Examples:

$$x = \log_5 (2 \cdot 3)$$

$$= \log_5 2 + \log_5 3 .$$

$$x = \log_a (y \cdot z)$$

$$a^x = y \cdot z$$

$$y = a^x / z$$

$$x = a^{\log_a y + \log_a z}$$

$$= a^{\log_a y} \cdot a^{\log_a z}$$

$$= y \cdot z$$

6.3.2 Logarithms of Quotients

When the argument of a logarithm is a quotient of two numbers, $\log_a (y/z)$, it can also be split apart, into the *difference* of two logarithms:

$$\log_a (y/z) = \log_a y - \log_a z$$

Explanation:

$$x = \log_a (y/z)$$

$$a^x = y/z$$

$$= a^{\log_a y} / a^{\log_a z}$$

$$= a^{\log_a y - \log_a z}$$

$$x = \log_a y - \log_a z$$

Examples:

$$x = \log_5 (2/3)$$
$$= \log_5 2 - \log_5 3 .$$

$$x = \log_a (y/z)$$
$$a^x = y/z$$
$$y = z \cdot a^x$$

$$x = a^{\log_a y - \log_a z}$$
$$= a^{\log_a y}/a^{\log_a z}$$
$$= y/z$$

6.3.3 Logarithms of Power Functions

When the argument of a logarithm is a power of two numbers, $\log_a b^y$, it can also be split apart, into a *multiple* of a logarithm:

$$\log_a b^y = y \log_a b$$

Explanation:

$$x = \log_a b^y$$
$$= \log_a (b \cdot b \cdots b) \underline{\hspace{5cm}} y \text{ times}$$
$$= \log_a b + \log_a b + ... + \log_a b \underline{\hspace{3cm}} y \text{ times}$$
$$= y \log_a b$$

Example:

$$x = \log_5 8$$
$$= \log_5 2^3$$
$$= 3 \log_5 2 .$$

$$x = \log_a (z \cdot b^y)$$
$$= \log_a z + \log_a b^y$$
$$x - \log_a \quad z = y \log_a b$$
$$y = (x - \log_a z)/\log_a b .$$

$$x = w \log_a y + \log_a z$$
$$= \log_a y^w + \log_a z$$
$$= \log_a (y^w \cdot z)$$
$$x = \log_a (b^y/b^z)$$
$$= \log_a b^{y-z}$$
$$= (y - z) \log_a b .$$

6.4 PARTICULAR ARGUMENT VALUES

Some arguments of the logarithm function have special characteristics.

6.4.1 Special Arguments

Because any number raised to the power of 1 is equal to itself, $a^1 = a$, the following identity will always be true for any argument equal to its base:

$$\log_a a = 1$$

Similarly, because any number raised to the power of 0 is equal to 1, $a^0 = 1$, the following identity will always hold for any base a:

$$\log_a 1 = 0$$

Because a power function a^x only reaches a value of 0 for very large exponents $x \to - \infty \, (a > 1)$ and $x \to + \infty \, (a < 1)$, an argument of zero will result in:

$$\log_a 0 = \pm \infty \, (a \lessgtr 1)$$

Logarithms of negative arguments generally result in complex values.

Examples:

$$x = \log_5 \log_3 3$$
$$= \log_5 1$$
$$= 0 \qquad .$$

$$x = \log_a (a^y/a^z)$$
$$= \log_a a^y - \log_a a^z \quad \text{Why?}$$
$$= y \log_a a - z \log_a a \quad \text{Why?}$$
$$= y - z \qquad .$$

$$x = \log_5 \log_3 1$$
$$= \log_5 0$$
$$= - \infty$$

6.5 PARTICULAR BASE VALUES

Some values of the base of a logarithm have particular use and importance.

6.5.1 Common Logarithms

Here each x is an integral power of 10. So, it is easy to find $\log x$. When x is not an integral power of 10, to find logarithm (to the base 10), first we write x as a product of an integral power of 10 and a number between 1 and 10. This is done because the logarithm tables have been prepared only for numbers between 1 and 10. It is convenient to find the logarithm of any positive number using this form.

In science, large and small numbers are often expressed in scientific notation, e.g.

$$1,000,000 = 10^6$$
$$3,512,000 = 3.512 \times 10^6 \, .$$

For this reason powers of 10 are very important, and we commonly want to calculate the exponent on such a power, e.g.,

$$x = \log_{10} 1,000,000$$

$$= 6 \qquad .$$

and

$$x = \log_{10} 3,512,000$$

$$= 6.5455545...$$

Again a scientific calculator may be required to evaluate such expressions (note that the $\log_{10}$ key is usually implemented as **inv** 10^x, or vice-versa).

Table 6.1: To Find Log x for Positive x

Number x	0.0001	0.001	0.01	0.1	1	10	100	1000
X written as power of 10	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3
Logarithm of x (to the base 10)	-4	-3	-2	-1	0	1	2	3

The base-10 logarithm is called the **common logarithm**, and the subscript is typically dropped:

$$x = \log_{10} y = \log y$$

6.5.2 Natural Logarithms

Another base that shows up naturally in calculus is the transcendental number $e = 2.718281...$, because the power of this base is equal to its own derivative.

The base-e logarithm, the inverse of e^x, is called the **natural logarithm**, and it is typically abbreviated **ln**:

$$x = \log_e y = \ln y$$

This function can also be found on a scientific calculator (and again note that the **ln** key is usually implemented as **inv** e^x, or vice-versa).

The exponential function can be used to define any other power function, because any base can be expressed as a power of e with a scaled exponent:

$$a^x = e^{x \cdot \ln a}$$

Explanation:

$$a = e^{\ln a}$$

$$a^x = (e^{\ln a})^x$$

$$= e^{x \cdot \ln a}$$

6.5.3 Characteristics and Mantissa of Logarithm

The internal part of the logarithm of a number is called its characteristic.

Case I: When the number is greater than 1 then the characteristic is one less than the number of digits in the left of the decimal point in the given number.

Case II: When the number is less than 1 then the characteristic is one more than the number of zeros between the decimal point and the first significant digit of the number and it is negative.

Let the standard form of a positive number n be $t \times 10^p$, where $1 < t < 10$ and p is an integer.

$$\log n = \log (t \times 10^p)$$

$$= \log t + \log 10^p$$

$$= \log t + p \log 10$$

$$\log t + p$$

Since $1 < t < 10$, we have $\log 1 < \log t < \log 10$. i.e. $0 < \log t < 1$. We note that $\log n = \log t + p$ consist of two parts: (1) p and (2) $\log t$.

Here p is called the **characteristic** and $\log t$ is called the **mantissa** of $\log n$.

Example:

Number	Characteristic	Number	Characteristic
654.24	2	0.6453	1
26.649	1	0.06134	2
8.3547	0	0.00123	3

For example:

$$83.628 = 8.3628 \times 10^1, p = 1$$

$$894.82 = 8.9482 \times 10^2, p = 2$$

$$0.0329 = 329 \times 10^{-2}, p = -2$$

$$0.000487 = 4.87 \times 10^{-4}, p = -4$$

$$279389 = 2.79389 \times 10^5, p = 5$$

6.6 APPLICATIONS OF LOGARITHM

Logarithms have many applications inside and outside mathematics. Some of these occurrences are related to the notion of scale invariance. For example, each chamber of the shell of a nautilus is an approximate copy of the next one, scaled by a constant factor. This gives rise to a logarithmic spiral. Logarithms are also linked to self-similarity. For example, logarithms appear in the analysis of algorithms that solve a problem by dividing it into two similar smaller problems and patching their solutions. The dimensions of self-similar geometric shapes, that is, shapes whose parts resemble the overall picture are also based on logarithms. Logarithmic scales are useful for quantifying the relative change of a value as opposed to its absolute difference. Moreover, because the logarithmic function $\log(x)$ grows very slowly for large x, logarithmic scales are used to compress large-scale scientific data. Logarithms also occur in numerous scientific formulas, such as the Tsiolkovsky rocket equation, the Fenske equation, or the Nernst equation.

We will discuss various fields where logarithm are used as given below.

Logarithmic Scale

Scientific quantities are often expressed as logarithms of other quantities, using a logarithmic scale. For example, the decibel is a unit of measurement associated with logarithmic-scale quantities.

Psychology

Logarithms occur in several laws describing human perception: Hick's law proposes a logarithmic relation between the time individuals take to choose an alternative and the number of choices they have.

Probability Theory and Statistics

Logarithms arise in probability theory: the law of large numbers dictates that, for a fair coin, as the number of coin-tosses increases to infinity, the observed proportion of heads approaches one-half. The fluctuations of this proportion about one-half are described by the law of the iterated logarithm.

Computational Complexity

Analysis of algorithms is a branch of computer science that studies the performance of algorithms (computer programs solving a certain problem). Logarithms are valuable for describing algorithms that divide a problem into smaller ones, and join the solutions of the sub-problems.

For example, to find a number in a sorted list, the binary search algorithm checks the middle entry and proceeds with the half before or after the middle entry if the number is still not found.

Fractals

Logarithms occur in definitions of the dimension of fractals. Fractals are geometric objects that are self-similar: small parts reproduce, at least roughly, the entire global structure.

Music

Logarithms are related to musical tones and intervals. In equal temperament, the frequency ratio depends only on the interval between two tones, not on the specific frequency, or pitch, of the individual tones.

Number Theory

Natural logarithms are closely linked to counting prime numbers (2, 3, 5, 7, 11, ...), an important topic in number theory. For any integer x, the quantity of prime numbers less than or equal to x is denoted $\pi(x)$. The prime number theorem asserts that $\pi(x)$ is approximately given by $\dfrac{x}{\ln(x)}$.

6.7 GRAPHS OF LOGARITHM FUNCTIONS

Since we have defined logarithm functions for all real positive arguments, we can draw graphs of them to see how they behave in general:
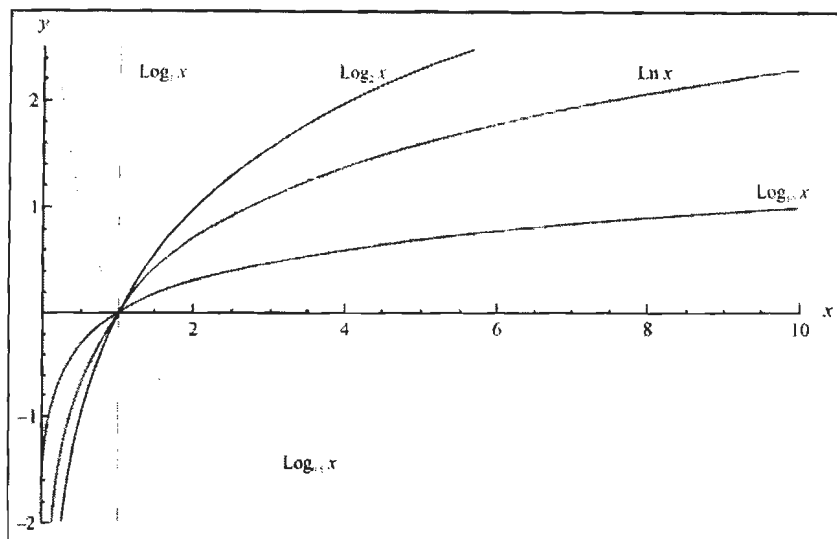
Figure 6.1: Graphs of Logarithm Functions

Note the special base $a = 1$, as well as the function values at the special arguments $x = 1$ and $x = a$.

Note also the up-down symmetry of $\log_2 x$ and $\log_{0.5} x$.

This graph should help you understand why logarithmic growth is commonly used to describe a slow increase.

6.7.1 Use of Logarithmic Tables

Ready tables of logarithm and antilogarithms shortly called logtable and antilogtables are available. The logtables consist of three parts: In the first part, there is one column, the first column from left, which contains two digit numbers from 10 to 99. Next there are ten columns headed by numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The last part called 'mean difference' has nine columns headed by numbers from 1 to 9.

The antilogtables are of the same type, except that the first column contains numbers from 0.00 to 0.99.

Suppose we start with a two digit number 81 and wish to find log 81. Here $81 = 81 + 0$. Its characteristic is 1. The mantissa can be obtained from logtables. Look for the number formed by first two digits in the first column. For this, find 81 in the first column and look at row against it. At the intersection of this row and the column headed by 0 is the number 9085. The mantissa of log 81 is 0.9085. Hence, log 81 = $1 + 0.9085 = 1.9085$.

To obtain the mantissa of the logarithm of a three digit number, first find the number formed by the first two digits of the given number in the column to the extreme left of the logtables. Look at the row against this number. In this row, the number in the column headed by the third digit of the given number gives the mantissa. For example, to find mantissa of log 723, look at the row against 72 in the first column and in the column headed by 3. The number 8591 appears there. Hence mantissa of log 723 is 0.8591. Since the characteristic of log 723 is 2, we have log 723 = 2.8591.

For finding the logarithm of a number with four digits, the columns of mean difference will also be used. For examples suppose we want to find the mantissa of log 3986. The number 3986 is divided into three parts 39, 8 and 6. Now look for 39 in the first column. Then find the number in the row against 39 in the column headed by 8. This is 5999. Finally look for the number in the same row in the column headed by 6 among the columns of mean differences. This number is 7. Adding 7 to 5999, we get 6006. Hence the mantissa of log 3986 is 0.6006. Since the characteristic of 3986 is 3, log 3986 = 3.6006.

Note that the logtables are used to find the mantissa of the logarithm of a number. Our logtables are four digits tables and so for finding the mantissa of the logarithm of a number with more than four digits. We approximate the number to a four digit number. For this, form the number formed by first four digits of the given number. If the fifth digit of the given number is less than 5, this four digit number is the required approximation. If the fifth digit is 5 or greater, then add 1 to the last digit of the four digit number obtained by truncation. The characteristic of the logarithm of a given number is obtained in the usual way. The mantissa is the mantissa of the logarithm of the four digit number which approximates the given number.

For example, let $x = 5.79881$. Then the characteristic of log x is 0. The four digit approximation of x is 5.799. Hence the mantissa of log x = the mantissa of log 5.799 = 0.7634. Hence log 5.79881 = 0.7634.

When the characteristic of a logarithm is a negative number $-n$ it is denoted by (read as n bar). For example, log (0.002675) = $\overline{3}$.4273.

COMMON LOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7

Contd...

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

6.8 ANTILOGARITHMS

Antilogarithms are the number of which a given number is the logarithm (to a given base). If "x" is the logarithm of "y", then "y" is the antilogarithm of "x".

If $\log y = x$, then y is called the antilogarithm of x and is written as $y = $ antilog x.

If the logarithmic value of a number be given then the number can be determined from the antilog-table. Antilog-table is similar to log-table; only difference is in the extreme left-hand column which ranges from .00 to .99.

For example, if log 39.2 = 1.5933, then antilog 1.5933 = 39.2.

To find the antilog of a number in a given base, raise the base to the number result.

$\log_2 8 = 3$

Anti $\log_2 3 = 2^3 = 8$

If the logarithm is known, a calculator can be used to find the antilog by pressing the $10x$ key. This is usually the second function of the log key.

$\log x = 2.4572$

$x = 10^{2.4572} = 286.55$

6.8.1 Use of Antilogarithmic Tables

The antilogarithm is used to get the number from its logarithm. The first column from the left of the antilogtables contains numbers from 0.00 to 0.99. In all other respects,

antilogtables are similar to logtables. The antilogs are also used in the same way as logtables.

Since the logtable gives only the mantissa part of the logarithm of a number, the antilog table will give a number corresponding to the mantissa part only. Then by using characteristic the actual number for the given logarithm can be obtained. For example, suppose we want to find antilog (1.5278). From antilogtables, we find that antilog 0.5278 = 3.371 (Meaning that log 3.371 = 0.5278). Hence, antilog 1.5278 = $3.371 \times 10^1 = 33.71$. Also antilog $\overline{3}.5278 = 3.371 \times 10^{-3} = 0.003371$. Note that power of 10 is (–1) means no zero between decimal point and first non-zero digit. (–3) means two zeroes between decimal point and first non-zero digit etc.

In fact antilog is obtained from the first four digits after decimal point (the truncated four digit number). If the characteristic is p. we multiply antilog obtained by 10^p.

COMMON ANTILOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	Mean difference								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7

Contd...

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4396	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	5	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	4	5	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	3	4	5	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6803	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Check Your Progress

Fill in the blanks:

1. _____ is the internal part of the logarithm of a number.

2. The idea of logarithms is to reverse the operation of _____ which is, raising a number to a power.

3. _____ were introduced by John Napier in the early 17th century as a means to simplify calculations.

6.9 LET US SUM UP

- Many natural phenomena vary as a power function a^x, so that they display exponential change over widely measurable ranges of the exponent x. Scientists are therefore often interested in the value of this exponent, which is called a logarithm.

- Quite often when analyzing a power function $y = a^x$, it is y that is the known quantity, and the exponent x is the value that one wishes to find.

- When the argument of a logarithm is a product of two numbers, $\log_a (y \cdot z)$, it can be split apart into the *sum* of two logarithms:

$$\log_a (y \cdot z) = \log_a y + \log_a z$$

- When the argument of a logarithm is a quotient of two numbers, $\log_a (y/z)$, it can also be split apart, into the *difference* of two logarithms:

$$\log_a (y/z) = \log_a y - \log_a z$$

- When the argument of a logarithm is a power of two numbers, $\log_a b^y$, it can also be split apart, into a *multiple* of a logarithm:

$$\log_a b^y = y \log_a b$$

- Logarithms have many applications inside and outside mathematics. Some of these occurrences are related to the notion of scale invariance.

- Antilogarithms are the number of which a given number is the logarithm (to a given base). If "x" is the logarithm of "y", then "y" is the antilogarithm of "x".

6.10 LESSON END ACTIVITY

Make a list of the properties of logarithms.

6.11 KEYWORDS

Antilogarithm: The number produced when a base number is taken to the power indicated by the logarithm.

Characteristic: It is the internal part of the logarithm of a number.

Common Logarithms: Logarithms to the base 10 are known as common logarithms.

Logarithm: If a is a positive real number, other than 1 and $a^m = x$, then we write: $m = \log_a x$ and we say that the value of $\log x$ to the base a is m.

Mantissa: The decimal part of the logarithm of a number is known is its mantissa.

6.12 QUESTIONS FOR DISCUSSION

1. Calculate $x = \log_a y$ for $\{a, y\}$ [substitute simple numerical values such as $\{2, 16\}$, $\{3, 27\}$, $\{1/9, 1/81\}$, etc. — no ones or zeros]

2. Using logarithms, calculate the value of x.

   (i) $x = \sqrt[5]{493}$

   (ii) $\dfrac{\sqrt[3]{0.3688}}{22.958^5}$

3. Simplify the following equation:

$$\log_3 \left(\frac{17}{25}\right) + \log_3 \left(\frac{600}{119}\right) - \log_3 \left(\frac{8}{7}\right)$$

4. Simplify:

$$\log_a \left(\frac{x^2}{yz}\right) + \log_a \left(\frac{y^2}{xz}\right) + \log_a \left(\frac{z^2}{xy}\right)$$

5. Find the value using logtable and antilogtables:

   (i) 49.673×9.4891 (ii) $\dfrac{(329)^{\frac{5}{2}} \times 9826}{(67.891)^3}0$ (iii) $\sqrt{\dfrac{(8432)^2 \times 0.1259}{(27.478)^5}}$

6. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$. What will be the value of $\log_5 512$?

7. If $\log_{10} 5 + \log_{10}(5x + 1) = \log_{10}(x + 5) + 1$, then what is the value of x?

8. If $\log_x y = 100$ and $\log_2 x = 10$, then what will be the value of y?

9. If $\log_{10} 7 = a$, then what will be the value of $\log_{10} \dfrac{1}{70}$?

10. If $\log \dfrac{a}{b} + \log \dfrac{b}{a} = \log(a + b)$, then what will be the value of $(a + b)$?

Check Your Progress: Model Answer

1. Characteristic

2. Exponentiation

3. Logarithms

6.13 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

Robert T. Smith, *Calculus: Single Variable*, Tata McGraw Hill.

LESSON

7

INTRODUCTION OF CALCULUS

7.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Define the term function
- Understand the concept of domain and range of a function
- Find value of function at a given point
- Understand the different types of functions
- Describe limits and its fundamental theorems

7.1 INTRODUCTION

In the previous lesson, we studied about the logarithm and antilogarithm. In this lesson, we will discuss the introduction of calculus. As you know that different variables are associated with each other and analysing the relationship of these variables is of paramount importance in business analysis. This lesson will take you around the relationships among variables and the concept of functions and limits or in simple words, boundaries.

7.2 FUNCTIONS

7.2.1 Definition, Domain and Range of a Function

A function is a technical term used to symbolise relationship between variables.

Notations for Function

Before proceeding for exploring the meaning of function, let us understand few notations for functions:

1. *Variables:* It is generally denoted by symbol x. If x can have any value from a specific set of numbers, then x is said to be a variable.

    Example: Say you want to know the weight of all the students of your class. Further suppose that you know that weights of your classmates vary between 32 kg to 80 kg.

    So, numbers from 32 to 80 can be termed as 'specific set of numbers'.

    Weights can be denoted by x, which can be any number from specific set of numbers (from 32 to 80). x is a variable here.

    It is customary to use last alphabets x, y, z, as symbols for variables.

2. *Constants:* The symbols which retain the same value throughout a set of mathematical operations are called constants.

    It is customary to use initial alphabets $a, b, c, \ldots$ as symbols for constants.

3. *A Domain Interval:* If a variable x which can take only those numerical values which lie between two given numbers 'a' and 'b' then all the numerical values between 'a' and 'b' is called Domain or interval of the variable x and is usually denoted by (a, b).

    In the example of weights of students of your class, (32, 80) is Domain or Interval of the weights.

Following are some examples in which two aspects (variables) have some relationship:

Example 1:
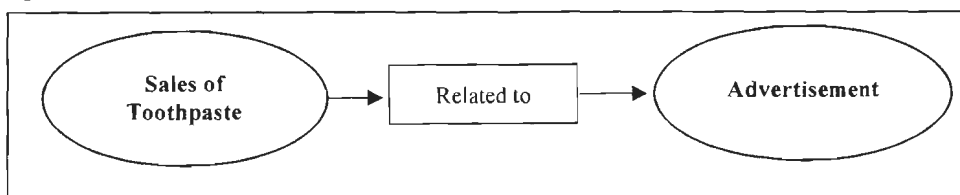
Example 2:
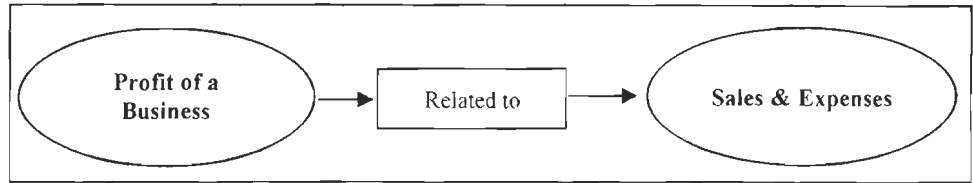
Example 3:
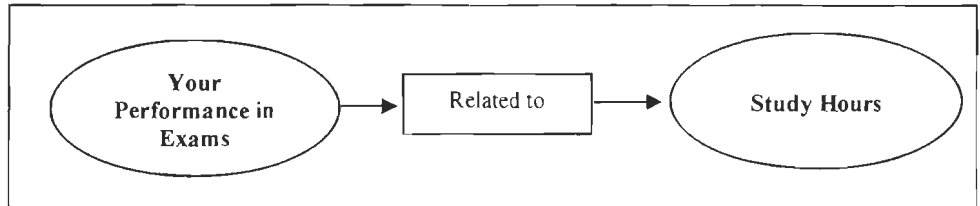
Thus, a function explains the nature of correspondence between variables indicated by some:

● Formula

● Graph

● Mathematical Equation

Function in the Form of a Formula

Consider a circle of radius *r*.
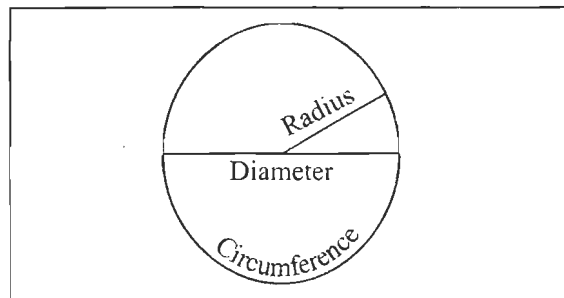
Figure 7.1: Relationship between Radius and Area

Area (which is a variable, say '*y*') is having correspondence with another variable, radius say, '*x*'.

Here, correspondence means that if radius increases, radius asks or makes a correspondence to area, to increase itself (area).

The relationship between both the variables (i) Area and (ii) Radius is expressed by the following formula:

$$A = \pi r^2$$

Note, here a constant, which is a ratio of Circumference to Diameter of a circle and its value is 22/7.

Function in the Form of a Graph

Refer Example 1: A sale of toothpaste is one variable, *y* and advertisement of the same toothpaste is another variable say, *x*.

One way of showing the correspondence between advertisement and sales is shown below in the form of a graph.
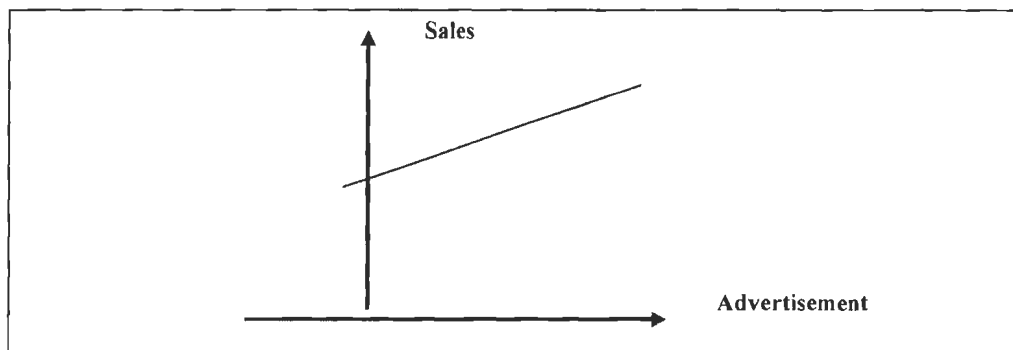
Figure 7.2: Relationship between Advertisement and Sales

Function in the Form of an Equation

Refer Example 1: Sale of toothpaste is one variable, *y* and advertisement of the same toothpaste is another variable say, *x*.

One way of showing the correspondence between advertisement and sales is shown below in the form of an equation.

$$y = a + bx$$

In the above equation, '*a*' and '*b*' are constants. Specifically, '*b*' is termed as 'Coefficient' of '*x*'.

You can clearly see that variable *y* (Area or Sale) depends upon *x* (Radius, Advertisement). So you can say that:

'*y*' is a function of '*x*'

The same is expressed symbolically by equations:

$$y = f(x)$$
$$y = F(x)$$
$$y = \Psi(x)$$

So, the idea of a function can be expressed as follows:

'The relationship between two real variables say *x* and y which are so related that corresponding to every value '*a*' of *x*, we get a finite value '*b*' of *y* then *y* is said to be the function of *x*'.

You can easily notice that, Area depends on Radius. Hence, Area can be termed as Dependent Variable, and Radius can be termed as Independent Variable. Same holds true, in case of Sales and Advertisement.

Sales can be termed as Dependent Variable and Advertisement can be termed as Independent Variable.

A function *f* from a set *X* (domain) to a set *Y* (range), a subset of *X* × *Y*, is a rule that associates a unique element of set *Y* (target) to *each* element in *X*. The unique element *y* in *Y* corresponding to an element *x* in *X* is denoted as *f*(*x*) and is called the image of

x. Alternatively, we write a functional relationship by writing *f*: *X* → *Y*. We sometimes also use the words *transformation* or *map* or *mapping* instead of *function*.

Domain of a Function

A Domain Interval: If a variable *x* which can take only those numerical values which lie between two given numbers '*a*' and '*b*' then all the numerical values between *a* and *b* is called Domain or interval of the variable *x* and is usually denoted by (*a*, *b*).

In the example of weight of students of your class, (32, 80) is Domain or Interval of the weights.

Range of a Function

The range of a real function *f* (*x*) of a real variable is the set of all values attained by *f*(*x*) at different points (values) in its domain.

Say, Sales is a function of Advertisement.

Domain of advertisement is say, ₹ 10,000 to 20,000.

Range will be all the values of *f*(*x*), sales, for each and every value of advertisement.

Diagrammatically, a functional relationship can be represented as shown in Figure 7.3.
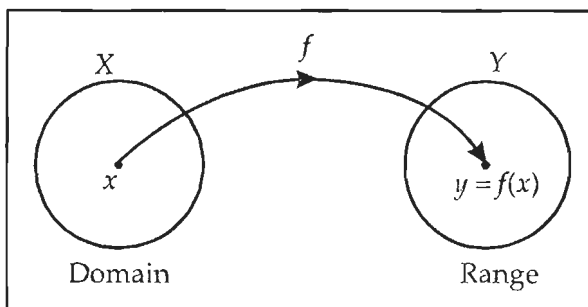
Figure 7.3: Functional Relationship

One-to-One Mapping: If different elements of the set *X* have different images in the set *Y* or each element of the set *Y* is the image of at the most one element of the set *X*, the mapping is termed as *one-to-one mapping*.

On to Mapping: If every element of the set *Y* be an image of the set *X* and no element of *Y* remains unused, the mapping is termed as *on to mapping*.

The set {*f*(*x*) : *x* ∈ *X*} is called the *range* of the function. It is the set of all possible values of *f*(*x*) in *Y*. When we write a function as *y* = *f*(*x*), *x* is termed as the *independent variable or argument or input* of the function and *y* is termed as the *dependent variable or output*.

Alternative Definitions of a Function

I.   A function is a set of ordered pairs in which no value of the first component *x* is repeated.

II. A function is a rule that associates exactly one value (*y*) of range corresponding to each value (*x*) of the domain.

III. An equation in two variables is said to define a function if to each value of the independent variable we have exactly one value of the dependent variable.

A function can be described either (i) by a set of ordered pairs, or (ii) by one or more algebraic equations or (iii) in words.

(i) *Using a set of ordered pairs:* We can write a function as the set of ordered pairs $S = \{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$, which denotes a function from the set $X = \{x_1, x_2 ......, x_n\}$ to the set $Y = \{y_1, y_2, ..., y_n\}$. Here, first element of each pair belongs to domain and the second element belongs to the range of the function. Alternatively, the set of ordered pairs can also be presented in the form of a table.

(ii) *Using one or more equations:* A function can also be described by using one or more equations. For example, $y = f(x) = ax^2 + bx + c$, where $x \in R$ and $y \in R$ are its domain and range respectively. Similarly, we can write a function by using different equations for different parts of the domain, e.g.,

$$y = 2x + 3 \qquad \text{when } x < 5$$
$$= 5x^2 + 3x - 100 \qquad \text{when } 5 \leq x < 16$$
$$= 4 \qquad \text{when } x \geq 16$$

(iii) *Using words:* A function can also be expressed in words. Let X be the set of students in a college and Y the set of colonies in a city. We have, of course, to assume here that every student has a unique name.

From the definition of a function, we should note that a function is also a relation but a relation may or may not be a function. Considering the subset given by $\{(x, y) : y = x^2\}$. We note that corresponding to each value of x, we can associate only one value of y. Thus, $y = x^2$ is a function.

Example: Find the domain of each of the following function given by:

(i) $f(x) = x/(x^2 - 3x + 2)$

(ii) $f(x) = (x^3 - 5x + 3)/(x^2 - 1)$

Solution:

(i) $f(x) = x/(x^2 - 3x + 2)$

Carefully observe the function (right side of the equation).

You can observe that $f(x)$ can assume real values for x, but 'cannot have those values which makes right hand side or the function, meaningless'.

Meaningless form of right hand side of the equation will evolve, when the denominator becomes '0'.

So, you are required to find those real values of x, for which denominator of right hand side becomes '0'. Obviously, these values cannot be a part of the Domain.

For this, you should equate the denominator of right hand side to '0' and solve for 'x', as shown below.

$$(x^2 - 3x + 2) = 0$$
$$(x - 1)(x - 2) = 0$$

So, either $x = 1$

Or, $x = 2$

Hence, you can conclude that Domain is all the real values, except (excluding) 1 and 2.

The same can be mathematically expressed as:

Domain $(f) = R - [1, 2]$

Where, R represents all real values.

(ii) $f(x) = (x^3 - 5x + 3)/(x^2 - 1)$

Again have a careful look at function (right side of the equation).

You can observe that $f(x)$ can assume real values for x, but 'cannot have those values which makes right hand side or the function, meaningless'.

And meaningless form of right hand side of the equation will evolve, when the denominator becomes '0'.

So, you are required to find those real values of x, for which denominator of right hand side becomes '0'. Obviously, these values cannot be a part of the Domain.

For this, you should equate the denominator of right hand side to '0' and solve for 'x', as shown below.

$$x^2 - 1 = 0$$
$$x^2 = 1$$

So, either, $x = 1$

Or $x = -1$

Hence, you can conclude that Domain is all the real values, except (excluding) 1 and −1.

The same can be mathematically expressed as

Domain $(f) = R - [1,-1]$

Where, R represents all real values.

7.2.2 Value of a Function at a Point

Another important point to be noted is that although the definition of a function requires that there exists only one value of y corresponding to a given value of x, the converse of this is not true, i.e., it is possible to associate more than one value of x to a single value of y. This type of situation is shown in Figure 7.4.
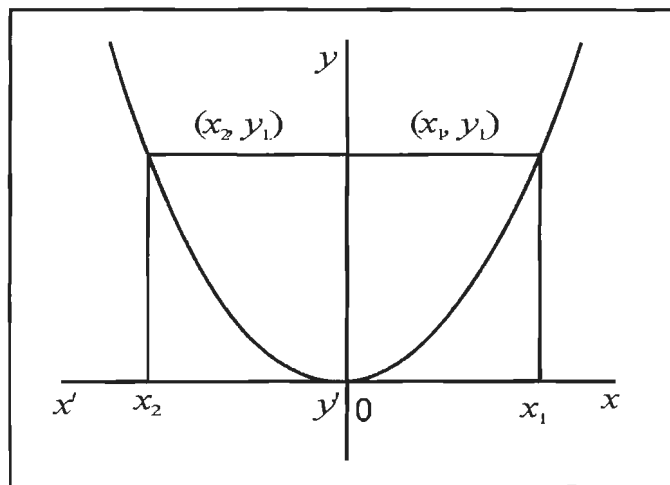

Figure 7.4: Value of a Functions at a Point

7.3 TYPES OF FUNCTIONS

One-to-One Function

If the ordered pairs (x, y) of a function $y = f(x)$ are such that no two ordered pairs have the same second component corresponding to different first components, the function

is said to be one-to-one. Thus if x_1 and x_2 are two elements in the domain such that $x_1 \neq x_2$, then $f(x)$ is said to be one-to-one if $f(x_1) \neq f(x_2)$.

Alternatively, a function $y = f(x)$ is said to be one-to-one if for each element of the range, we can associate a unique element of the domain.

The function $y = x^2$ is not a one-to-one function. Refer to Figure 7.4, we note that corresponding to two values of x (i.e., x_1 and x_2) we have only one value of y (i.e., y_1) or in other words, the ordered pairs (x_1, y_1) and (x_2, y_1) have same second component corresponding to different first components.

Horizontal Line Test for a One-to-One Function

Given the graph of a function, if we can draw a horizontal line that intersects the graph at more than one point, then the function is not a one-to-one function. For example, $y = x^2$ is not a one-to-one function, since a horizontal line cuts the curve at more than one point, as shown in Figure 7.4.

Vertical Line Test for a Function

An algebraic equation defines a function if it is not possible to draw a vertical line that intersects its graph at more than one point.

For example, for the graph given in Figure 7.4, it is not possible to draw a vertical line that intersects the curve at more than one point. Thus $y = x^2$ is a function.

Examples of Functions from Economics

(i) Demand function, written as $x = D(p)$ where p, the price, is an independent variable and x, the quantity demanded, is a dependent variable.

(ii) Cost function, written as $C = F(x)$ where x, the level of output, is an independent variable and C, the cost of production, is a dependent variable.

(iii) Total revenue function, written as $R = f(x)$ where x, the level of output, is an independent variable and R, the total revenue, is a dependent variable.

(iv) Consumption function $C = f(Y)$, where Y, the level of national income, is an independent variable and C, the level of national consumption, is a dependent variable.

7.3.1 General Characteristics of a Function

Functions can be classified into different categories according to the nature of their definition or of symbolic expressions. To facilitate this, we first define the following general characteristics of a function.

1. **Increasing or Decreasing Function:** Let $y = f(x)$ be a function defined in an interval I and x_1, x_2 be two points of the interval:

    (a) If $f(x_2) \geq f(x_1)$ when $x_1 < x_2$, then $f(x)$ is *increasing*.

    (b) If $f(x_2) \leq f(x_1)$ when $x_1 < x_2$, then $f(x)$ is *decreasing*.

    (c) If $f(x_2) = f(x_1)$ for all values of x_1 and x_2 in I, then $f(x)$ is *constant*.

    If, however, the strict inequality holds in the above statements, then $f(x)$ is strictly increasing (or decreasing) function.

2. **Monotonic Function:** A function $y = f(x)$ is said to be monotonic if y is either increasing or decreasing over its domain, as x increases.

If the function is increasing (decreasing) over its domain, it is called *monotonically increasing (decreasing) function*.

A *one-to-one function* is also a *monotonic function*.

3. **Implicit and Explicit Function:** When a relationship between x and y is written as $y = f(x)$, it is said to be an explicit function. If the same relation is written as $F(x, y) = 0$, it is said to be an implicit function. Production possibility function or the transformation function is often expressed as an implicit function.

4. **Inverse Function:** If a function $y = f(x)$ is such that for each element of the range we can associate a unique element of the domain (i.e., one-to-one function), then the inverse of the function, denoted as $x = f^{-1}(y) = g(y)$, is obtained by solving $y = f(x)$ for x in terms of y. The functions $f(x)$ and $g(y)$ are said to be inverse of each other and can be written as either $g[f(x)] = x$ or $f[g(y)] = y$. We note here that an implicit function $F(x, y) = 0$, can be expressed as two explicit functions that are inverse of each other.

5. **Symmetry of a Function:** Symmetry of a function is often helpful in sketching its graph. Following types of symmetry are often useful:

   (i) *Symmetry about y-axis:* A function $y = f(x)$ is said to be symmetric about y-axis if $f(-x) = f(x)$ for all x in its domain. For example, the function $y = x^2$ is symmetric about y-axis. Such a function is also known as *even function*.

   Similarly, if $g(y) = g(-y)$, then the function $x = g(y)$ is said to be symmetric about x-axis.

   (ii) *Symmetry about the line $x = h$:* A function $y = f(x)$ is said to be symmetric about the line $x = h$ if $f(h - k) = f(h + k)$ for any real value of k.

   (iii) *Symmetry about origin:* A function $y = f(x)$ is said to be symmetric about origin if $f(-x) = -f(x)$, for all values of x in its domain. For example, the function $y = x^3$ is symmetric about origin. Such a function is also known as *odd function*.

   (iv) *Symmetry about the line $y = x$ (45° line):* Two functions are said to be symmetrical about the line $y = x$ (45° line), if the interchange of x and y in one function gives the other function. This type of symmetry implies that y as an explicit function of x is exactly of the same form as x as an explicit function of y.

   Two points with coordinates (a, b) and (b, a) are said to be reflections of one another (or symmetrical) about the line $y = x$.

   Since the inverse function $x = g(y)$ is obtained simply by solving $y = f(x)$ for x, the graphs of these functions remain unaltered. However, when we interchange the role of x and y in the function $x = g(y)$ and write as $y = g(x)$, the graph of $y = f(x)$ gets reflected about the $y = x$ line to get the graph of $y = g(x)$.

   To illustrate this, we consider $y = f(x) = 2x + 5$ and $y = g(x) = \dfrac{1}{2}(x - 5)$. Note that $(1, 7)$ is a point on the graph of $y = 2x + 5$ and $(7, 1)$ is a point on the graph of $y = \dfrac{1}{2}(x - 5)$. The graphs of these functions are shown in Figure 7.5.
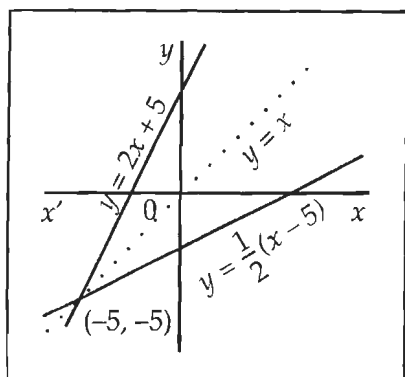
Figure 7.5

The point of intersection of the two functions, that are symmetric about the 45° line, occurs at this line.

An implicit function $F(x, y) = 0$ is said to be symmetric about the 45° line if an interchange of x and y leaves the function unchanged. For example, the function $xy = \alpha$ is symmetric about the 45° line.

6. **Composite Function:** If y is a function of u and u is a function of x, then y is said to be a composite function of x. For example, if $y = f(u)$ and $u = g(x)$, then $y = f[g(x)]$ is a composite function of x. A composite function can also be written as $y = (f \circ g)(x)$, where $f \circ g$ is read as f of g.

The domain of $f\{g(x)\}$ is the set of all real numbers x in the domain of g for which $g(x)$ is in the domain of f.

The rules for the sum, difference, product and quotient of the functions f and g are defined below:

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\therefore \qquad = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

The domain of each of the resultant function is given by the intersection of the domains of f and g. In the case of quotient, the value of x at which $g(x) = 0$ must be excluded from the domain.

7.3.2 Polynomial Functions

Depending upon the nature of their symbolic expressions, various functions can be classified into the different categories. A brief description of some common types of functions is given in the following sections. A function of the form $y = a_0 + a_1 x + a_2 x^2$ + ... + $a_n x^n$, where n is a positive integer and $a_n \neq 0$, is called a polynomial function of degree n.

(i) If $n = 0$, we have $y = a_0$, a constant function.

(ii) If $n = 1$, we have $y = a_0 + a_1 x$, a linear function.

(iii) If $n = 2$, we have $y = a_0 + a_1 x + a_2 x^2$, a quadratic or parabolic function.

(iv) If $n = 3$, we have $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, a cubic function.

Constant Functions

A function of the form $y = f(x) = a_0$ for all real values of x, is a constant function. Graph of such a function is a horizontal straight line with equation $y = a_0$, as shown in Figure 7.6.
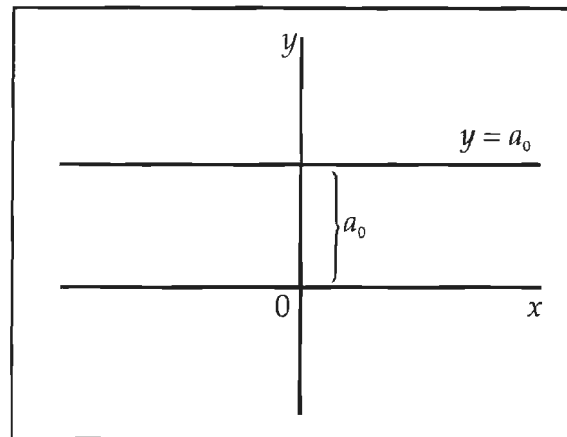
Figure 7.6

Linear Functions

$y = a_0 + a_1 x$ $(a_1 \neq 0)$ is a linear function. The graph of the linear function is a straight line. Here a_0 is the value of y when $x = 0$, known as the intercept of the line on y-axis and a_1 is the slope of the line. If $a_1 > 0$, the line slopes upward and when $a_1 < 0$, the line slopes downward, as shown in Figure 7.7 (a) and 7.7 (b) respectively.

(a)   (b)
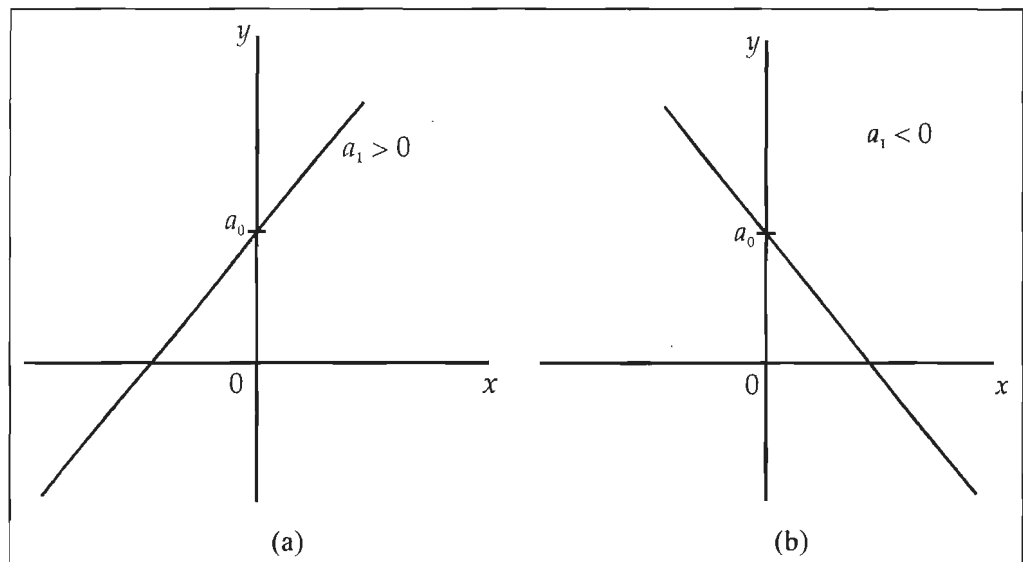
Figure 7.7

Alternatively, a linear function or the equation of a straight line can be written as:

(i) $y = mx + c$, where m is the slope and c is the intercept, or

(ii) $y - y_1 = m(x - x_1)$ is the equation of a line passing through the point (x_1, y_1) with slope m, or

(iii) $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a line passing through two points

(x_1, y_1) and (x_2, y_2).

Where, $\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line.

Linear Models in Economics

Linear equations are very frequently used in economic analysis. Very often, even when a relationship between economic variables is not linear, we use their linear approximation to comprehend it better. Some applications of linear equations are illustrated by the following examples.

Example: The population of a country was 80 crores in the year 2005 and it became 92 crore in the year 2013. Assuming that the population growth is linear:

(i) Find the relation between population and time.

(ii) Predict the population for the year 2015.

Solution:

(i) Assuming base year as 2005, we take $x = 0$ for the year 2005. Thus, $x = 8$ for 2013.

Let y denote population, thus,

When $x = 0$, $y = 80$ and when $x = 8$, $y = 92$. The equation of a line passing through the two points is $y - 80 = \dfrac{92 - 80}{8}(x - 0)$. Thus, $y - 80 = \dfrac{3}{2}x$ or $y = \dfrac{3}{2}x + 80$.

(ii) For the year 2015, $x = 10$.

$\therefore \quad y = \dfrac{3}{2} \times 10 + 80 = 95$ crores will be the population in the year 2015.

Quadratic Functions

A quadratic function is a special case of a polynomial when $n = 2$. Thus, $y = a_0 + a_1 x + a_2 x^2$ $(a_2 \neq 0)$ is a quadratic function.

Very often, we write a quadratic function as $y = ax^2 + bx + c$ $(a \neq 0)$.

Since $ax^2 + bx + c$ represents a real number for all real values of x, the *domain of a quadratic function is the set of real numbers.*

To find range, we first transform the equation as given below:

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c.$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

If $a > 0$, then $a\left(x+\dfrac{b}{2a}\right)^2 > 0$ for all values of x. When $x \neq -\dfrac{b}{2a}$, $a\left(x+\dfrac{b}{2a}\right)^2$ is added to $\dfrac{4ac-b^2}{4a}$ thus making y larger. Hence $y = f\left(-\dfrac{b}{2a}\right) = \dfrac{4ac-b^2}{4a}$ is the minimum value of the function.

Therefore, range of the function is $y \geq \dfrac{4ac-b^2}{4a}$ or $\left[\dfrac{4ac-b^2}{4a}, \infty\right)$.

Similarly, when $a < 0$, y will be maximum when $x = -\dfrac{b}{2a}$ and the range of the function will be $y \leq \dfrac{4ac-b^2}{4a}$ or $\left(-\infty, \dfrac{4ac-b^2}{4a}\right]$.

Note that $y = ax^2 + bx + c$ is symmetrical about the line $x = -\dfrac{b}{2a}$. The point at which the function is minimum or maximum is called the vertex of the parabola.

A quadratic function is also known as a parabola with axis vertical. The diagrams of a quadratic function when a $>$ (or $<$) 0 are shown in Figure 7.8 [(a), (b)].

$x = -\dfrac{b}{2a}$ is the equation of the axis of parabola.
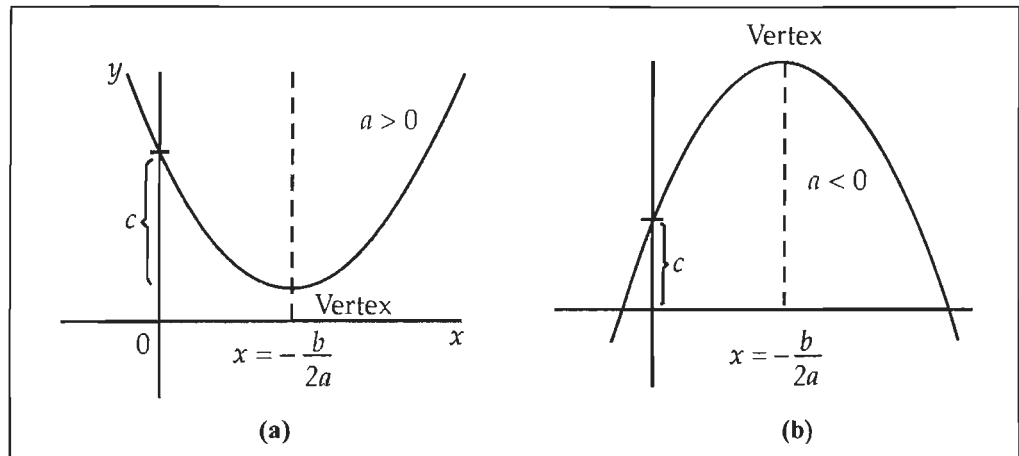
Figure 7.8

Quadratic Functions in Economics

The curves of quadratic functions can be used to represent various types of functions in economics.

Example: A man derives $\bar{x}$ from his business this year and $\bar{y}$ next year. By alternative use of resources, he can vary x and y according to the relation:
$y = b - \dfrac{x^2}{a}, (a, b > 0).$

(i)   What is this function known as?

(ii) What type of curve does it represent?

(iii) Show that this curve represents a normal case provided that $b > a$.

Solution:

(i) The given function is a quadratic function, also known as a parabola with vertex at (0, b).

The relevant arc of the curve is shown in figure.

(ii) It represents a transformation curve.
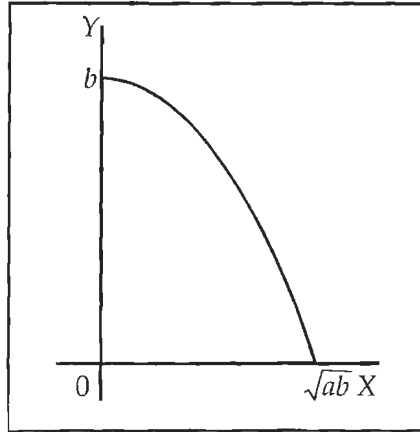
(iii) When nothing is transformed into future, i.e. $y = 0$, we have maximum present income, i.e., $x = \sqrt{ab}$. Similarly, when the whole of present income is transformed into future, the maximum future income is b. The transformation curve will be normal if the income transformed into future is greater than its equivalent income in current year, i.e. $b > \sqrt{ab}$ or $b > a$.

Roots of a Quadratic Polynomial

A number r is said to be a root of the polynomial $y = P(x)$ if $P(r) = 0$.

The root of a polynomial is also known as solution or zero of the polynomial.

Let $P(x) = ax^2 + bx + c$

We find x so that $P(x) = ax^2 + bx + c = 0$

On completing square, we can write,

$$P(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$$

or

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

or

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the two roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(i) If $b^2 - 4ac > 0$, the two roots are real and distinct.

(ii) If $b^2 - 4ac = 0$, there is only one real root.

(iii) If $b^2 - 4ac < 0$, the two roots are imaginary, conjugate of each other.

Cubic Functions

A cubic function is a special case of a polynomial when $n = 3$. Thus $y = a_3x^3 + a_2x^2 + a_1x + a_0$, $(a_3 \neq 0)$, is a cubic polynomial or cubic function.

As before, the domain of this function is $-\infty < x < \infty$.

Example: Find an integer root of the following polynomials:

(a) $P(x) = x^3 - 2x^2 - 5x + 6$

(b) $P(x) = x^4 - 2x^3 - 2x^2 + 8x - 8$

Solution:

(a) Let $x = r$ be an integer root, then

$r^3 - 2r^2 - 5r + 6 = 0$

or $r(r^2 - 2r - 5) = -6$

So r is a factor of -6

Possible factors of -6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Taking $r = 1$, we find that $P(r) = 1 - 2 - 5 + 6 = 0$

$\therefore x = 1$ is an integer root.

(b) Proceeding as above, the integer roots are the possible factors of 8. These factors are $\pm 1, \pm 2, \pm 4, \pm 8$.

$P(1) = 1 - 2 - 2 + 8 - 8 = -3$

$P(-1) = 1 + 2 - 2 - 8 - 8 = -15$

$P(2) = 16 - 16 - 8 + 16 - 8 = 0$

$\therefore x = 2$ is an integer root.

If the coefficients of $P(x)$ are not integers, then we can always write the polynomial as a constant time the polynomial with integer coefficients.

Every polynomial $P(x)$ of degree $n > 0$ can be expressed as the product of n linear factors, where one or more of these may be repeated. Hence, $P(x)$ has at the most n distinct roots.

A polynomial $P(x)$ with real coefficients may have imaginary roots which occur in conjugate pairs.

A polynomial $P(x)$ of odd degree with real coefficients always has at least one real root.

In case of the limit of a function, we are concerned with the manner in which a function, say $y = f(x)$, approaches a value as the independent variable, x, approaches a particular value a (say), which may be finite or infinite. When a is finite, x can approach a either through values which are less than a, to be denoted as $x \to a-$, or through values which are greater than a, to be denoted as $x \to a+$. The limit of a function describes the behaviour of the function $y = f(x)$ as x behaves in a particular way.

7.4.1 Definition

A function $y = f(x)$ is said to have a limit b (a finite number) as $x \to a$ if

$$\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = b.$$

Here, $\lim_{x \to a-} f(x)$ is termed as left-hand limit (LHL) and $\lim_{x \to a+} f(x)$ as the right-hand limit (RHL) of the function. Alternatively, the existence of limit can also be expressed as $\lim_{x \to a} f(x) = b$. The left-hand limit (LHL) and right-hand limit (RHL) are individually called one-sided limits as well. A more precise definition of the limit can be given as follows:

Let $f(x)$ be defined in an open interval about $x = a$, except possibly at a. If $f(x)$ gets arbitrarily close to b (a finite number) for x sufficiently close to a, we say that $f(x)$ approaches the limit b as x approaches a, and write $\lim_{x \to a} f(x) = b$.

Note that when a is not finite, we need to consider only one side limit; the left-hand limit when $x \to \infty$ and the right-hand limit when $x \to -\infty$.

Further, if $\lim_{x \to a} f(x)$ is not a finite number, we say that the function does not have a limit. This case, however, is often written as $\lim_{x \to a} f(x) = +\infty$ (or $-\infty$) and we often say that limit of the function is infinity.

A formal definition of the limit of a function can be given as follows:

A function $f(x)$ is said to approach the limit b as $x \to a$ if for any positive number , howsoever small, we can find a number δ such that $|f(x) - b| < \epsilon$, for all values of x satisfying $|x - a| < \delta$ except possibly at $x = a$.

It may be noted here that the function need not be defined at $x = a$. Even if the function is defined, $f(a)$ need not be equal to b.

7.4.2 Fundamental Theorems on Limits

Theorem 1: Suppose that f and g are functions such that $f(x) = g(x)$ for all x in some open interval interval containing a except possibly for a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

Theorem 2: Suppose that f and g are functions such that the two limits

$$\lim_{x \to a} f(x) \ and \ \lim_{x \to a} g(x)$$

exist, suppose that k is a constant and suppose that n is a positive integer. Then

(a) $\lim\limits_{x \to a} k = k$

(b) $\lim\limits_{x \to a} x = a$

(c) $\lim\limits_{x \to a} k \cdot f(x) = k \cdot \lim\limits_{x \to a} f(x)$

(d) $\lim\limits_{x \to a}[f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(e) $\lim\limits_{x \to a}[f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(f) $\lim\limits_{x \to a}[f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

(g) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ provided $\lim\limits_{x \to a} g(x) \neq 0$

(h) $\lim\limits_{x \to a} x^n = a^n$

(i) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$

(j) $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ provided $a > 0$

(k) $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ provided $\lim\limits_{x \to a} f(x) > 0$ when n is even

(l) If $f(x) \leq g(x)$ for all $x \neq a$ then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$

(m) If f is a polynomial then $\lim\limits_{x \to a} f(x) = f(a)$

(n) If f is a rational function then, for all a in the domain of f, $\lim\limits_{x \to a} f(x) = f(a)$

Theorem 3: The limit $\lim\limits_{x \to a} f(x) = M$

If and only if the right-hand and left-hand limits exist and are equal to M:

$$\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a-} f(x) = M$$

Theorem 4: (Squeeze Theorem) Suppose that f, g and h are three functions such that $f(x) \leq g(x) \leq h(x)$ for all x. If

$$\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = M$$

Then

$$\lim\limits_{x \to a} g(x) = M$$

Theorem 5: Suppose that f and g are two functions such that

$\lim\limits_{x \to a} f(x) = L \neq 0$ and $\lim\limits_{x \to a} g(x) = 0$ then the limit $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ does not exist.

The above method of finding limit of a function may become very cumbersome particularly when the function $f(x)$ is not linear.

Alternatively, the limit of a function can be determined by following certain rules, given below:

1. Limit of a constant is constant itself.

2. If $y = x$, then $\lim_{x \to a} y = a$.

   For two functions, $f(x)$ and $g(x)$, such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$,

   where l and m are real numbers.

Example: Find the limit of $y = 2x + 5$ as $x \to 2$.

Solution:

Here we have to determine the behaviour of y values as x becomes nearer to 2. To find LHL, we make the following table:

x	0	1	1.5	1.9	1.99
y	5	7	8	8.8	8.98

From the above table, it is obvious that $\lim_{x \to 2-} (2x + 5) = 9$.

Similarly, to find the RHL, we make the following table:

x	4	3	2.5	2.1	2.01
y	13	11	10	9.2	9.02

Thus we, have $\lim_{x \to 2+} (2x + 5) = 9$.

Since LHL = RHL, we can write $\lim_{x \to 2} (2x + 5) = 9$.

7.4.3 Algebra of Limits

Sum or Difference Rule: $\lim_{x \to a} \{ f(x) \pm g(x) \} = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l \pm m$

We note that if l and m are infinite numbers then $\infty + \infty = \infty$, but $\infty - \infty$ is indeterminate.

Product Rule: $\lim_{x \to a} \{ f(x) \cdot g(x) \} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = l.m$

Here again $\infty \times \infty = \infty$ but is $0 \times \infty$ is indeterminate.

Using the above rules, the limit of a polynomial of degree n can be written as given below:

If $\quad \phi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots\ldots + \alpha_n x^n$,

Then $\quad \lim_{x \to a} \phi(x) = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \ldots\ldots + \alpha_n a^n = \phi(a)$

It follows from the above that **the limit of a polynomial $\phi(x)$, as $x \to a$, is given by the value of the polynomial $\phi(a)$.**

Quotient Rule: $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{l}{m}$ (l and m are finite and $m \neq 0$)

It may be noted here that $\dfrac{0}{0}$ and $\dfrac{\pm\infty}{\pm\infty}$ are indeterminate.

Power Rule: If p and q are integers, then $\lim\limits_{x\to a}\left[f(x)\right]^{p/q} = l^{p/q}$, provided $l^{p/q}$ is a real number.

The case, when $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}\left(a\ \dfrac{0}{0}\ \text{form}\right)$ exists, is of great practical importance because, as we shall see later, it forms the basis for the development of differential calculus.

Most of the problems regarding the finding of limits involve the evaluation of the indeterminate forms mentioned above.

7.4.4 Standard Limits

Let us consider some examples to have the better understanding of the concept.

Example: Evaluate each of the following limits:

(i) $\lim\limits_{x\to 4}\left(\dfrac{3x+5}{2x-3}\right)$

(ii) $\lim\limits_{x\to 0}\left(2x^2+5x-10\right)$

(iii) $\lim\limits_{x\to 1}\left(\dfrac{x^2+3x+2}{x^3+2x^2-x+1}\right)$

(iv) $\lim\limits_{x\to 3}\left(\dfrac{2x^2-x-15}{x^2-5x+6}\right)$

(v) $\lim\limits_{x\to 0}\dfrac{1}{x}\left\{1-\sqrt{1-x}\right\}$

Solution:

(i) Let $f(x) = 3x + 5$ and $g(x) = 2x - 3$

Using polynomial rule, we have (finite)

and $\lim\limits_{x\to 4} g(x) = 2\times 4 - 3 = 5$

$\therefore\quad \lim\limits_{x\to 4}\dfrac{3x+5}{2x-3} = \dfrac{17}{5} = 3.4.$

(ii) $\lim\limits_{x\to 0}\left(2x^2+5x-10\right) = -10.$

(iii) Note that $\lim\limits_{x\to 1}\left(x^2+3x+2\right) = 6$ and $\lim\limits_{x\to 1}\left(x^3+2x^2-x+1\right) = 3$

$\therefore\quad \lim\limits_{x\to 1}\dfrac{x^2+3x+2}{x^3+2x^2-x+1} = \dfrac{6}{3} = 2.$

(iv) Since $\lim\limits_{x\to 3}\left(2x^2-x-15\right) = 0$ and $\lim\limits_{x\to 3}\left(x^2-5x+6\right) = 0$,

This implies that $x = 3$ is a root and hence $(x - 3)$ is a factor of both the polynomials.

$\therefore\quad \lim\limits_{x\to 3}\dfrac{2x^2-x-15}{x^2-5x+6} = \lim\limits_{x\to 3}\dfrac{(x-3)(2x+5)}{(x-3)(x-2)} = \lim\limits_{x\to 3}\dfrac{2x+5}{x-2} = 11.$

(v) We note that the ratio is a $\dfrac{0}{0}$ form. In this case, we can find limit by rationalisation.

We can write

$$\frac{1-\sqrt{1-x}}{x} = \frac{1-\sqrt{1-x}}{x} \times \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}} = \frac{1-1+x}{x\left(1+\sqrt{1-x}\right)} = \frac{1}{1+\sqrt{1-x}}$$

$$\therefore \qquad \lim_{x\to 0}\frac{1}{x}\left\{1-\sqrt{1-x}\right\} = \lim_{x\to 0}\frac{1}{1+\sqrt{1-x}} = \frac{1}{2}.$$

Do not conclude that left hand side limit and right hand side limit will always be same. There can be instances when both are not same. It is also possible that either of the two doesn't exist. It is also possible that BOTH limits do not exist.

The case, when $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}\left(a\ \dfrac{0}{0}\ \text{form}\right)$ exists, is of great practical importance because, as we shall see later, it forms the basis for the development of differential calculus.

Most of the problems regarding the finding of limits involve the evaluation of the indeterminate forms mentioned above.

7.4.5 Limits at Infinity

Evaluation of a $\dfrac{0}{0}$ *or* $\dfrac{\pm\infty}{\pm\infty}$ *Forms*

As a general rule, to find $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$, the ratio $\dfrac{f(x)}{g(x)}$ should first be simplified or manipulated algebraically as mentioned below:

(a) If $\lim\limits_{x\to a} f(x) = 0$, where $f(x)$ is a polynomial, then it implies that $x = a$ is a root of $f(x)$. Similarly, $x = a$ will be root of the polynomial $g(x)$ if $\lim\limits_{x\to a} g(x) = 0$. This suggests that whenever $\dfrac{f(x)}{g(x)}$ is of $\dfrac{0}{0}$ form as $x \to a$, than $(x-a)$ will be factor of $f(x)$ and $g(x)$ which can be cancelled before evaluating the limit.

(b) To evaluate $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}$ when it is a $\dfrac{\pm\infty}{\pm\infty}$ form, we divide the numerator and the denominator by the highest power of x in $g(x)$ and simplify.

Example: Evaluate each of the following limits:

(i) $\lim\limits_{x\to\pm\infty}\dfrac{a_1 x^2 + b_1 x + c_1}{a_2 x^2 + b_2 x + c_2}$
    (ii) $\lim\limits_{x\to\pm\infty}\dfrac{3x+2}{x^2-1}$
    (iii) $\lim\limits_{x\to\pm\infty}\dfrac{2x^2+5}{x+3}$

Solution:

(i) Dividing the numerator and denominator by x^2, we get

$$\lim_{x\to\pm\infty}\frac{a_1 + \dfrac{b_1}{x} + \dfrac{c_1}{x^2}}{a_2 + \dfrac{b_2}{x} + \dfrac{c_2}{x^2}} = \frac{a_1}{a_2}.$$

(ii) Dividing the numerator and denominator by x^2, we get

$$\lim_{x \to \pm\infty} \frac{\dfrac{3}{x} + \dfrac{2}{x^2}}{1 - \dfrac{1}{x^2}} = 0$$

(iii) Dividing the numerator by denominator, we get

$$\lim_{x \to \pm\infty} \frac{2x^2 + 5}{x + 3} = \lim_{x \to \pm\infty} \left(2x - 6 + \frac{23}{x + 3} \right) = \pm\infty$$

7.4.6 Limits of Trigonometric Functions

Theorem 1: For each point c in functions domain:

$$\lim_{x \to c} \sin x = \sin c$$

$$\lim_{x \to c} \tan x = \tan c$$

$$\lim_{x \to c} \csc x = \csc c$$

$$\lim_{x \to c} \cos x = \cos c$$

$$\lim_{x \to c} \cot x = \cot c$$

$$\lim_{x \to c} \sec x = \sec c$$

Example: Evaluate limit

$$\lim_{x \to \pi/4} \theta \tan(\theta)$$

Since $\theta = \pi/4$ is in the domain of the function $\theta \tan(\theta)$ we use Substitution Theorem to substitute $\pi/4$ for q in the limit expression:

$$\lim_{x \to \pi/4} \theta \tan(\theta) = \frac{\pi}{4} \tan\left(\frac{\pi}{4} \right) = \frac{\pi}{4} . 1 = \frac{\pi}{4}$$

Special Trigonometric Limits

$\text{Sin}(x)/x \to ?$ as $x \to 0$

Theorem 2

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Theorem 3

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Example: Evaluate limit

$$\lim_{t \to 0} \frac{\tan t}{t}$$

Recalling tan t = sin t/cost t, and using 2:

$$\lim_{t \to 0} \frac{\sin t}{(\cos t)t} = \left[\lim_{t \to 0} \frac{\sin t}{t} \right] \lim_{t \to 0} \frac{1}{\cos t} = 1 . \frac{1}{1} = 1$$

7.4.7 Logarithmic Functions

$$\lim_{x \to +\infty} \log_a x = +\infty \quad \textit{if } a > 1$$

$$\lim_{x \to 0^+} \log_a x = +\infty \quad \textit{if } a < 1$$

$$\lim_{x \to +\infty} \log_a x = +\infty \quad \textit{if } a > 1$$

$$\lim_{x \to +\infty} \log_a x = -\infty \quad \textit{if } a < 1$$

$$\lim_{x \to 0^+} x^b \log_a x = 0 \quad \textit{if } b > 0$$

$$\lim_{x \to +\infty} \frac{1}{x^b} \log_a x = 0 \quad \textit{if } b > 0$$

7.4.8 Exponential Functions

To calculate the limit of an exponential function, we must note that:

If a > 0

Source: http://www.ditutor.com/limits/limit_exponential.html

$$\lim_{x \to \infty} a^x = \infty$$
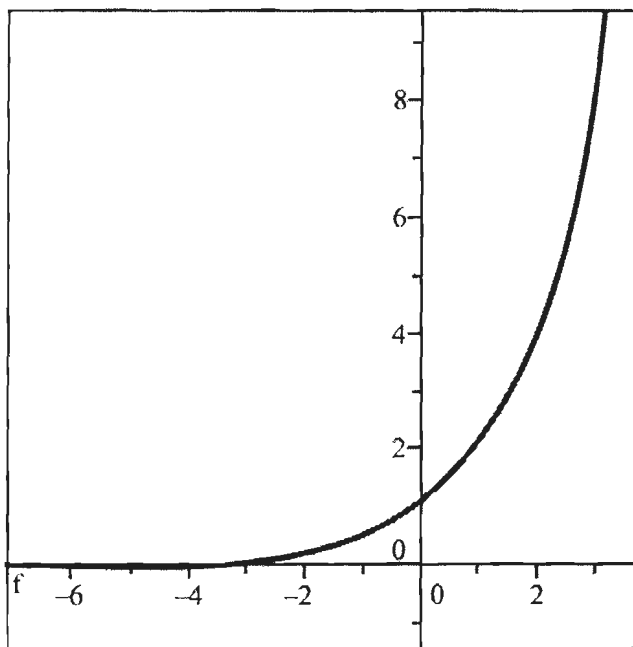
$$\lim_{x \to -0} a^x = 0$$

If 0 < a < 1
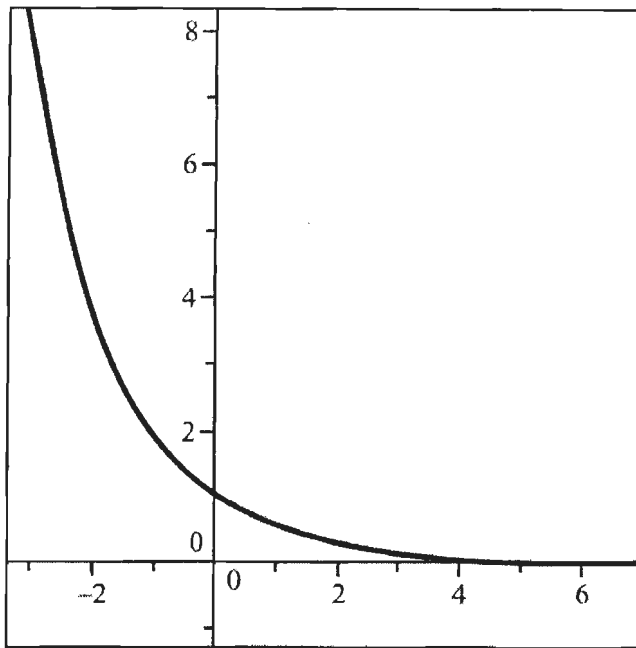
Source: http://www.ditutor.com/limits/limit_exponential.html

$$\lim_{x \to \infty} a^x = \infty$$

$$\lim_{x \to -\infty} a^x = 0$$

$$\lim_{x \to \infty} 3^{x+2} = \infty$$

$$\lim_{x \to -\infty} 3^{x+2} = \lim_{x \to \infty} 3^{-x+2} = \lim_{x \to \infty} 3^{-(x-2)} = \lim_{x \to \infty} \left(\frac{1}{3^{x-2}}\right) = \frac{1}{\infty} = 0$$

$$\lim_{x \to \infty} \left(\frac{1}{3}\right)^{x+2} = 0$$

$$\lim_{x \to -\infty} \left(\frac{1}{3}\right)^{x+2} = \lim_{x \to \infty} \left(\frac{1}{3}\right)^{-x+2} = \lim_{x \to \infty} \left(\frac{1}{3}\right)^{-(x-2)} = \lim_{x \to \infty} 3^{x-2} = \infty$$

It is important to appreciate the behavior of exponential functions as the input to them becomes a large positive number, or a large negative number. This behavior is different from the behavior of polynomials or rational functions, which behave similarly for large inputs regardless of whether the input is large positive or large negative. By contrast, for exponential functions, the behavior is radically different for large positive or large negative.

$$\lim_{x \to c} x^r = c^r \qquad \text{if } r \text{ is a positive integer}$$

$$\lim_{x \to 0+} \frac{1}{x^r} = +\infty$$

$$\lim_{x \to 0-} \frac{1}{x^r} = \begin{cases} -\infty, & \text{if } r \text{ is odd} \\ +\infty, & \text{if } r \text{ is even} \end{cases}$$

State whether the following statements are True or False:

1. Left hand side limit and Right hand side limit will always be same.

2. It is important to appreciate the behaviour of exponential functions as the input to them becomes a large positive number or a large negative number.

3. By contrast, for exponential functions, the behaviour is radically different for large positive or large negative.

Fill in the blanks:

4. The _____ of a real function f (x) of a real variable is the set of all values attained by f(x) at different points (values) in its domain.

5. An Interval of the variable x between a and b is usually denoted by _____.

7.5 LET US SUM UP

- A function is a technical term used to symbolise relationship between variables.

- A function can be expressed by formula, graph or by equation.

- If the power of every variable involved in the equation (or function) is one, the function is termed as linear function.

- If a variable x which can take only those numerical values which lie between two given numbers a and b then all the numerical values between a and b is called domain or interval of the variable x and is usually denoted by (a, b).

- The range of a real function $f(x)$ of a real variable is the set of all values attained by $f(x)$ at different points (values) in its domain.

- The limit of a function is that fixed value (l) to which a function $\{y = f(x)\}$ approaches as the independent variable approaches a given value (a).

- There are left hand side limit and right hand limit.

- It is not necessary that the left hand limit and right hand limit are the same, always.

7.6 LESSON END ACTIVITY

Make a list of the limit of an exponential function and logarithmic functions.

7.7 KEYWORDS

Function: It is the relationship between variables.

Real Numbers: The system of rational and irrational numbers together forms the system of real numbers. The set of all real numbers is denoted by R.

Domain: Set of all numbers (values) can be occupied by independent variable.

Range: Set of all numbers (values) can be occupied by dependent variable.

Limit: A value which can be approached by dependent variable when independent variable approaches a specific value or number.

7.8 QUESTIONS FOR DISCUSSION

1. Find the domain and range of the following functions:

   (i) $y = 4x^3 + 2x^2 + x - 5$ (ii) $y = -x^4 + 7x^3 - 2x^2 + 4$

2. If $f(x) = \dfrac{x-2}{x+1}$, find (i) $f(-x)$, (ii) $f(2x)$ and (iii) $f(x+5)$.

3. Find the value of $f\left(\dfrac{1}{a}\right)$ and show that $f\left(\dfrac{1}{a}\right) = f(a)$, when

   $$f(x) = \frac{(2x-1)(x-2)}{1-2x+x^2}$$

4. Find the equation of a line passing through the following pairs of points:

   (i) $P(-3, 10)$ and $Q(-5, 18)$.

   (ii) $P(-2, -4)$ and $Q(-6, -20)$

   (iii) $P(a, -b)$ and $Q(-a, b)$

   (iv) $P(a, b)$ and $Q(c, d)$.

5. Find the slope if it exists, and intercept of the following equations and draw their graphs:

   (i) $4x + 3y = 24$

   (ii) $6x - 7y + 49 = 0$

   (iii) $y = -5$

   (iv) $x = 2$

6. Show that the points $(1, -1)$, $(-2, 8)$ and $(0, 2)$ are collinear.

7. (i) Find the values of a, b and c so that $y = ax^2 + bx + c$ passes through the point $(1, 2)$, $(2, 3)$ and $(3, 6)$.

   (ii) If $y = k - kx^2$, where k is a constant, draw a family of curves by taking $k = 1$, 2, 3.

8. Show that the equation $x^3 + 3x + 1 = 0$ has at least one real root.

9. Find the indicated limits of the following functions:

   (i) $\displaystyle\lim_{x \to 2} \frac{3x+2}{2x+5}$ (ii) $\displaystyle\lim_{x \to -2} \frac{x+3}{4x+8}$ (iii) $\displaystyle\lim_{x \to 1} \frac{x^3-1}{x-1}$

10. Evaluate:

    (i) $\displaystyle\lim_{x \to 1} \frac{x^4-1}{x-1}$ (ii) $\displaystyle\lim_{x \to a} \frac{x^3-a^3}{x-a}$ (iii) $\displaystyle\lim_{x \to 3}\left[\frac{3}{x-3} - \frac{x}{x-3}\right]$

11. Prove that $\displaystyle\lim_{x \to \infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \frac{2}{3}$.

12. Show that $\displaystyle\lim_{x \to 0} \frac{10^x - 2^x - 5^x + 1}{x^2} = \log_e 2 \cdot \log_e 5$

Check Your Progress: Model Answer

1. False

2. True

3. True

4. Domain

5. (a, b)

7.9 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

Robert T. Smith, *Calculus: Single Variable*, Tata McGraw Hill.

LESSON

8

METHODS OF DIFFERENTIATION

8.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Define the term differentiation
- Describe the methods of differentiation
- Understand the function of a composite function
- Describe implicit functions & logarithmic differentiation

8.1 INTRODUCTION

In the previous lesson, we studied about the introduction to calculus. In this lesson, we will discuss the methods of differentiation and partial derivative. As you know the process of finding a derivative is called differentiation. The reverse process is called anti-differentiation. The fundamental theorem of calculus states that anti-differentiation is the same as integration. Differentiation and integration constitute the two fundamental operations in single-variable calculus.

8.2 DIFFERENTIATION

The derivative of a function of a real variable measures the sensitivity to change of the function (output) value with respect to a change in its argument (input value). Derivatives are a fundamental tool of calculus. For example, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the position of the object changes when time advances.

8.2.1 Definition

A function $f(x)$ is said to be differentiable at $x = a$ if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists. This limit is denoted by $f'(a)$.

$$\therefore \quad f'(a) = \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Note: $f'(a)$ exists if the $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists as $x \to a$ through values $< a$ (left hand limit) and $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ through values $> a$ (right hand limit) exist and further they are equal.

8.2.2 Notation

Two distinct notations are commonly used for the derivative, one deriving from Leibniz and the other from Joseph Louis Lagrange.

In Leibniz's notation, an infinitesimal change in x is denoted by dx, and the derivative of y with respect to x is written $\dfrac{dy}{dx}$, suggesting the ratio of two infinitesimal quantities. (The above expression is read as "the derivative of y with respect to x", "dy by dx", or "dy over dx". The oral form "$dy\ dx$" is often used conversationally, although it may lead to confusion.)

8.3 METHODS OF DIFFERENTIATION

Methods of differentiation explores various rules including the product, quotient, chain, power, exponential and logarithmic rules.

8.3.1 Derivative of a Function – Method of First Principles

If $y = f(x)$ is a function then as x changes y also changes.

A change in x is called the increment in x and is denoted by Δx. Corresponding change in y is called increment in y and is denoted by Δy.

$\therefore$ as x changes to $x + \Delta x$, y changes to $y + \Delta y$.

First Principles

Let $\quad y = f(x)$ ...(1)

$\therefore \quad y + \Delta y = f(x + \Delta x)$ ...(2)

Subtracting (1) from (2), we get

$$\Delta y = f(x + \Delta x) - f(x)$$

Divide both sides by Δx

$\therefore \quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Taking limits as $\Delta x \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If this limit exists then it is called the derivative of y w.r.t., x and is denoted by $\dfrac{dy}{dx}$.

$\therefore \quad \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$\dfrac{dy}{dx}$ is also called the differential coefficient of y w.r.t., x.

Note: $\dfrac{dy}{dx}$ should not be read as the product of d and y divided by the product of d and x.

In fact, $\dfrac{d}{dx}$ is the symbol for the derivative w.r.t. x or differential coefficient w.r.t. x.

8.3.2 General Theorems on Derivatives (Without Proof)

Theorem 1: $\dfrac{d(k)}{dx} = 0$ where k is a constant.

Theorem 2: $\dfrac{d}{dx}[kf(x)] = k\dfrac{d}{dx}[f(x)]$ where k is a constant and $f(x)$ is a function of x.

or

$$\frac{d}{dx}(ku) = k\frac{du}{dx}$$

Addition Rule

Theorem 3: $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{d}{dx}[f(x)] + \dfrac{d}{dx}[g(x)]$

or

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Substraction Rule

Theorem 4: $\dfrac{d}{dx}[f(x) - g(x)] = \dfrac{d}{dx}[f(x)] - \dfrac{d}{dx}[g(x)]$

or

$$\frac{d}{dx}[u - v] = \frac{du}{dx} - \frac{dv}{dx}$$

Product Rule

Theorem 5: $\dfrac{d}{dx}[f(x) \cdot g(x)] = f(x)\dfrac{d}{dx}[g(x)] + g(x)\dfrac{d}{dx}[f(x)]$

This is called the product rule.

or $\qquad \dfrac{d}{dx}[(uv)] = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

This can be remembered as:

$$\frac{d}{dx}[I \text{ function} \times II \text{ function}]$$

$$= (I \text{ function})\frac{d}{dx}(II \text{ function}) + (II \text{ function})\frac{d}{dx}(I \text{ function})$$

Note: $\dfrac{d}{dx}(uvw) = uv\dfrac{dw}{dx} + uw\dfrac{dv}{dx} + vw\dfrac{du}{dx}$

Quotient Rule

Theorem 6: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}[f(x)] - f(x) \cdot \dfrac{d}{dx}[g(x)]}{[g(x)]^2}$

This is called the Quotient Rule.

or $\qquad \dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

This can be remembered as:

$$\frac{d}{dx}\left[\frac{Nr}{Dr}\right] = \frac{Dr\dfrac{d}{dx}(Nr) - Nr\dfrac{d}{dx}(Dr)}{(Dr)^2}$$

where Nr = Numerator, Dr = Denominator

Notes:

1. While doing problems on differentiation, the above theorems should be strictly followed.

2. The above theorems can be proved using the method of first principles. Since there is no mention of proofs of the theorems in the syllabus, the proofs are not given here. Only statements are given because they have to be used in the problems.

8.3.3 Derivatives of Standard Functions

1. $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$

2. $\dfrac{d}{dx}(x) = 1$

3. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$

4. $\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$

5. $\dfrac{d}{dx}\left(e^x\right) = e^x$

6. $\dfrac{d}{dx}\left(a^x\right) = a^x \log_e a$, where $a > 0$ and $a \neq 1$

7. $\dfrac{d}{dx}\left(\log_e x\right) = \dfrac{1}{x}$ where $x > 0$.

8. $\dfrac{d}{dx}\left(\log_a x\right) = \dfrac{1}{x \log_e a}$

9. $\dfrac{d}{dx}\left(ax + b\right)^n = na\left(ax + b\right)^{n-1}$

10. $\dfrac{d}{dx}\left(\dfrac{1}{ax + b}\right) = -\dfrac{1}{a} \cdot \dfrac{1}{\left(ax + b\right)^2}$

11. $\dfrac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$

12. $\dfrac{d}{dx} \log_e \left(ax + b\right) = a \cdot \dfrac{1}{ax + b}$

8.4 FUNCTION OF A FUNCTION (OR COMPOSITE FUNCTION)

If a function is made up of more than one function then it is called a composite function. A composite function is denoted by the symbol, $f(g(x))$, $f(g(h(x)))$, etc.

8.4.1 To Find the Derivative of a Composite Function

Chain Rule

To find the derivative of $f(g(x))$, we use a rule called chain rule.

Let $y = f(g(x))$, $u = g(x)$

$\therefore$ $y = f(u)$ and $u = g(x)$

By differentiating y w.r.t. u, we get $\dfrac{dy}{du}$ and by differentiating u w.r.t. x, we get $\dfrac{du}{dx}$.

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

This is called the chain rule.

Similarly, let $y = f(g(h(x)))$, $u = g(h(x)), v = h(x)$

$\therefore \qquad\qquad y = f(u), u = g(v), v = h(x)$

By differentiating y w.r.t. u, we get $\dfrac{dy}{du}$, by differentiating u w.r.t. v, we get $\dfrac{du}{dv}$ and

by differentiating v w.r.t. x, we get $\dfrac{dv}{dx}$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$ which is chain rule.

General Rules

1. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\cdot\dfrac{d}{dx}f(x)$

2. $\dfrac{d}{dx}[e^{f(x)}] = e^{f(x)}\cdot\dfrac{d}{dx}f(x)$

3. $\dfrac{d}{dx}[\log_e f(x)] = \dfrac{1}{f(x)}\cdot\dfrac{d}{dx}f(x)$ etc.

Differentiate the following functions w.r.t. x:

Example 1: $\sqrt{x^2 + 1}$

Solution:

Let $y = \sqrt{x^2 + 1}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x^2+1}}\cdot\dfrac{d}{dx}(x^2+1)$

$\qquad = \dfrac{1}{2\sqrt{x^2+1}}\cdot 2x$

$\qquad = \dfrac{x}{\sqrt{x^2+1}}$

Example 2: $(ax^2 + bx + c)^5$

Solution:

Let $y = (ax^2 + bx + c)^5$

$\therefore \quad \dfrac{dy}{dx} = 5(ax^2 + bx + c)^{5-1}\cdot\dfrac{d}{dx}(ax^2 + bx + c)$

$\qquad = 5(ax^2 + bx + c)^4\cdot(2ax + b)$

Example 3: e^{2x+3}

Solution:

Let $y = e^{2x+3}$

$$\therefore \quad \frac{dy}{dx} = e^{2x+3} \cdot \frac{d}{dx}(2x+3)$$

$$= e^{2x+3} \cdot (2)$$

$$= 2e^{2x+3}$$

Example 4: $\log(2x^2 - 5x + 7)$

Solution:

Let $y = \log(2x^2 - 5x + 7)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2x^2 - 5x + 7} \cdot \frac{d}{dx}(2x^2 - 5x + 7)$$

$$= \frac{1}{2x^2 - 5x + 7} \cdot (4x - 5)$$

Example 5: $(x \log x)^{\frac{1}{5}}$

Solution:

Let $y = (x \log x)^{\frac{1}{5}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{5}(x \log x)^{\frac{1}{5}-1} \cdot \frac{d}{dx}(x \log x)$$

$$= \frac{1}{5}(x \log x)^{\frac{-4}{5}} \cdot \left(x \cdot \frac{1}{x} + \log x \cdot 1 \right)$$

$$= \frac{1}{5}(x \log x)^{\frac{-4}{5}} \cdot (1 + \log x)$$

Example 6: $(x^2 e^x)^{-2}$

Solution:

Let $y = (x^2 e^x)^{-2}$

$$\therefore \quad \frac{dy}{dx} = (-2)(xe^x)^{-2-1} \cdot \frac{d}{dx}(x^2 e^x)$$

$$= (-2)(x^2 e^x)^{-3} \cdot (x^2 e^x + e^x \cdot 2x)$$

$$= (-2)(x^2 e^x)^{-3} \cdot e^x (x^2 + 2x)$$

$$= (-2)x^{-6} e^{-3x} \cdot e^x (x^2 + 2x)$$

$$= -2e^{-2x} (x^{-4} + 2x^{-5})$$

Example 7: $\sqrt{\dfrac{x^2+1}{x^2-1}}$

Solution:

Let $y = \sqrt{\dfrac{x^2+1}{x^2-1}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\dfrac{x^2+1}{x^2-1}}} \cdot \dfrac{d}{dx}\left(\dfrac{x^2+1}{x^2-1}\right)$

$\qquad = \dfrac{1}{2\sqrt{\dfrac{x^2+1}{x^2-1}}} \cdot \dfrac{\left(x^2-1\right)2x - \left(x^2+1\right)2x}{\left(x^2-1\right)^2}$

$\qquad = \dfrac{\sqrt{x^2-1}\cdot(-4x)}{2\sqrt{x^2+1}\sqrt{x^2-1}\sqrt{x^2-1}\left(x^2-1\right)}$

$\qquad = \dfrac{-2x}{\sqrt{x^2+1}\sqrt{x^2-1}\left(x^2-1\right)}$

$\qquad = \dfrac{-2x}{\sqrt{\left(x^2+1\right)\left(x^2-1\right)}\cdot\left(x^2-1\right)}$

$\qquad = \dfrac{-2x}{\left(x^2-1\right)\sqrt{x^4-1}}$

Example 8: $\log\left(\dfrac{2+3x}{2-3x}\right)$

Solution:

Let $y = \log\left(\dfrac{2+3x}{2-3x}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{2+3x}{2-3x}\right)} \cdot \dfrac{(2-3x)(3) - (2+3x)(-3)}{(2-3x)^2}$

$\qquad = \dfrac{(2-3x)}{(2+3x)} \cdot \dfrac{12}{(2-3x)^2}$

$\qquad = \dfrac{12}{(2+3x)(2-3x)}$

$\qquad = \dfrac{12}{4-9x^2}$

Example 9: $\log(\log x)$

Solution:

Let $y = \log(\log x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\log x} \cdot \dfrac{d}{dx}(\log x)$

$\qquad\qquad = \dfrac{1}{\log x} \cdot \dfrac{1}{x} = \dfrac{1}{x \log x}$

Example 10: $\sqrt[3]{2+3x}$

Solution:

Let $y = \sqrt[3]{2+3x}$

$\Rightarrow \quad y = (2x+3)^{\frac{1}{3}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{3}(2+3x)^{\frac{1}{3}-1} \cdot \dfrac{d}{dx}(2+3x)$

$\qquad\qquad = \dfrac{1}{3}(2+3x)^{\frac{-2}{3}} \cdot (3)$

$\qquad\qquad = \dfrac{1}{\sqrt[3]{(2+3x)^2}}$

Example 11: $\log\left(x + \sqrt{1+x^2}\right)$

Solution:

Let $y = \log\left(x + \sqrt{1+x^2}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{x+\sqrt{1+x^2}} \cdot \dfrac{d}{dx}\left(x + \sqrt{1+x^2}\right)$

$\qquad\qquad = \dfrac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \dfrac{1}{2\sqrt{1+x^2}} \cdot \dfrac{d}{dx}(1+x^2)\right)$

$\qquad\qquad = \dfrac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \dfrac{1}{2\sqrt{1+x^2}} \cdot (2x)\right)$

$\qquad\qquad = \dfrac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \dfrac{x}{\sqrt{1+x^2}}\right)$

$\qquad\qquad = \dfrac{1}{x+\sqrt{1+x^2}} \cdot \left(\dfrac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}\right)$

$\qquad\qquad = \dfrac{\left(x + \sqrt{1+x^2}\right)}{\left(x+\sqrt{1+x^2}\right)\sqrt{1+x^2}}$

$\qquad\qquad = \dfrac{1}{\sqrt{1+x^2}}$

8.5 IMPLICIT FUNCTIONS

If a function is in the form $y = f(x)$, then the function is said to be in the explicit form. Instead of this, if the variables x and y are related by means of an equation, then the function is said to be in the implicit form. In general, an implicit function is given by $f(x, y) = c$ where c is a constant.

e.g., $y^2 = 4ax$, $x^2 + y^2 = a^2$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

8.5.1 To Find the Derivative of the Implicit Function f(x, y) = c

Differentiate $f(x, y) = c$ using the rules of differentiation. Collect all the terms containing $\dfrac{dy}{dx}$ on the left hand side and the remaining terms on the right hand side.

Take the common factor $\dfrac{dy}{dx}$ on the left hand side. Divide both sides by the coefficient of $\dfrac{dy}{dx}$ to get $\dfrac{dy}{dx}$.

Find $\dfrac{dy}{dx}$, if

Example 1: $y^2 = 4ax$

Solution:

Differentiate both sides w.r.t. x

$$2y \frac{dy}{dx} = 4a(1)$$

$\therefore \qquad \dfrac{dy}{dx} = \dfrac{4a}{2y}$ i.e., $\dfrac{dy}{dx} = \dfrac{2a}{y}$

Example 2: $x^2 + y^2 = 2xy$

Solution:

Differentiate w.r.t. x

$\therefore \qquad 2x + 2y\dfrac{dy}{dx} = 2\left(x\dfrac{dy}{dx} + y\right)$

Cancelling 2 on both sides, we get

$$x + y\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$\Rightarrow \qquad y\dfrac{dy}{dx} - x\dfrac{dy}{dx} = y - x$

$\Rightarrow \qquad (y - x)\dfrac{dy}{dx} = (y - x)$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{y - x}{y - x}$

$\Rightarrow \qquad \dfrac{dy}{dx} = 1$

Example 3: $x^2 + y^2 = a^2$

Solution:

Differentiate w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow$
$$2y\frac{dy}{dx} = -2x$$

$\Rightarrow$
$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$\Rightarrow$
$$\frac{dy}{dx} = \frac{-x}{y}$$

Example 4: $x^2 + y^2 + 2x - 4y + 6 = 0$

Solution:

Differentiate w.r.t. x

$$2x + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$$

Cancelling 2, we get

$$x + y\frac{dy}{dx} + 1 - 2\frac{dy}{dx} = 0$$

$\therefore$
$$\frac{dy}{dx}(y - 2) = -x - 1$$

$\therefore$
$$\frac{dy}{dx} = \frac{-(x + 1)}{y - 2}$$

Example 5: $e^{x+y} = e^x$

Solution:

Differentiate w.r.t. x

$$e^{x+y} \cdot \frac{d}{dx}(x + y) = e^x$$

i.e.,
$$e^{x+y}\left(1 + \frac{dy}{dx}\right) = e^x$$

$\therefore$
$$1 + \frac{dy}{dx} = \frac{e^x}{e^{x+y}} = \frac{1}{e^y}$$

i.e.,
$$\frac{dy}{dx} = \frac{1 - e^y}{e^y}$$

Example 6: $x^{2/3} + y^{2/3} = a^{2/3}$

Solution:

Differentiate w.r.t. x

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1} \cdot \frac{dy}{dx} = 0$$

Dividing throughout by $\frac{2}{3}$, we get

$$x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$\therefore$

$$y^{-\frac{1}{3}}\frac{dy}{dx} = -x^{-\frac{1}{3}}$$

$\therefore$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$$

i.e.,

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^{1/3}$$

Example 7: $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\ldots\text{to}\infty}}}$

Solution:

$$y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\ldots\text{to}\infty}}}$$

$\Rightarrow$

$$y = \sqrt{\log x + y}$$

Squaring, we get

$$y^2 = \log x + y$$

Differentiate w.r.t. x

$$2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$\Rightarrow$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \frac{1}{x}$$

$\Rightarrow$

$$\frac{dy}{dx}(2y-1) = \frac{1}{x}$$

$\therefore$

$$\frac{dy}{dx} = \frac{1}{x(2y-1)}$$

Example 8: If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ where $x \neq y$, prove that $\dfrac{dy}{dx} = \dfrac{-1}{(1+x)^2}$.

Solution:

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\Rightarrow \qquad x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring,

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \qquad x^2 + x^2 y = y^2 + y^2 x$$

$$\Rightarrow \qquad x^2 - y^2 = -x^2 y + y^2 x$$

$$\Rightarrow \qquad (x+y)(x-y) = -xy(x-y)$$

$$\Rightarrow \qquad x + y = -xy \quad \text{(cancelling } (x-y)) \qquad \qquad \text{...(1)}$$

Differentiating w.r.t. x

$$1 + \frac{dy}{dx} = -\left(x\frac{dy}{dx} + y\right)$$

i.e.,
$$\frac{dy}{dx} + x\frac{dy}{dx} = -1 - y$$

$$(1+x)\frac{dy}{dx} = -(1+y)$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{(1+y)}{1+x} \qquad \qquad \text{...(2)}$$

From (1), $x = -y - xy$

$$\Rightarrow \qquad x = -y(1+x)$$

$$\Rightarrow \qquad y = \frac{-x}{1+x}$$

Substituting this in (2), we get

$$\frac{dy}{dx} = \frac{-\left(1 - \dfrac{x}{1+x}\right)}{1+x}$$

$$= -\left[\frac{1+x-x}{(1+x)(1+x)}\right]$$

$$= \frac{-1}{(1+x)^2}$$

8.6 LOGARITHMIC DIFFERENTIATION

To differentiate a function of the form $\left[f(x)\right]^{g(x)}$ or $a^{f(x)}$, we use a method called Logarithmic differentiation.

To find the derivative of the functions $\left[f(x)\right]^{g(x)}$ and $a^{f(x)}$.

(i) Let $y = \left[f(x)\right]^{g(x)}$

Taking logarithms, we get

$$\log y = g(x)\log\left[f(x)\right]$$

Differentiate w.r.t. x

$$\frac{1}{y}\cdot\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot f'(x) + \log\left[f(x)\right].g'(x)$$

$\therefore$
$$\frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}\cdot f'(x) + g'(x)\cdot\log\left(f(x)\right)\right]$$

i.e.,
$$\frac{dy}{dx} = \left[f(x)\right]^{g(x)}\left[\frac{g(x)}{f(x)}\cdot f'(x) + g'(x)\cdot\log\left[f(x)\right]\right]$$

This method of finding $\frac{dy}{dx}$ is called Logarithmic differentiation.

(ii) Let $y = a^{f(x)}$

Taking logarithms, we get

$$\log y = f(x)\log a$$

Differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \left(\log a\right)f'(x)$$

$\therefore$
$$\frac{dy}{dx} = y\left[\left(\log a\right)f'(x)\right] = a^{f(x)}\left[\left(\log a\right)f'(x)\right]$$

Example: Differentiate the following w.r.t. x:

1. x^x 2. x^{x^x} 3. $\left(x^x\right)^x$

4. 3^x 5. $x^{x^{x^{\log x}}}$

Solution:

1. Let $y = x^x$

Taking logs, we get

$$\log y = x\log x$$

Differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1$$

$\therefore$
$$\frac{dy}{dx} = y\left(1 + \log x\right)$$

i.e.,
$$\frac{dy}{dx} = x^x\left(1 + \log z\right)$$

2. Let $y = x^{x^x}$

Taking logs, we get

$$\log y = x^x \log x$$

Taking logs again, we get

$$\log(\log y) = x \log x + \log(\log x)$$

Differentiate w.r.t. x

$$\frac{1}{\log y} \cdot \frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 + \frac{1}{\log x} \cdot \frac{1}{x}$$

$\therefore \qquad \dfrac{dy}{dx} = y \log y \left[1 + \log x + \dfrac{1}{x \log x} \right]$

i.e., $\qquad \dfrac{dy}{dx} = x^{x^x} \cdot x^x \log x \left[1 + \log x + \dfrac{1}{x \log x} \right]$

3. Let $y = \left(x^x \right)^x$

$\Rightarrow \qquad y = x^{x^2}$

Taking logs, we get

$$\log y = x^2 \log x$$

Differentiate w.r.t. x

$$\frac{1}{y} \frac{dy}{dx} = x^2 \cdot \frac{1}{x} + (\log x) 2x$$

$$\frac{d}{dx} (x+1)^3 = \frac{d}{dx} \left(x^3 + 3x^2 + 3x + 1 \right)$$

i.e., $\qquad \dfrac{dy}{dx} = \left(x^x \right)^x \left[x + 2x \log x \right]$

4. Let $y = 3^x$

Taking logs, we get

$$\log y = x \log 3$$

Differentiating w.r.t. x

$$\frac{1}{y} \frac{dy}{dx} = (\log 3) \cdot 1$$

$\therefore \qquad \dfrac{dy}{dx} = y (\log 3)$

i.e., $\qquad \dfrac{dy}{dx} = 3^x (\log 3)$

5. Let $y = x^{x^{x^{...^\infty}}}$

$\Rightarrow$ $\qquad y = x^y$

Taking logs, we get

$\qquad \log y = y \log x$

Differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$\therefore \qquad \dfrac{dy}{dx}\left(\dfrac{1}{y} - \log x\right) = \dfrac{y}{x}$

$\Rightarrow \qquad \dfrac{dy}{dx}\left(\dfrac{1 - y \log x}{y}\right) = \dfrac{y}{x}$

$\therefore \qquad \dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y \log x)}$

Example: Find $\dfrac{dy}{dx}$, if

1. $e^y = y^x$
2. $y = a^{x+y} \cdot y^x$
3. $x^m y^n = (x+y)^{m+n}$
4. $x^y = y^x$

Solution:

1. $e^y = y^x$

   Taking logs, we get

   $\qquad y \log e = x \log y$

   i.e., $\qquad y(1) = x \log y \quad (\because \log e = \log_e e = 1)$

   Differentiating w.r.t. x

   $$\frac{dy}{dx} = x \cdot \frac{1}{y}\frac{dy}{dx} + \log y \cdot (1)$$

   $$\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} = \log y$$

   $\therefore \qquad \dfrac{dy}{dx}\left(1 - \dfrac{x}{y}\right) = \log y$

   i.e., $\qquad \dfrac{dy}{dx}\left(\dfrac{y - x}{y}\right) = \log y$

   $\therefore \qquad \dfrac{dy}{dx} = \dfrac{y \log y}{y - x}$

2. $y = a^{x+y} \cdot y^x$

Taking logs, we get

$$\log y = (x + y)\log a + x \log y$$

Differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = (\log x)\left(1 + \frac{dy}{dx}\right) + x \cdot \frac{1}{y}\frac{dy}{dx} + (\log y)1$$

i.e.,

$$\left(\frac{1}{y} - \log a - \frac{x}{y}\right)\frac{dy}{dx} = \log a + \log y$$

$\therefore$

$$\frac{dy}{dx} = \frac{(\log a + \log y)\,y}{(1 - y\log a - x)}$$

3. $x^m y^n = (x + y)^{m+n}$

Taking logs, we get

$$m\log x + n\log y = (m+n)\log(x+y)$$

Differentiating w.r.t. x

$$m\left(\frac{1}{x}\right) + n\left(\frac{1}{y}\frac{dy}{dx}\right) = (m+n)\frac{1}{x+y}\cdot\left(1 + \frac{dy}{dx}\right)$$

i.e.,

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\cdot\frac{dy}{dx}$$

$\therefore$

$$\frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

i.e.,

$$\frac{dy}{dx}\left(\frac{nx + ny - my - ny}{y(x+y)}\right) = \frac{mx + nx - mx - my}{x(x+y)}$$

i.e.,

$$\frac{dy}{dx}\left(\frac{nx - my}{y(x+y)}\right) = \frac{(nx - my)}{x(x+y)}$$

i.e.,

$$\frac{dy}{dx}\frac{(nx - my)}{y(x+y)} = \frac{(nx - my)}{x(x+y)}$$

i.e.,

$$\frac{dy}{dx}\cdot\frac{1}{y} = \frac{1}{x}$$

$\therefore$

$$\frac{dy}{dx} = \frac{y}{x}$$

4. $x^y = y^x$

Taking logs, we get

$$y\log x = x\log y$$

Differentiating w.r.t. x

$$y\left(\frac{1}{x}\right) + \left(\log x\right)\frac{dy}{dx} = x \cdot \frac{1}{y}\frac{dy}{dx} + \left(\log y\right)1$$

i.e.,
$$\frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \frac{y}{x}$$

i.e.,
$$\frac{dy}{dx}\left(\frac{y\log x - x}{y}\right) = \frac{x\log y - y}{x}$$

$\therefore$
$$\frac{dy}{dx} = \frac{y\left(x\log y - y\right)}{x\left(y\log x - x\right)}$$

8.7 PARTIAL DERIVATIVE

Partial derivative is a derivative of a function of two or more variables with respect to one variable, the other(s) being treated as constant.

In mathematics, a partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary). Partial derivatives are used in vector calculus and differential geometry.

The partial derivative of a function $f(x, y, ...)$ with respect to the variable x is variously denoted by:

$$f'_x, f_x, \partial_x f, D_x f, D_1 f, \frac{\partial}{\partial x}f, or \frac{\partial f}{\partial x}$$

Since in general a partial derivative is a function of the same arguments as was the original function, this functional dependence is sometimes explicitly included in the notation, as in:

$$f_x\left(x, y, ...\right), \frac{\partial f}{\partial x}\left(x, y, ...\right)$$

The partial-derivative symbol is ∂. One of the first known uses of the symbol in mathematics is by Marquis de Condorcet from 1770, who used it for partial differences.

Suppose that f is a function of more than one variable. For instance,

$$z = f\left(x, y\right) = x^2 + xy + y^2$$

The graph of this function defines a surface in Euclidean space. To every point on this surface, there is an infinite number of tangent lines. Partial differentiation is the act of choosing one of these lines and finding its slope. Usually, the lines of most interest are those that are parallel to the xz-plane, and those that are parallel to the yz-plane (which result from holding either y or x constant, respectively.)

To find the slope of the line tangent to the function at $P(1, 1)$ that is parallel to the xz-plane, the y variable is treated as constant. The graph and this plane are shown on the right. On the graph below it, we see the way the function looks on the plane $y = 1$. By finding the derivative of the equation while assuming that y is a constant, the slope of f at the point (x, y) is found to be:

$$\frac{\partial z}{\partial x} = 2x + y$$

So at $(1, 1)$, by substitution, the slope is 3. Therefore,

$$\frac{\partial z}{\partial x} = 3$$

at the point $(1, 1)$. That is, the partial derivative of z with respect to x at $(1, 1)$ is 3, as shown in the graph.

Example: Differentiate the following w.r.t. x.

1. x^{100}

2. $x^{\frac{2}{3}}$

3. $x^{-\frac{4}{5}}$

4. $(2x+3)^2$

5. $(ax+b)^2$

6. $(x+1)^3$

7. $\dfrac{x^2 - 5x + 1}{\sqrt{x}}$

8. e^{3x}

9. $\log_e(2x+3)$

10. $\log_{10} x$

11. xe^x

12. $x^2 \log x$

13. $\dfrac{x+1}{x-1}$

14. $\dfrac{2x+3}{x-7}$

15. 3^x

16. $3x^2 + e^x - \log x$

17. $2x^e + e^x + e^e$

18. $(1+x^2)e^x$

19. $(x^2 + x - 1)\log x$

20. $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

21. $\dfrac{e^x + 1}{e^x - 1}$

22. $\dfrac{e^x}{1 - x^2}$

23. $a^{\frac{1}{5}}$

24. $\dfrac{1}{x^3}$

25. e^{2x-5}

26. $\log(4x+5)$

27. $(x^2 + 1)(x^2 - 1)$

28. $(x+1)(x+2)(x+3)$

29. $e^x(x-5)\log x$

30. $\dfrac{e^x \log x}{(1+x^2)}$

Solution:

1. $\dfrac{d}{dx}(x^{100}) = 100x^{100-1} = 100x^{99}$

2. $\dfrac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{\frac{2-3}{3}} = \frac{2}{3}x^{-\frac{1}{3}}$

3. $\dfrac{d}{dx}\left(x^{-\frac{4}{5}}\right) = -\frac{4}{5}x^{-\frac{4}{5}-1} = -\frac{4}{5}x^{\frac{-4-5}{5}} = -\frac{4}{5}x^{\frac{-9}{5}}$

4. $\dfrac{d}{dx}(2x+3)^2 = \dfrac{d}{dx}(4x^2 + 12x + 9) = 4\dfrac{d}{dx}(x^2) + 12\dfrac{d}{dx}(x) + \dfrac{d}{dx}(9)$

$$= 4(2x) + 12(1) + 0 = 8x + 12$$

5. $\dfrac{d}{dx}(ax+b)^2 = \dfrac{d}{dx}(a^2x^2 + 2abx + b^2)$

$\qquad = a^2\dfrac{d}{dx}(x^2) + 2ab\dfrac{d}{dx}x + \dfrac{d}{dx}(b^2)$

$\qquad = a^2(2x) + 2ab(1) + 0$

$\qquad = 2a^2x + 2ab$

6. $\dfrac{d}{dx}(x+1)^3 = \dfrac{d}{dx}(x^3 + 3x^2 + 3x + 1)$

$\qquad = \dfrac{d}{dx}(x^3) + 3\dfrac{d}{dx}(x^2) + 3\dfrac{d}{dx}(x) + \dfrac{d}{dx}(1)$

$\qquad = 3x^{3-1} + 3(2x) + 3(1) + 0$

$\qquad = 3x^2 + 6x + 3$

7. $\dfrac{d}{dx}\left(\dfrac{x^2 - 5x + 1}{\sqrt{x}}\right) = \dfrac{d}{dx}\left(\dfrac{x^2}{\sqrt{x}} - \dfrac{5x}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right)$

$\qquad = \dfrac{d}{dx}\left(x^{2-\frac{1}{2}} - 5x^{1-\frac{1}{2}} + x^{-\frac{1}{2}}\right)$

$\qquad = \dfrac{d}{dx}\left(x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)$

$\qquad = \dfrac{d}{dx}(x^{\frac{3}{2}}) - 5\dfrac{d}{dx}(x^{\frac{1}{2}}) + \dfrac{d}{dx}(x^{-\frac{1}{2}})$

$\qquad = \dfrac{3}{2}x^{\frac{3}{2}-1} - 5\cdot\dfrac{1}{2}x^{\frac{1}{2}-1} + \left(-\dfrac{1}{2}\right)x^{-\frac{1}{2}-1}$

$\qquad = \dfrac{3}{2}x^{\frac{1}{2}} - \dfrac{5}{2}x^{-\frac{1}{2}} - \dfrac{1}{2}x^{-\frac{3}{2}}$

8. $\dfrac{d}{dx}(e^{3x}) = 3e^{3x}$

9. $\dfrac{d}{dx}\log_e(2x+3) = 2\cdot\dfrac{1}{2x+3} = \dfrac{2}{2x+3}$

10. $\dfrac{d}{dx}(\log_{10}x) = \dfrac{d}{dx}\left(\dfrac{\log_e x}{\log_e 10}\right)$ (Using change of base in logarithms)

$\qquad = \dfrac{1}{\log_e 10}\cdot\dfrac{d}{dx}(\log_e x)$

$\qquad = \dfrac{1}{\log_e 10}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\log_e 10}$

11. $\dfrac{d}{dx}(xe^x) = x\dfrac{d}{dx}e^x + e^x\dfrac{d}{dx}(x)$ (Product rule)

$\qquad = xe^x + e^x(1)$

$\qquad = (x+1)e^x$

12. $\dfrac{d}{dx}\left(x^2 \log x\right) = x^2 \dfrac{d}{dx}\left(\log x\right) + \left(\log x\right)\dfrac{d}{dx}\left(x^2\right)$ (Product rule)

$$= x^2 \cdot \dfrac{1}{x} + \log x \cdot 2x$$

$$= x + 2x \log x$$

13. $\dfrac{d}{dx}\left(\dfrac{x+1}{x-1}\right) = \dfrac{(x-1)\dfrac{d}{dx}(x+1) - (x+1)\dfrac{d}{dx}(x-1)}{(x-1)^2}$ (Quotient rule)

$$= \dfrac{(x-1)(1+0) - (x+1)(1-0)}{(x-1)^2}$$

$$= \dfrac{(x-1) - x - 1}{(x-1)^2}$$

$$= \dfrac{-2}{(x-1)^2}$$

14. $\dfrac{d}{dx}\left(\dfrac{2x+3}{x-7}\right) = \dfrac{(x-7)\dfrac{d}{dx}(2x+3) - (2x+3)\dfrac{d}{dx}(x-7)}{(x-7)^2}$ (Quotient rule)

$$= \dfrac{(x-7)(2+0) - (2x+3)(1-0)}{(x-7)^2}$$

$$= \dfrac{2x - 14 - 2x - 3}{(x-7)^2}$$

$$= -\dfrac{17}{(x-7)^2}$$

15. $\dfrac{d}{dx}\left(3^x\right) = 3^x \log_e 3$

16. $\dfrac{d}{dx}\left(3x^2 + e^x - \log x\right) = 3\dfrac{d}{dx}\left(x^2\right) + \dfrac{d}{dx}\left(e^x\right) - \dfrac{d}{dx}\left(\log x\right)$

$$= 3(2x) + e^x - \dfrac{1}{x}$$

$$= 6x + e^x - \dfrac{1}{x}$$

17. $\dfrac{d}{dx}\left(2x^e + e^x + e^e\right) = 2\dfrac{d}{dx}\left(x^e\right) + \dfrac{d}{dx}e^x + \dfrac{d}{dx}\left(e^e\right)$

$$= 2 \cdot ex^{e-1} + e^x + 0 = 2ex^{e-1} + e^x$$

18. $\dfrac{d}{dx}\left[\left(1+x^2\right)e^x\right] = \left(1+x^2\right)\dfrac{d}{dx}e^x + e^x \dfrac{d}{dx}\left(1+x^2\right)$

$$= \left(1+x^2\right)e^x + e^x (2x)$$

$$= e^x \left(1 + 2x + x^2\right)$$

19. $\dfrac{d}{dx}\left[\left(x^2+x-1\right)\log x\right]$

$$=\left(x^2+x-1\right)\dfrac{d}{dx}\log x+\log x\cdot\dfrac{d}{dx}\left(x^2+x-1\right)$$

$$=\left(x^2+x-1\right)\dfrac{1}{x}+\left(\log x\right)\left(2x+1\right)$$

$$=x+1-\dfrac{1}{x}+\left(2x+1\right)\log x$$

20. $\dfrac{d}{dx}\left[\dfrac{x^2+x+1}{x^2-x+1}\right]$

$$=\dfrac{\left(x^2-x+1\right)\dfrac{d}{dx}\left(x^2+x+1\right)-\left(x^2+x+1\right)\dfrac{d}{dx}\left(x^2-x+1\right)}{\left(x^2-x+1\right)^2}$$

$$=\dfrac{\left(x^2-x+1\right)\left(2x+1\right)-\left(x^2+x+1\right)\left(2x-1\right)}{\left(x^2-x+1\right)^2}$$

$$=\dfrac{2x^3-2x^2+2x+x^2-x+1-2x^3-2x^2-2x+x^2+x+1}{\left(x^2-x+1\right)^2}$$

$$=\dfrac{-2x^2+2}{\left(x^2-x+1\right)^2}=\dfrac{2\left(-x^2+1\right)}{\left(x^2-x+1\right)^2}$$

21. $\dfrac{d}{dx}\left(\dfrac{e^x+1}{e^x-1}\right)=\dfrac{\left(e^x-1\right)\dfrac{d}{dx}\left(e^x+1\right)-\left(e^x+1\right)\dfrac{d}{dx}\left(e^x-1\right)}{\left(e^x-1\right)^2}$

$$=\dfrac{\left(e^x-1\right)\left(e^x\right)-\left(e^x+1\right)\left(e^x\right)}{\left(e^x-1\right)^2}$$

$$=\dfrac{e^{2x}-e^x-e^{2x}-e^x}{\left(e^x-1\right)^2}$$

$$=\dfrac{-2e^x}{\left(e^x-1\right)^2}$$

22. $\dfrac{d}{dx}\left(\dfrac{e^x}{1-x^2}\right)=\dfrac{\left(1-x^2\right)\dfrac{d}{dx}e^x-e^x\dfrac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}$

$$=\dfrac{\left(1-x^2\right)e^x-e^x\left(-2x\right)}{\left(1-x^2\right)^2}$$

$$=\dfrac{e^x\left(1-x^2+2x\right)}{\left(1-x^2\right)^2}$$

23. $\dfrac{d}{dx}\left(a^{\frac{1}{5}}\right) = 0 \quad \because a^{\frac{1}{5}}$

$\qquad \because \qquad a^{\frac{1}{5}}$ is a constant.

24. $\dfrac{d}{dx}\left(\dfrac{1}{x^3}\right) = \dfrac{d}{dx}\left(x^{-3}\right)$

$\qquad\qquad = (-3)x^{-3-1} = -3x^{-4}$

$\qquad\qquad = (-3)\cdot\dfrac{1}{x^4}$

$\qquad\qquad = \dfrac{-3}{x^4}$

25. $\dfrac{d}{dx}\left(e^{2x-5}\right) = 2e^{2x-5}$

26. $\dfrac{d}{dx}\log(4x+5) = \dfrac{4}{4x+5}$

27. $\dfrac{d}{dx}\left[\left(x^2+1\right)\left(x^2-1\right)\right] = \dfrac{d}{dx}\left(x^4-1\right)$ $\qquad$ (On multiplication)

$\qquad\qquad = \dfrac{d}{dx}\left(x^4\right) - \dfrac{d}{dx}(1)$

$\qquad\qquad = 4x^3 - 0 = 4x^3$

Aliter:

$\dfrac{d}{dx}\left[\left(x^2+1\right)\left(x^2-1\right)\right]$

$\qquad\qquad = \left(x^2+1\right)\dfrac{d}{dx}\left(x^2-1\right) + \left(x^2-1\right)\dfrac{d}{dx}\left(x^2+1\right)$

$\qquad\qquad = \left(x^2+1\right)(2x) + \left(x^2-1\right)(2x)$

$\qquad\qquad = 2x^3 + 2x + 2x^3 - 2x = 4x^3$

28. $\dfrac{d}{dx}\left[(x+1)(x+2)(x+3)\right]$

$\qquad\qquad = \dfrac{d}{dx}\left[x^3 + 6x^2 + 11x + 6\right]$

$\qquad\qquad = 3x^2 + 6(2x) + 11(1) + 0$

$\qquad\qquad = 3x^2 + 12x + 11$

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

$$\therefore \quad \frac{d}{dx}\left[(x+1)(x+2)(x+3)\right]$$

$$= (x+1)(x+2)\frac{d(x+3)}{dx} + (x+1)(x+3)\frac{d(x+2)}{dx} + (x+2)(x+3)\frac{d(x+1)}{dx}$$

$$= (x+1)(x+2)(1) + (x+1)(x+3)(1) + (x+2)(x+3)(1)$$

$$= x^2 + 3x + 2 + x^2 + 4x + 3 + x^2 + 5x + 6$$

$$= 3x^2 + 12x + 11$$

29. $\dfrac{d}{dx}\left[e^x(x-5)\log x\right]$

$$= e^x(x-5)\frac{d}{dx}(\log x) + e^x\log x\frac{d}{dx}(x-5) + (x-5)\log x\frac{d}{dx}(e^x)$$

$$= e^x(x-5)\frac{1}{x} + e^x\log x(1) + (x-5)\log x(e^x)$$

$$= e^x\left[1 - \frac{5}{x} + \log x + x\log x - 5\log x\right]$$

$$= e^x\left[1 - \frac{5}{x} + \log x + x\log x - 5\log x\right]$$

$$= e^x\left[1 - \frac{5}{x} - 4\log x + x\log x\right]$$

30. $\dfrac{d}{dx}\left[\dfrac{e^x\log x}{1+x^2}\right]$

$$= \frac{(1+x^2)\dfrac{d}{dx}\left[e^x\log x\right] - \left(e^x\log x\right)\dfrac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\left[e^x\dfrac{d}{dx}(\log x) + \left(\log x\dfrac{d}{dx}(e^x)\right)\right] - \left(e^x\log x\right)(0+2x)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\left(e^x\cdot\dfrac{1}{x} + \log x\cdot e^x\right) - e^x(\log x)(2x)}{(1+x^2)^2}$$

$$= \frac{e^x\left[\dfrac{1}{x} + x + (1+x^2)\log x - 2x\log x\right]}{(1+x^2)^2}$$

Example: Find $\dfrac{dy}{dx}$ if

1. $y = \dfrac{2x+1}{1-x^2}$

2. $y = \dfrac{x^3 - 2x}{x+2}$

3. $y = \dfrac{(x+1)(x+2)}{(x+3)}$

4. $y = \dfrac{x-1}{x+1}$

5. $y = (x^2 + 1)(e^x)$

6. $y = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

7. $y = (x^2 - 1)e^x \log x$

8. $y = \dfrac{e^x - x^2}{1 - \log x}$

9. $y = \dfrac{x^3 - 4x^2 + 1}{x^3 - 1}$

10. $y = \dfrac{(x^2 - 1)\log x}{x^2 e^x}$

Solution:

1. $y = \dfrac{2x+1}{1-x^2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(1-x^2)\dfrac{d}{dx}(2x+1) - (2x+1)\dfrac{d}{dx}(1-x^2)}{(1-x^2)^2}$

$= \dfrac{(1-x^2)(2) - (2x+1)(-2x)}{(1-x^2)^2}$

$= \dfrac{2 - 2x^2 + 4x^2 + 2x}{(1-x^2)^2}$

$= \dfrac{2 + 2x + 2x^2}{(1-x^2)^2}$

2. $y = \dfrac{x^3 - 2x}{x+2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(x+2)\dfrac{d}{dx}(x^3 - 2x) - (x^3 - 2x)\dfrac{d}{dx}(x+2)}{(x+2)^2}$

$= \dfrac{(x+2)(3x^2 - 2) - (x^3 - 2x)(1+0)}{(x+2)^2}$

$= \dfrac{3x^3 + 6x^2 - 2x - 4 - x^3 + 2x}{(x+2)^2}$

$= \dfrac{2x^3 + 6x^2 - 4}{(x+2)^2}$

3. $y = \dfrac{(x+1)(x+2)}{(x+3)}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(x+3)\dfrac{d}{dx}\left[(x+1)(x+2)\right] - (x+1)(x+2)\dfrac{d}{dx}(x+3)}{(x+3)^2}$

$= \dfrac{(x+3)\left[(x+1)\dfrac{d}{dx}(x+2) + (x+2)\dfrac{d}{dx}(x+1)\right] - (x+1)(x+2)(1)}{(x+3)^2}$

$= \dfrac{(x+3)\left[(x+1)(1) + (x+2)(1)\right] - (x+1)(x+2)}{(x+3)^2}$

$= \dfrac{(x+3)\left[(2x+3)\right] - (x+1)(x+2)}{(x+3)^2}$

$= \dfrac{2x^2 + 9x + 9 - x^2 - 3x - 2}{(x+3)^2}$

$= \dfrac{x^2 + 6x + 7}{(x+3)^2}$

4. $y = \dfrac{x-1}{x+1}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(x+1)\dfrac{d}{dx}(x-1) - (x-1)\dfrac{d}{dx}(x+1)}{(x+1)^2}$

$= \dfrac{(x+1)(1-0) - (x-1)(1+0)}{(x+1)^2}$

$= \dfrac{x+1-x+1}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2}$

5. $y = (x^2 + 1)(e^x)$

$\therefore \quad \dfrac{dy}{dx} = (x^2 + 1)\dfrac{d}{dx}(e^x) + e^x \dfrac{d}{dx}(x^2 + 1)$

$= (x^2 + 1)e^x + e^x(2x)$

$= e^x(x^2 + 1 + 2x)$

$= e^x(x+1)^2$

6. $y = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

$\quad = \dfrac{x^2 - 3x + 2}{x^2 - 7x + 12}$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(x^2 - 7x + 12\right)\dfrac{d}{dx}\left(x^2 - 3x + 2\right) - \left(x^2 - 3x + 2\right)\dfrac{d}{dx}\left(x^2 - 7x + 12\right)}{\left(x^2 - 7x + 12\right)^2}$

$\quad = \dfrac{\left(x^2 - 7x + 12\right)(2x - 3) - \left(x^2 - 3x + 2\right)(2x - 7)}{\left(x^2 - 7x + 12\right)^2}$

$\quad = \dfrac{2x^3 - 14x^2 + 24x - 3x^2 + 21x - 36 - 2x^3 + 6x^2 - 4x + 7x^2 - 21x + 14}{\left(x^2 - 7x + 12\right)^2}$

$\quad = \dfrac{-2x^3 + 6x^2 - 4x + 7x^2 - 21x + 14}{\left(x^2 - 7x + 12\right)^2}$

$\quad = \dfrac{-4x^2 + 20x - 22}{\left(x^2 - 7x + 12\right)^2}$

7. $y = \left(x^2 - 1\right)e^x \log x$

$\therefore \dfrac{dy}{dx} = \left(x^2 - 1\right)e^x \dfrac{d}{dx}(\log x) + \left(x^2 - 1\right)\log x \dfrac{d}{dx}e^x + e^x \log x \dfrac{d}{dx}\left(x^2 - 1\right)$

$\quad = \left(x^2 - 1\right)e^x \cdot \dfrac{1}{x} + \left(x^2 - 1\right)\log x \cdot e^x + e^x \log x \cdot 2x$

$\quad = e^x\left[x - \dfrac{1}{x} + \left(x^2 - 1\right)\log x + 2x \cdot \log x\right]$

8. $y = \dfrac{e^x - x^2}{1 - \log x}$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(1 - \log x\right)\dfrac{d}{dx}\left(e^x - x^2\right) - \left(e^x - x^2\right)\dfrac{d}{dx}\left(1 - \log x\right)}{\left(1 - \log x\right)^2}$

$\quad = \dfrac{(1 - \log x)(e^x - 2x) - (e^x - x^2)\left(0 - \dfrac{1}{x}\right)}{(1 - \log x)^2}$

$\quad = \dfrac{e^x(1 - \log x) + 2x(1 - \log x) + \dfrac{e^x}{x} - x}{(1 - \log x)^2}$

$\quad = \dfrac{e^x\left(1 - \log x + \dfrac{1}{x}\right) + x - 2x\log x}{(1 - \log x)^2}$

9. $y = \dfrac{x^5 - 4x^2 + 1}{x^3 - 1}$

$\therefore \dfrac{dy}{dx} = \dfrac{(x^3 - 1)\dfrac{d}{dx}(x^5 - 4x^2 + 1) - (x^5 - 4x^2 + 1)\dfrac{d}{dx}(x^3 - 1)}{(x^3 - 1)^2}$

$= \dfrac{(x^3 - 1)(5x^4 - 8x) - (x^5 - 4x^2 + 1)(3x^2)}{(x^3 - 1)^2}$

$= \dfrac{5x^7 - 5x^4 - 8x^4 + 8x - 3x^7 + 12x^4 - 3x^2}{(x^3 - 1)^2}$

$= \dfrac{2x^7 - x^4 - 3x^2 + 8x}{(x^3 - 1)^2}$

10. $y = \dfrac{(x^2 - 1)\log x}{x^2 e^x}$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(x^2 e^x\right)\dfrac{d}{dx}\left[(x^2 - 1)\log x\right] - (x^2 - 1)\log x \dfrac{d}{dx}\left(x^2 e^x\right)}{\left(x^2 e^x\right)^2}$

$= \dfrac{\left(x^2 e^x\right)\left[(x^2 - 1)\dfrac{1}{x} + \log x(2x)\right] - (x^2 - 1)\log x\left(x^2 e^x + e^x 2x\right)}{\left(x^2 e^x\right)^2}$

$= \dfrac{x^2 e^x\left(x - \dfrac{1}{x}\right) + 2x^3 e^x \log x - x^2(x^2 - 1)e^x \log x - e^x\left(2x^3 - 2x\right)}{x^4 e^{2x}}$

$= \dfrac{-x^2 e^x + e^x + 2x^2 e^x \log x - x(x^2 - 1)e^x \log x}{x^3 e^{2x}}.$

Check Your Progress

Fill in the blanks:

1. To differentiate a function of the form $\left[f(x)\right]^{g(x)}$ or $a^{f(x)}$, we use a method called _____ .

2. If a function is in the form $y = f(x)$, then the function is said to be in the _____ form.

3. If the variables x and y are related by means of an equation, then the function is said to be in the _____ .

8.8 LET US SUM UP

- Differentiation is all about finding rates of change of one quantity compared to another. We need differentiation when the rate of change is not constant.

- If $y = e^x$, then $\dfrac{dy}{dx} = e^x$

- If a function is made up of more than one function then it is called a composite function.

- If a function is in the form $y = f(x)$, then the function is said to be in the explicit form.

- If the variables x and y are related by means of an equation, then the function is said to be in the implicit form.

- To differentiate a function of the form $a^{f(x)}$ we use a method called logarithmic differentiation.

- If $y = a^x$, then $\dfrac{dy}{dx} = a^x \log_e a$

- Partial derivative is a derivative of a function of two or more variables with respect to one variable, the other(s) being treated as constant.

8.9 LESSON END ACTIVITY

Differentiate: $(x-1)(x-2)(x-3)e^x \log(1+e^x)^2$.

8.10 KEYWORDS

Composite Functions: If a function is made up of more than one function then it is called a composite function.

Explicit Functions: If a function is in the form $y = f(x)$, then the function is said to be in the explicit form.

Implicit Functions: If the variables x and y are related by means of an equation, then the function is said to be in the implicit form.

Logarithmic Differentiation: To differentiate a function of the form $a^{f(x)}$ we use a method called logarithmic differentiation.

8.11 QUESTIONS FOR DISCUSSION

1. Find derivative of the following functions:

   (i) $y = x^2 - 5x + 10$

   (ii) $y = (x+1).e^x$

   (iii) $y = 10\ (x+1)(4-x)$

   (iv) $y = \log x^3$

2. Differentiate the following w.r.t. x:

   (i) $(ax+b)$ (ii) $(ax+b)^3$

   (iii) $ax^2 + bx + c$ (iv) $(x-1)^2$

   (v) $(x+1)^2$ (vi) $(x-2)^3$

   (vii) $(x-1)^3$ (viii) $\dfrac{x^4+1}{x}$

(ix) $\dfrac{3x^2 - 5x + 1}{\sqrt{x}}$ (x) $x^4 \log x$

(xi) $x^{-3} e^x$ (xii) $x e^{4x}$

(xiii) $x^{-5} \log x$ (xiv) $x^{-6} e^{-4x}$

(xv) $(x^2 + 1)(x^3 + 1)$ (xvi) $(x + 1)(x + 2)(x + 3)(x + 4)$

(xvii) $(x - 1)(x - 4)(x + 7)$ (xviii) $\dfrac{x^2 + 1}{x^2 - 1}$

(xix) $\dfrac{2x - 1}{2x + 1}$ (xx) $\dfrac{(x + 1)(x^2 - 5)}{(x - 1)(x^2 + 4)}$

(xxi) $\dfrac{3e^x \log x}{1 + x^2}$ (xxii) $\dfrac{e^{-2x}(1 - x^2)}{1 + x + x^2}$

(xxiii) $\dfrac{x^3 + 1}{x + 1}$ (xxiv) $\dfrac{x^3 + 4x}{x - 1}$

(xxv) $\dfrac{x^2 - 5x + 1}{x^2 + 6x - 7}$ (xxvi) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

(xxvii) $\dfrac{e^x - 1}{e^x + 1}$ (xxviii) $e^{5x - 7}$

(xxix) $\log(3x - 1)$ (xxx) $\dfrac{5^x(x + 2)}{(x - 1)(x - 3)}$

(xxxi) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$ (xxxii) $\dfrac{x^2 + x + 1}{\sqrt{x}}$

3. Find $\dfrac{dy}{dx}$ if $y =$

(i) $(3x - 1)^2$ (ii) $e^{3x}(x - 2)^3$

(iii) $\dfrac{9e^{4x}}{\log(7x - 1)}$ (iv) $4^x \log_{10} x$

(v) $\dfrac{(x + 2)^3}{(2x + 1)^2}$ (vi) $\dfrac{x^3 + 1}{x^3 - 1}$

(vii) $\dfrac{ax^2 + bx + c}{dx^2 + ex + f}$ (viii) $\dfrac{3x^2 - 5x}{7x + 1}$

(ix) $\dfrac{5x - 3}{2x^2 - x + 1}$ (x) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(xi) $\dfrac{(x - 1)(x + 1)}{x - 2)\,(x + 2)}$ (xii) $\dfrac{x^2 - x + 1}{x^2 + x + 1}$

4. Find $\dfrac{dy}{dx}$ when $y = u^2 + 5$, $u = v^2 + 2$ and $v = x^2 - x$.

5. Find the elasticity of demand of the following demand function

$$x = 200 - 4P \text{ at } x = 40$$

6. Differentiate the following functions w.r.t. x:

(i) $\left(x^3 + 1\right)^{-2}$

(ii) $\sqrt{1 - x^2}$

(iii) $\dfrac{1}{\left(x^2 - x + 1\right)^4}$

(iv) $\left(\dfrac{x^2 + x + 1}{x^2 - x + 1}\right)^2$

(v) $\log\left(xe^x\right)$

(vi) e^{e^x}

(vii) $e^{\sqrt{1 - x^2}}$

(viii) $\log\left(x + \sqrt{1 - x^2}\right)$

(ix) $\log\left(\log\left(\log x\right)\right)$

(x) $\sqrt{\dfrac{1 + x}{1 - x}}$

(xi) $\left(x - 1\right)^2 e^{3x + 7}$

(xii) 2^{3x-1}

(xiii) $\left(\dfrac{1 - x^2}{1 + x^2}\right)^{-1}$

(xiv) $\sqrt{\dfrac{2x}{1 + x^2}}$

(xv) $\dfrac{e^{\log x}}{\log x}$

(xvi) $\dfrac{1 - e^{2x}}{\left(x^2 + 1\right)^2}$

(xvii) $\dfrac{\sqrt{1 + x^2}}{x}$

(xviii) $e^{\sqrt{x}} \cdot \log\left(4x - 5\right)$

(xix) $\sqrt{1 - \sqrt{x}}$

(xx) $\dfrac{1}{x^2 e^x}$

(xxi) $x \log\left(\log x\right)$

(xxii) $\dfrac{1 - e^{2x}}{1 + e^{2x}}$

(xxiii) $\left(2 + 3x\right)^2 \cdot e^{3x}$

(xxiv) $\log\left(\sqrt{\dfrac{1 + x^2}{1 - x^2}}\right)$

(xxv) $\dfrac{1}{x\sqrt{1 + x^2}}$

7. Find $\dfrac{dy}{dx}$, if

(i) $xy = c^2$

(ii) $x^2 + 2xy + 3y^2 = 0$

(iii) $x^2 + y^2 = 9$

(iv) $\left(x + y\right)^2 = 4y$

(v) $ax^2 + 2hxy + by^2 = 0$

(vi) $27ay^2 = 4\left(x - 2a\right)^3$

(vii) $\left(e^{y}+1\right)\log x=\left(e^{x}+1\right)\log y$ (viii) $x+y=1-xy$

(ix) $\left(x^{2}+y^{2}\right)^{2}=a^{2}\left(y^{2}-x^{2}\right)$ (x) $\dfrac{x^{2}}{16}-\dfrac{y^{2}}{9}=1\,\text{at}\,(0,4)$

(xi) $x^{2}+6xy+3y^{2}=8$ (xii) $e^{xy}-4xy=2$

(xiii) $\sqrt{x}+\sqrt{y}=1\,\text{at}\left(\dfrac{1}{4},\dfrac{1}{4}\right)$ (xiv) $e^{xy}+4xy=2x$

(xv) $x^{3}+y^{3}=3axy$

8. Find $\dfrac{dy}{dx}$, if

(i) If $y=\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots\ldots\ldots+\infty}}}}$, prove that $\dfrac{dy}{dx}=\dfrac{1}{2y-1}$

(ii) If $y=\sqrt{e^{x}\sqrt{e^{x}+\sqrt{e^{x}+\ldots\ldots+\infty}}}$, prove that $\dfrac{dy}{dx}=\dfrac{e^{x}}{2y-1}$

(iii) If $y=\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots\ldots to\,\infty}}}$, prove that $\dfrac{dy}{dx}=\dfrac{-1}{\left(x+y\right)^{2}+1}$

(iv) If $y=\sqrt{f\left(x\right)+\sqrt{f\left(x\right)+\sqrt{f\left(x\right)+\ldots\ldots+\infty}}}$, prove that $\dfrac{dy}{dx}=\dfrac{f'\left(x\right)}{2y-1}$.

(v) If $x^{2}-7xy+3y^{2}=5$, prove that $\dfrac{dy}{dx}=\dfrac{2x-7y}{7x-6y}$.

9. Differentiate the following w.r.t. x

(i) $\left(\log x\right)^{x}$ (ii) $\left(e^{x}\right)^{e^{x}}$

(iii) $\left(\log x\right)^{x^{2}}$ (iv) $\left(\sqrt{x}\right)^{\log x}$

(v) $x^{\log x}+\left(\log x\right)^{x}$ (vi) $x^{1+x+x^{2}}$

(vii) $\left(x^{2}+1\right)^{x}$ (viii) $\left(1-x+x^{2}\right)^{x}$

(ix) $\left(x^{2}+1\right)e^{\left(x^{2}-1\right)}$ (x) $e^{e^{x}}\cdot e^{x}$

10. Find $\dfrac{dy}{dx}$, if

(i) $x^{y}+y^{x}+2a=0$ (ii) $x^{y}=e^{x-y}$

(iii) $x^{y}-y^{x}+2ax=0$ (iv) $\left(e^{x}\right)^{x}-y=0$

11. Calculate $\partial z/\partial x$ and $\partial z/\partial y$ when $z=x^{2}+3xy+y-1$.

Check Your Progress: Model Answer

1. Logarithmic differentiation

2. Explicit

3. Implicit form

8.12 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory Manual - Mathematics, secondary stage - NCERT Publication.

Robert T. Smith, *Calculus: Single Variable*, Tata McGraw Hill.

UNIT V

LESSON

9

SIMPLE AND COMPOUND INTEREST

CONTENTS

9.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Explain the concept of simple interest
- Discuss compound interest
- Learn about compound interest including half yearly and quarterly calculations
- Elaborate annuities percentages

9.1 INTRODUCTION

In the previous lesson, we studied about the differentiation and partial derivatives. Interest is the charge against the use of money by the borrower. The same is profit earned by the lender of money. The amount which is invested in a bank in order to earn interest is called principal. The interest rate is normally expressed in percentage and represents the dollar interest earned per ₹100 of principal in a specific time, usually a year. Simple interest and compound interest are the two types of interest

based on the way they are calculated. Simple interest is charged only on the principal amount.

In this lesson, we will discuss simple interest, compound interest including half yearly and quarterly calculations, annuities percentages.

9.2 SIMPLE INTEREST

When we borrow some amount of money as loan from somebody for a fixed period of time, an extra amount charged is called the interest. This interest depends on three factors.

(a) Amount borrowed,

(b) Period of time, and

(c) Rate of interest.

The amount borrowed is called the *principal* and is denoted by P. The sum of the principal and the interest is called the *amount* and is denoted by A.

Thus, we have,

Principal : P

Amount : A

Period of time : T or N

Rate of Interest : R

(per annum)

A = Principal + Interest = $P + N$

The simple interest on the principal P for the number of years N at the rate of interest R is calculated by using the formula:

$$I = \frac{PNR}{100}$$

For solving problems on the simple interest, we have the following formulae:

1. To find I when P, N, R are given

$$I = \frac{PNR}{100}$$

2. To find P when I, N, R are given

$$P = \frac{100I}{NR}$$

3. To find N when I, P, R are given

$$N = \frac{100I}{PR}$$

4. To find R when I, P, N are given

$$R = \frac{100I}{PN}$$

5. To find A when P and I are given

$$A = P + I$$

6. To find P when A, R, N are given

$$P = \frac{100A}{100 + RN}$$

Keep in mind always when you make a payment on a simple interest loan, the payment first goes towards that month's interest, and the remainder goes towards the principal. Each month's interest is paid in full so it never accrues.

Example: Find the simple interest on ₹ 2,000 for 10 years at the rate of 10% per annum.

Solution:

Here $P = 2000$, $N = 10$, $R = 10$

$$\therefore \quad SI = \frac{PNR}{100}$$

$$= \frac{2000 \times 10 \times 10}{100}$$

$$= 2000$$

$\therefore$ Simple interest is ₹ 2000

Example: Calculate the simple interest on ₹ 4,000 for 5 years at the rate of 6% per annum.

Solution:

Here $P = 4000$, $N = 5$, $R = 6$

$$\therefore \quad SI = \frac{PNR}{100}$$

$$= \frac{4000 \times 5 \times 6}{100}$$

$$= 1200$$

$\therefore$ Simple interest is ₹ 1200

Example: Find the amount on ₹ 15000 at 5% for 10 years.

Solution:

Here $P = 15000$, $N = 10$, $R = 5$

$$SI = \frac{PNR}{1000}$$

$$= \frac{15000 \times 10 \times 5}{100} = ₹ \ 7500$$

Example: If a principal amount is charged the simple interest for ₹ 750 for 5 years at the rate of 6% per annum, what is the principal?

Solution:

$$SI = \frac{PNR}{100}$$

$$750 = \frac{P \times 5 \times 6}{100} \qquad \therefore P = \frac{750 \times 100}{5 \times 6}$$

i.e., $\qquad P = 25 \times 100$

$$= ₹ \ 2500$$

Example: If a person who takes a loan of ₹ 3600 is asked to repay ₹ 6000 at the end of 3 years, what is the rate of interest charged as per simple interest?

Solution:

Here $A = 6000, P = 3600, N = 3$

$$SI = A - P$$
$$= 6000 - 3600$$
$$= 2400$$

Now,

$$SI = \frac{PNR}{100}$$

$$2400 = \frac{3600 \times 3 \times R}{100}$$

$$R = \frac{2400 \times 100}{3600 \times 3}$$

$$= \frac{2400}{36 \times 3}$$

$$= 22\frac{2}{9}\%$$

Example: An amount of loan borrowed from a bank totals to ₹ 5000 at the end of 4 years as per the rate of $6\frac{1}{4}\%$ simple interest. What is the loan amount borrowed?

Solution:

Let the loan amount be ₹

$$P = \frac{100A}{100 + RN}$$

$$= \frac{(100 \times 5000)}{100 + (25/4) \times 4}$$

$$= \frac{500000}{125}$$

$$= 4000$$

$\therefore$ Loan amount = ₹ 4000

Example: A man deposits ₹ 100 on the first of every month in a savings bank account in which simple interest is 2% per annum on the minimum monthly balances, what is the amount accumulated at the end of $2\frac{1}{2}$ years?

Solution:

Total Interest = interest on ₹ 100 for 30 months

$\qquad\qquad$ + interest on ₹ 100 for 29 months

$\qquad\qquad$ + ...

$\qquad\qquad$ + interest on ₹ 100 for 1 month

$$= \left(100 \times \frac{2}{100} \times \frac{30}{12}\right) + \left(100 \times \frac{2}{100} \times \frac{29}{12}\right) + ... + \left(100 \times \frac{2}{100} \times \frac{1}{12}\right)$$

$$= \frac{1}{6}(30 + 29 + \ldots + 1)$$

$$= \frac{1}{6}\left(\frac{30}{2}(1 + 30)\right)$$ (Using the formula for the sum to n terms of an A.P. $S_n = \frac{n}{2}[a + l]$)

$$= 77.50$$

The amount deposited $= 100 \times 30 = 3000$

$\therefore$ The amount accumulated at the end of $2\frac{1}{2}$ years is ₹ 3000 + ₹ 77.50 = ₹ 3077.50.

Example: A money lender lent ₹ 250 and demanded ₹ 340 towards the amount due at the end of 4 years. Find the rate of interest (simple).

Solution:

$I = $ ₹ $340 - $ ₹ $250 = 110$

$$R = \frac{100\,I}{PN}$$

$$= \frac{100 \times 110}{250 \times 4}$$

$$= 11$$

$\therefore$ The rate of interest is 11%

9.3 COMPOUND INTEREST

Compound interest is interest calculated on the initial principal and also on the accumulated interest of previous periods of a deposit or loan. Compound interest can be thought of as "interest on interest," and will make a deposit or loan grow at a faster rate than simple interest, which is interest calculated only on the principal amount. The rate at which compound interest accrues depends on the frequency of compounding; the higher the number of compounding periods, the greater the compound interest. Thus, the amount of compound interest accrued on ₹100 compounded at 10% annually will be lower than that on ₹100 compounded at 5% semi-annually over the same time period. Compound interest is also known as compounding.

Compound interest adds some of the monthly interest back onto the loan; in each succeeding month, you pay new interest on old interest.

9.3.1 Elements of Compound Interest

The elements of compound interest are as follows:

Principal Sum

It is a sum of money which is borrowed or lended or invested. In compound interest calculation, the Principal amount changes after every compounding period such as the principal amount and interest earned after the first unit of time becomes the principal for the second unit, the principal amount and interest earned after the second unit of time becomes the principal for third time and so on. After that certain period, the difference between the amount and the money borrowed is the compound interest.

Total Time Period

It is a time period to which the principal amount is borrowed or lended or invested at a certain rate of interest. At the end of this period, the total amount comprises of principal and interest should be returned.

Compounding Period or Frequency

It is a certain period of time after which an interest earned on principal during that period added to the principal is called as compounding period.

Interest Rate

Interest rate in percentage is the rate at which the money is borrowed or saved or lended or invested.

Compound Interest = Total amount of Principal and Interest in future (or Future Value) less Principal amount at present (or Present Value)

9.3.2 Formula for Yearly Compounding

If r% per annum is the rate of simple interest for a sum of at the end of 1 year, the amount A_1 will be

$$A_1 = P\left(1 + \frac{r}{100}\right)$$

Now, if A_1 is taken as the principal amount and interest calculated on this amount at r% interest, at the end of 2 years, the amount will become:

$$A_2 = P\left(1 + \frac{r}{100}\right) + P\left(1 + \frac{r}{100}\right)\frac{r}{100}$$

$$= P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right) \quad \text{(by taking } P\left(1 + \frac{r}{100}\right) \text{ as common factor)}$$

$$= P\left(1 + \frac{r}{100}\right)^2$$

Similarly, the amount at the end of 3 years will be

$$A_3 = P\left(1 + \frac{r}{100}\right)^3 \text{ etc.,}$$

Continuing this way, we get the amount at the end of *n* years is

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

∴ Total interest gained on ₹ is

$$A_n - P = P\left(1 + \frac{r}{100}\right)^n - p$$

For solving problems concerning compound interest we use the formula:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

9.3.3 Formula for Half-yearly Compounding

This below half-yearly compounding formula is one of the few formulas used in this calculator to find the total compound interest payable based on the semi-annual compounded period,

$$CI_{half\text{-}yearly} = P\left(1 + \frac{R/2}{100}\right)^{2n}$$

where

$CI_{half\text{-}yearly}$ is the compound interest payable based on 6 months compounding frequency

P is the initial principal sum of money

R is the interest rate in percentage at which interest accrued over time

n is the time period in years

9.3.4 Formula for Quarterly Compounding

This below quarterly compounding formula is one of the few formulas used in this calculator to find the total compound interest payable based on 3 months compounded period,

$$CI_{quarterly} = P\left(1 + \frac{R/4}{100}\right)^{4n}$$

where

$CI_{quarterly}$ is the compound interest payable based on 3 months compounding frequency

P is the initial principal sum of money

R is the interest rate in percentage at which interest accrued over time

n is the time period in years

9.3.5 Formula for Monthly Compounding

This below monthly compounding formula is one of the few formulas used in this calculator to find the total compound interest payable based on the monthly compounded period,

$$CI_{monthly} = P\left(1 + \frac{R/12}{100}\right)^{12n}$$

where

$CI_{monthly}$ is the compound interest payable based on every months compounding frequency

P is the initial principal sum of money

R is the interest rate in percentage at which interest accrued over time

n is the time period in years

Example: Find the compound interest on ₹ 4000 at the end of 5 years at the rate of 8% per annum.

Solution:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 4000\left(1 + \frac{8}{100}\right)^5$$

$$= 4000 \times (1.08)^5$$

Taking log on both sides,

$$\log A = \log 4000 + 5\log 1.08$$

$$= 3.6021 + 5 \times 0.0334$$

$$= 3.6021 + 0.1670$$

$$= 3.7691$$

$\therefore \qquad A = \text{Antilog} 3.7691$

$$= 5876.00$$

$\therefore$ At the end of 5 years, the amount $A = ₹ 5876.00$

The principal $P = ₹ 4000.00$

Compound interest $= ₹ 5876 - ₹ 4000 = ₹ 1876.00$

Example: What sum amounts to ₹ 6000 after 4 years at 5% compound interest?

Solution:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$\therefore \quad 6000 = P\left(1 + \frac{5}{100}\right)^4$

$$6000 = P(1.05)^4$$

$\therefore \quad P = \dfrac{6000}{(1.05)^4}$

Taking logs on both sides

$$\log P = \log 6000 - 4\log 1.05$$

$$= 3.7782 - 4 \times 0.0212$$

$$= 3.7782 - 0.0348$$

$$= 3.6934$$

$\therefore \qquad P = \text{antilog} 3.6934$

$$= 4937$$

$\therefore$ Principal $= ₹ 4937$

Example: In how many years, ₹ 1000 will be doubled at 6% compound interest?

Solution:

Here $A = 2000, P = 1000, r = 6$

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$2000 = 1000\left(1 + \frac{6}{100}\right)^n$$

$$2 = (1.06)^n$$

Taking logs,

$$\log 2 = n \log 1.06$$

$$0.3010 = n(0.0212)$$

∴ $\quad n = 0.3010/0.0212$

i.e., $\quad n = 3010/212$

Taking logs on both sides,

$$\log n = \log 3010 - \log 212$$

$$= 3.4786 - 2.3263$$

$$= 1.1523$$

∴ $\quad$ antilog $1.1523 = 14.20$

Hence in 14.2 years, ₹ 1000 will be doubled.

Example: A man borrowed ₹ 6250 from a bank and after 2 years paid ₹ 6760 in full settlement of his debt. Find the rate of compound interest charged by the bank.

Solution:

$A = 6760, P = 6250, n = 2$

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$6760 = 6250\left(1 + \frac{r}{100}\right)^2$$

∴ $\quad \left(1 + \frac{r}{100}\right)^2 = \frac{6760}{6250}$

∴ $\quad 1 + \frac{r}{100} = \sqrt{\frac{676}{625}}$

Taking log on both sides,

$$\log\left(1 + \frac{r}{100}\right) = \frac{1}{2}[\log 676 - \log 625]$$

$$= \frac{1}{2}[2.8299 - 2.7959]$$

$$= \frac{1}{2}[0.0340]$$

$$= 0.0170$$

$$\therefore \qquad 1 + \frac{r}{100} = \text{antilog}\, 0.0170$$

$$= 1.040$$

$$\therefore \qquad \frac{r}{100} = 1.040 - 1$$

$$= 0.040$$

$$\therefore \qquad r = 0.040 \times 100$$

$$= 4$$

$\therefore$ Rate of compound interest = 4% p.a.

Example: Find the difference between compound interest and simple interest on ₹ 6000 for 5 years at 10%.

Solution:

$P = 6000, n = 5, R = 10$

$$I = \frac{PNR}{100}$$

$$= \frac{6000 \times 5 \times 10}{100}$$

$$= 3000$$

$\therefore \qquad SI = ₹\, 3000$ ...(1)

Here $P = 6000, n = 5, R = 10$

$$A = 6000\left(1 + \frac{10}{100}\right)^5$$

i.e., $\qquad A = 6000(1.01)^5$

Taking log on both sides,

$$\log A = \log 6000 + 5\log 1.1$$

$$= 3.7782 + 5(0.0414)$$

$$= 3.7782 + 0.2070$$

$$= 3.9852$$

$\therefore \qquad A = \text{Antilog}\, 3.9852$

$$= 9665.00$$

i. e., $\qquad CI = ₹\, 9665 - ₹\, 6000$

$\therefore \qquad CI = ₹\, 3665$

The difference between *CI* and *SI* is

$$₹\, 3665 - ₹\, 3000 = ₹\, 665$$

Example: What principal will amount to ₹ 1352 in two years at 4% compound interest?

Solution:

Let *P* be the principal amount

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$1352 = P\left(1 + \frac{4}{100}\right)^2$$

$$1352 = P(1.04)^2$$

$$\therefore \qquad P = 1352/(1.04)^2$$

Taking logs on both sides,

$$\log P = \log 1352 - 2\log(1.04)$$

$$= 3.1310 - 2 \times 0.0170$$

$$= 3.1310 - 0.0340$$

$$= 3.0970$$

$$\therefore \qquad P = \text{Antilog}\, 3.0970$$

$$= 1250.00$$

$\therefore$ Principal $= ₹\, 1250.00$

Example: Find the compound interest on ₹ 2400 for $2\frac{1}{2}$ years at 5% per annum interest being compounded annual.

Solution:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 2400\left(1 + \frac{5}{100}\right)^{5/2}$$

$$= 2400(1.05)^{5/2}$$

$$\therefore \qquad \log A = \log 2400 + \frac{5}{2}\log 1.05$$

$$= 3.3802 + \frac{5}{2} \times 0.0212$$

$$= 3.3802 + 5 \times 9.0106$$

$$= 3.3802 + 0.0530$$

$$= 3.4332$$

$$\therefore \qquad A = \text{Antilog}\, 3.4332$$

$$= 2711.00$$

$$\therefore \qquad CI = ₹\, 2711 - ₹\, 2400$$

$$= ₹\, 311$$

9.4 ANNUITIES

"An annuity is a fixed sum paid at a regular interval under certain stated conditions", the period may be one year or half year or one month.

Example: Interest, house rent, salaries, pension, etc.

9.4.1 Types of Annuity

1. *Annuity Certain:* An annuity payable for a fixed number of years is called annuity certain.

2. *Annuity Inmidiate:* An annuity payments are made at the end of each period. Then the annuity is called annuity inmidiate.

3. *Annuity Contigent:* An annuity payment or duration of annuity payments or happening of any event is known as annuity contigent.

4. *Perpetuity:* Annuity which is for ever (i.e., for an infinite period) is called perpetuity.

5. *Deferred Annuity:* When the annuity payment starts after lapse of a certain specific period, the annuity is called deferred annuity.

6. *Annuity Due:* If the annuity payments are made in advance at the beginning of each stipulated time then it is called annuity due.

9.4.2 Formulae for Annuity

1. Future Value of Annuity $F = \dfrac{A\left[(1+r)^n - 1\right]}{r}$

   A = Annuity, r = Interest rate, n = number of years

   F = Future value of annuity.

2. Present Value of Annuity (PVA) $= \dfrac{A\left[(1+r)^n - 1\right]}{r(1+r)^n}$

3. Future Value of Annuity due $F = \dfrac{A\left[(1+r)^n - 1\right]}{r}(1+r)$

4. Present Value of Annuity due $P = \dfrac{A\left[(1+r)^n - 1\right]}{r(1+r)^n}(1+r)$

5. Perpetuity $P_\infty = \dfrac{A}{r}$.

Example: A father wants to send his child for higher studies after 15 years. He expects the cost of higher studies then to be ₹ 1,00,000. How much should he save annually to have ₹ 1,00,000 after 15 years if the interest rate is 12% p.a.?

Solution:

$F = 1,00,000, r = 12\%$ p.a. $= 0.12, n = 15$ years, $A = ?$

$$A = \frac{Fr}{\left[(1+r)^n - 1\right]}$$

$$= \frac{1,00,000 \times 0.12}{\left[(1+0.12)^{15} - 1\right]}$$

$$= \frac{1,00,000 \times 0.12}{(1.12)^{15} - 1}$$

Let $x = (1.12)^{15}$

$$\log x = 15 \log 1.12$$

$$= 15 \times 0.0492$$

$$= 0.7380$$

$$x = \text{Antilog } 0.7380$$

$$= 5.470$$

$\therefore$
$$A = \frac{1,00,000 \times 0.12}{5.470 - 1}$$

$$= \frac{1,00,000 \times 0.12}{4.470}$$

$$= \frac{12,000}{4.470}$$

$\therefore$
$$A = ₹ \, 2,684.5/-$$

Example: Find the future value of ₹ 1,000 deposited annually for 12 years gathering compound interest 16% p.a.

Solution:

$A = 1,000, n = 12$ years, $r = 16\% = 0.16, F = ?$

$$F = \frac{A\left[(1+r)^n - 1\right]}{r}$$

$$= \frac{1,000\left[(1+0.16)^{12} - 1\right]}{0.16}$$

Let $x = (1.16)^{12}$

$$\log x = \log (1.16)^{12}$$

$$= 12 \log 1.16$$

$$= 12 \times 0.0645$$

$$= 0.7740$$

$$x = \text{Antilog } (0.7740)$$

$$= 5.9343$$

$\therefore$
$$F = \frac{1,000(5.943 - 1)}{0.16}$$

$$F = \frac{1,000[4.943]}{0.16}$$

$$= \frac{4943}{0.16} = 30,893.75$$

$\therefore$
$$F = 30,893.75$$

Example: You will deposit ₹ 1,000 into your recurring deposit account with a bank. At the end of each month for the next 24 months. The bank allows interest of 12% p.a. How much will you receive at the end of 24 months?

Solution:

$$A = 1,000, n = 24 \text{ months}, r = \frac{0.12}{12} = 0.01, F = ?$$

$$F = \frac{A\left[(1+r)^n - 1\right]}{r}$$

$$= \frac{1,000\left[(1+0.01)^{24} - 1\right]}{0.01}$$

$$= \frac{1,000\left[(1.01)^{24} - 1\right]}{0.01}$$

Let $x = (1.01)^{24}$

$$\log x = 24 \log(1.01)$$

$$= 24 \times 0.0043$$

$$= 0.269$$

$$x = \text{Antilog } (0.269)$$

$$= 1.269$$

$\therefore$ $$F = \frac{1,000[1.269 - 1]}{0.01}$$

$$F = \frac{1,000(0.269)}{0.01}$$

$$= \frac{269}{0.01} = 26,900/-$$

$\therefore$ $$F = ₹ 26,900/-$$

Example: Four equal quarterly payments of ₹ 2,000 are made into a deposit account that pays 10% interest p.a. What is the future value of this annuity after 4 deposits?

Solution:

Given the nominal rate $R = 10\% = 0.10$

$$r = \left[1 + \frac{R}{q}\right]^q - 1$$

$$= \left[1 + \frac{0.10}{r}\right]^4 - 1$$

$$= (1 + 0.024)^4 - 1$$

$$= (1.025)^4 - 1$$

$$\therefore \quad r = 0.1038 \times 100$$

$$r = 10.38\% = 0.1038$$

$$F = \frac{A\left[(1+r)^n - 1\right]}{r}$$

$$= \frac{2,000\left[(1+0.1038)^4 - 1\right]}{0.1038}$$

Let $x = (1.1038)^4$

$$\log x = 4\log 1.1038$$

$$= 4 \times 0.425$$

$$= 0.170$$

$$x = \text{Antilog } (0.170)$$

$$= 1.4844$$

$$\therefore \quad F = \frac{2,000[1.4844 - 1]}{0.1038}$$

$$= 2,000[1.4844 - 1]$$

$$\therefore \quad F = ₹ 9,334/-$$

Example: A company wants to create a Sinking Fund to replace a Machine after 8 years which then is expected to cost ₹ 50,00,000. That present machine will not have any scrap value. How much depreciation should be provided annually to be invested at 15% p.a. so as to raise enough funds for replacement?

Solution:

$F = 50,00,000, r = 15\% = 0.15, n = 8$ years, $A = ?$

$$A = \frac{Fr}{\left[(1+r)^n - 1\right]} = \frac{50,00,000 \times 0.15}{\left[(1+0.15)^8 - 1\right]}$$

$$= \frac{7,50,000}{2.059} = 3,07,935$$

$$A = ₹ 3,07,935/-$$

Example: Calculate the present value of an annuity of ₹ 3,000 in 6 years at 14% p.a.

Solution:

$A = 3,000, n = 6$ years, $r = 14\% = 0.14, P = ?$

$$P = \frac{A\left[(1+r)^n - 1\right]}{r(1+r)^n}$$

$$= \frac{3,000\left[(1+0.14)^6 - 1\right]}{(0.14)(1+0.14)^6}$$

$$= \frac{3,000\left[(1.14)^6 - 1\right]}{(0.14)(1.14)^6}$$

Let $x = (1.14)^6$

$$\log x = 6 \log 1.14$$

$$= 6 \times 0.0569$$

$$= 0.3414$$

$$x = \text{Antilog}\ (0.3414)$$

$$x = 2.195$$

$\therefore$
$$P = \frac{3,000\left[2.195 - 1\right]}{(0.14)(2.195)}$$

$$= \frac{3,000(1.195)}{(0.14)(1.195)} = \frac{3585}{0.3073}$$

$\therefore$
$$P = 11,666/-$$

Example: A person wants to provide for his daughter's college studies which will require ₹ 10,000 p.a. for 5 years the money will be needed one year from now. How much should he deposit in his bank account now in such a way that the account will be closed at the end of 5 years? Bank allows compound interest at the rate of 12% p.a..

Solution:

$A = 10,000, n = 5$ years, $r = 12\% = 0.12, P = ?$

$$P = \frac{A\left[(1+r)^n - 1\right]}{r(1+r)^n}$$

$$= \frac{10,000\left[(1+0.12)^5 - 1\right]}{(0.12)(1+0.12)^5}$$

$$= \frac{10,000\left[(1.12)^5 - 1\right]}{(0.12)(1.12)^5}$$

Let $x = (1.12)^5$

$$\log x = 5 \log(1.12)$$

$$= 5 \times 0.0492$$

$$= 0.2460$$

$$x = \text{Antilog}\ (0.2460)$$

$$= 1.762$$

$\therefore$
$$P = \frac{10,000\left[1.762 - 1\right]}{(0.12)(1.762)}$$

$$= \frac{7.620}{0.2144}$$

$$\therefore \qquad P = ₹\,36,038.6/-$$

Example: In what time will a sum of money double itself at 10% p.a. compounded interest payable half-yearly? Given $\log 2 = 0.3010$, $\log 1.05 = 0.0212$.

Solution:

Let the sum be ₹. Let the sum will be doubled in n years.

$$A = P\left(1 + \frac{r}{100}\right)^{2n}$$

$$\therefore \qquad P\left(1 + \frac{5}{100}\right)^{2n} = 2p \Rightarrow (1 + 0.05)^n = 2$$

Taking logs,

$$2n\log(1.05) = \log 2$$

$$2n = \frac{\log 2}{\log(1.05)} = \frac{0.3010}{0.0212} = 14.198$$

$$\therefore \qquad n = \frac{14.198}{2} = 7.09$$

$\therefore$ The sum will double itself in approximate 7 years.

Example: A man borrows ₹ 40,000 at 4% compound interest and agrees to pay both the principal and interest in 10 equal instalments at the end of each year. Find the amount of these instalments.

Solution:

Let ₹ y be amount of each instalment.

$$\frac{y}{1.04} + \frac{y}{(1.04)^2} + \ldots\ldots + \frac{y}{(1.04)^{10}} = 40,000$$

$$\frac{y}{1.04}\left[1 + \frac{1}{1.04} + \ldots\ldots\ldots + \frac{1}{(1.04)^9}\right] = 40,000$$

$$\frac{y}{1.04}\left[\frac{1\left(\frac{1}{1.04}\right)^{10}}{1 - \frac{1}{1.04}}\right] = 40,000$$

$$= \frac{y}{0.04}\left[1 - \left(\frac{1}{1.04}\right)^{10}\right] = 40,000$$

$$\Rightarrow \qquad y = ₹\,4932$$

Check Your Progress

Fill in the blanks:

1. When we borrow some amount of money as loan from somebody for a fixed period of time, an extra amount charged is called the _____

2. The interest depends on three factors namely _____ the period of time, and the _____

3. The amount borrowed is called the _____ and is denoted by P. The sum of the principal and the interest is called the _____ and is denoted by A.

4. Simple interest is a fixed sum paid at a regular interval under certain stated conditions; the _____ may be one year or half year or one month.

9.5 LET US SUM UP

- The simple interest on the principal p for the number of years N at the rate of interest R is calculated as $I = \dfrac{PNR}{100}$

- Formula for compound interest is $A = P\left(1 + \dfrac{r}{100}\right)^n$

- Future Value of Annuity $F = \dfrac{A[(1+r)^n - 1]}{r}$

- Present Value of Annuity $PV = \dfrac{A[(1+r)^n - 1]}{r(1+r)^n}$

- Perpetuity $(P_\infty) = \dfrac{A}{r}$

9.6 LESSON END ACTIVITY

Two sums of money which differ by ₹ 2000 are lent out, the larger at 4% and smaller at 6% p.a. simple interest. If the difference between the amounts is ₹ 1840 at the end of 2 years, find the sum lent at each rate.

9.7 KEYWORDS

Amount: The sum of the principal and the interest is called the amount.

Annuity: A fix sum paid at a regular interval under certain stated conditions.

Perpetuity: Annuity which is forever, i.e., for an infinite period.

Principal: The amount borrowed is called the principal.

9.8 QUESTIONS FOR DISCUSSION

1. Calculate the present value and future value of annuity of ₹ 5,000 per annum for 2 years, the interest being 4% compounded annually.

2. If Mr. X borrows ₹ 20,000 at 4% compound interest and agrees to pay both principal and the interest in 10 equal instalments at the end of year. Find the amount of each instalment.

3. Find the simple interest on ₹ 7000 for 8 years at the rate of 7% per annum.

4. Find the principal if the simple interest is ₹ 750 for 10 years at the rate of 5% per annum.

5. Find the rate of interest if ₹ 4000 earns a simple interest of ₹ 600 for 4 years.

6. Find the amount after 8 years if the rate of interest 6% is charged on ₹ 10000.

7. A person who took a loan of ₹ 6000 was asked to repay a sum of ₹ 8200 after 6 years. What is the rate of interest charged on simple interest?

8. At what rate of compound interest will ₹ 400 amount to ₹ 441 in 2 years?

9. A manufacturer buys a machine for ₹ 44000 and writes off $12\frac{1}{2}$% depreciation per annum. At the end of 3 years, he sells the machine for ₹ 33,013.75. What is the percentage gain he has made on the book-value of the machine?

10. In how many years a principal will become three times if the rate of interest charged is 5% compound interest?

11. Rajesh deposits ₹ 1000 in savings bank account which earns a compound interest at the rate of 8% per annum. What is the amount at the end of 10 years?

Check Your Progress: Model Answer

1. Interest

2. Amount borrowed, rate of interest

3. Principal, amount

4. Period

9.9 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory manual - Mathematics, secondary stage - NCERT Publication.

LESSON

10

PROFIT AND LOSS

CONTENTS

10.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

- Learn the concept of cost price

- Understand the selling price

- Learn the concept of profit or gain

- Explain the formulas for profit/loss

- Learn about the true discount

10.1 INTRODUCTION

In the previous lesson, we studied about the simple and compound interest. In this lesson, we will discuss about profit and loss.

Profit and loss is the branch of basic mathematics which deals with the study of profit and loss made in a business transaction. The profit and loss account is fundamentally a summary of the trading transactions of a business and shows whether it has made a profit or loss during a particular period of account. Indeed, by deducting the total expenditure from total income the profit or loss of a business can be calculated. Along with the balance sheet, it is one of the key financial statements that make up a company's statutory accounts.

10.2 CONCEPTS OF PROFIT AND LOSS

Profit and loss is mainly used in finance and business transactions. Notations used in profit and loss:

S.P. – Selling Price

C.P. – Cost Price

M.P. – Marked Price

Cost Price

The price, at which an article is purchased, is called its cost price, abbreviated as *C.P.* The price at which article is purchased abbreviated as *C.P.* It is the original price of any item. The cost is the total outlay required to produce a product or carry out a service. Cost price is used in establishing profitability in the following ways:

- Selling price (excluding tax) less cost results in the profit in money terms.

- Profit/selling price (excluding tax) when expressed as a percentage produces (gross profit) or *G.P.%*.

- Expense/net sales yield a percentage that when used as the target margin will produce gross profit.

Selling Price

The price, at which an article is sold, is called its selling prices, abbreviated as *S.P.* It is the market value, or agreed exchange value, that will purchase a definite quantity, weight, or other measure of a good or service.

Profit or Gain

Profit is called a normal income. Gain is called an additional income. If *S.P.* is greater than *C.P.*, the seller is said to have a profit or gain.

Loss

If *S.P.* is less than *C.P.*, the seller is said to have incurred a loss.

10.3 IMPORTANT FORMULAE

There are various formulae used in profit and loss that given in this section. The overall summary of all these formulae in which we can calculate the gain, loss, *S.P* & *C.P* directly by using the below formula:

1. Gain = $(S.P.) - (C.P.)$

2. Loss = $(C.P.) - (S.P.)$

3. Loss or gain is always reckoned on *C.P.*

4. Gain Percentage (Gain %)

$$\text{Gain}\% = \left(\frac{\text{Gain} \times 100}{C.P.} \right)$$

5. Loss Percentage (Loss %)

$$\text{Loss}\% = \left(\frac{\text{Loss} \times 100}{C.P.} \right)$$

6. Selling Price (*S.P.*)

$$S.P. = \left[\frac{(100 + \text{Gain }\%)}{100} \times C.P.\right]$$

7. Selling Price (*S.P.*)

$$S.P. = \left[\frac{(100 - \text{Loss }\%)}{100} \times C.P.\right]$$

8. Cost Price (*C.P.*)

$$C.P. = \left[\frac{100}{(100 + \text{Gain }\%)} \times S.P.\right]$$

9. Cost Price (*C.P.*)

$$C.P. = \left[\frac{100}{(100 - \text{Loss }\%)} \times S.P.\right]$$

10. If an article is sold at a gain of say 35%, then *S.P.* = 135% of *C.P.*

11. If an article is sold at a loss of say, 35% then *S.P.* = 65% of *C.P.*

12. When a person sells two similar items, one at a gain of say *x*%, and the other at a loss of *x*%, then the seller always incurs a loss given by:

$$\text{Loss }\% = \left(\frac{\text{Common Loss and Gain }\%}{10}\right)^2 = \left(\frac{x}{10}\right)^2$$

13. If a trader professes to sell his goods at cost price, but uses false weights, then

$$\text{Gain }\% = \left[\frac{\text{Error}}{(\text{True Value}) - (\text{Error})} \times 100\right]\%.$$

Example: A man buys an article for ₹ 27.50 and sells it for ₹ 28.60. Find his gain %.

Solution:

C.P. = ₹ 27.50, *S.P.* = ₹ 28.60

Gain = ₹ (28.60 – 27.50) = ₹ 1.10

So gain% = {(1.10/27.50) × 100} = 4%

Example: If a radio is bought for ₹ 490 and sold for ₹ 465.50. Find loss%.

Solution:

C.P. = ₹ 490, *S.P.* = 465.50

Loss = ₹ (490 – 465.50) = ₹ 24.50.

Loss % = [(24.50/490) × 100]% = 5%

Example: Find *S.P.* when *C.P.* = 56.25, gain = 20%.

Solution:

S.P. = 20% of ₹ 56.25 = ₹ {(120/100) × 56.25} = ₹ 67.50.

Example: Find *S.P.* when *C.P.* = ₹ 80.40, loss = 5%

Solution:

S.P. = 85% of ₹ 80.40

$\qquad$ = ₹ {(85/100) × 80.40} = ₹ 68.34.

Example: Find *C.P.* when *S.P.* = ₹ 40.60, gain = 16%

Solution:

C.P. = ₹ {(100/116) × 40.60} = ₹ 35.

Example: Find *C.P.* when *S.P.* = ₹ 51.70, loss = 12%

Solution:

C.P. = ₹ {(100/88) × 51.87} = ₹ 58.75.

Example: A man bought 3 toffees for a rupee. How many for a rupee must he sell to gain 50%?

Solution:

C.P. of 3 toffees = ₹ 1; *S.P.* of 3 toffees = 150% of ₹ 1 = 3/2.

For ₹ 3/2, toffees sold = 3, for ₹ 1, toffees sold = [3 × (2/3)] = 2.

Example: A dishonest dealer professes to sell his goods at cost price but uses a weight of 960 gms. for a kg weight. Find his gain per cent.

Solution:

$$\text{Gain } \% = \left[\frac{(Error)}{(Error\ Value - Error)} \times 100 \right] \%$$

$$= [(40/1000) \times 100] \%$$

$$= 4 \%$$

10.4 TRUE DISCOUNT

True discount is the method of calculation and collection of interest upfront from the borrower while disbursing the loan at a specific rate of interest at the same time same rate of interest is considered for calculating interest on interest and transfer the same to customer.

$$TD = \text{Interest on } PW$$

$$\text{Amount} = (PW) + (TD).$$

Interest is reckoned on *PW* and true discount is reckoned on the amount.

Let rate = *R*% per annum and Time = *T* years. Then,

1. $P.W. = \dfrac{100 \times \text{Amount}}{100 + (R \times T)} = \dfrac{100 \times T.D.}{R \times T}$

2. $T.D. = \dfrac{(P.W.) \times (R \times T)}{100} = \dfrac{\text{Amount} \times R \times T}{100 + (R \times T)}$

3. $\text{Sum} = \dfrac{(S.I.) \times (T.D.)}{(S.I.) - (T.D.)}$

4. $(S.I.) - (T.D.) = S.I.$ on $T.D.$

5. When the sum is put at compound interest, then $P.W. = \dfrac{\text{Amount}}{\left(1 + \dfrac{R}{100}\right)^T}$

Example: A trader owes a merchant ₹ 10,028 due 1 year hence. The trader wants to settle the account after 3 months. If the rate of interest 12% per annum, how much cash should he pay?

Solution:

Required money = $P.W.$ of ₹ 10,028 due 9 month hence

$$= ₹ \left[\frac{10,028 \times 100}{100 + \left(12 \times \dfrac{9}{12}\right)} \right]$$

$$= ₹ 9,200$$

Example: A man wants to sell his scooter. There are two offers, one at ₹12,000 cash and the other a credit of ₹ 12,880 to be paid after 8 months, money being at 18% per annum. Which is the better offer?

Solution:

$P.W.$ of ₹ 12,880 due 8 months hence $= ₹ \left[\dfrac{12,880 \times 100}{100 + \left(18 \times \dfrac{8}{12}\right)} \right]$

$$= ₹ \left(\frac{12,880 \times 100}{112} \right)$$

$$= ₹ 11,500$$

Example: If ₹ 10 be allowed as true discount on a bill of ₹ 110 due at the end of a certain time, then calculate the amount of the discount allowed on the same sum due at the end of double the time.

Solution:

$S.I.$ on ₹ (110 – 10) for a certain time = ₹ 100

$S.I.$ on ₹ 100 for double the time = ₹ 20

$T.D.$ on ₹ 120 = ₹ (120 – 100) = ₹ 20

$T.D.$ on ₹ 1110 = ₹ $\left(\dfrac{20}{120} \times 110 \right)$ = ₹ 18.33

Check Your Progress

Fill in the blanks:

1. If *S.P.* is greater than *C.P.*, seller is said to have a _____ or gain.

2. The _____ is the total outlay required to produce a product or carry out a service.

3. If *S.P.* is less than *C.P.*, the seller is said to have incurred a _____

4. Interest is reckoned on *P.W.* and _____ is reckoned on the amount.

10.5 LET US SUM UP

- Profit is a financial gain, especially the difference between the amount earned and the amount spent in buying, operating, or producing something.

- The price at which article is purchased abbreviated as *C.P.* It is the original price of any item.

- The price, at which an article is sold, is called its selling prices, abbreviated as *S.P.* It is the market value, or agreed exchange value, that will purchase a definite quantity, weight, or other measure of a good or service.

- Profit is called a normal income. Gain is called an additional income. If *S.P.* is greater than *C.P.*, the seller is said to have a profit or gain.

$$\text{Gain} = (S.P.) - (C.P.)$$

- If *S.P.* is less than *C.P.*, the seller is said to have incurred a loss.

$$\text{Loss} = (C.P.) - (S.P.)$$

$$\text{Gain } \% = \left(\frac{\text{Gain} \times 100}{C.P.} \right)$$

- True discount is the method of calculation and collection of Interest upfront from the borrower while disbursing the loan at a specific rate of Interest at the same time same rate of interest is considered for calculating interest on interest and transfer the same to customer.

$$T.D. = \frac{(P.W.) \times R \times T}{100} = \frac{\text{Amount} \times R \times T}{100 + (R \times T)}$$

10.6 LESSON END ACTIVITY

The manufacturing cost of a watch is ₹180 and the transportation lost is ₹500 for 100 watches. If it is sold at 20% gain what will be the selling price?

10.7 KEYWORDS

Cost Price: The cost is the total outlay required to produce a product or carry out a service.

Selling Price: It is the market value, or agreed exchange value, that will purchase a definite quantity, weight, or other measure of a good or service.

Profit: If *S.P.* is greater than *C.P.*

Loss: If *S.P.* is less than *C.P.*

True Discount: If interest is deducted at the time a loan is obtained it is called true discount if the amount received plus the interest equals the amount to be paid at the maturity of the obligation.

10.8 QUESTIONS FOR DISCUSSION

1. A TV is purchased at ₹ 5000 and sold at ₹ 4000, what will be the loss?

2. By mixing two qualities of pulses in the ratio 2 : 3 and selling the mixture at the rate of ₹ 22 per kilogram, a shopkeeper makes a profit of 10 %. If the cost of the smaller quantity be ₹ 14 per kg, what will be the cost of the larger quantity per kg?

3. A shopkeeper expects a gain of 22.5% on his cost price. If in a week, his sale was of ₹ 392, calculate his profit.

4. A TV and a music system were sold for ₹ 12000 each. The TV was sold at a loss of 20% and the music system at gain of 20% and calculates the entire transaction.

5. Cost of 3 cricket balls equals to cost of 2 pairs of leg pads. Cost of 3 pairs of leg pads equals to cost of 2 pairs of gloves. Cost of 3 pairs of gloves equals to cost of 2 cricket bats. If a cricket bat costs ₹ 54, then what will be the cost of a cricket ball?

6. In a certain store, the profit is 320% of the cost. If the cost increases by 25% but the selling price remains constant, approximately what percentage of the selling price is the profit?

7. The percentage profit earned by selling an article for ₹ 1920 is equal to the percentage loss incurred by selling the same article for ₹ 1280. At what price should the article be sold to make 25% profit?

8. A man buys a cycle for ₹ 1400 and sells it at a loss of 15%. What is the selling price of the cycle?

9. Sam purchased 20 dozens of toys at the rate of ₹ 375 per dozen. He sold each one of them at the rate of ₹ 33. What was his percentage profit?

10. If selling price is doubled, the profit triples. Find the profit per cent.

11. What is the present value, true discount, banker's discount and banker's gain on a bill of ₹104500 due in 9 months at 6% per annum?

Check Your Progress: Model Answer

1. Profit

2. Cost

3. Loss

4. True Discount

10.9 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor, *Business Mathematics*.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*.

Laboratory Manual - Mathematics, secondary stage - NCERT Publication.

LESSON

11

LINEAR PROGRAMMING

11.0 AIMS AND OBJECTIVES

After studying this lesson, you should be able to:

● Explain the concept of linear programming and its formulations

● Solving the linear programming problems by graphical method

● Discuss general form of a linear programming problem

● Explain simplex method for maximisation of LPP in standard form

● Learn about linear programming problem not in standard form

11.1 INTRODUCTION

In the previous lesson, we studied about the profit and loss. In this lesson, we will discuss linear programming and its formulation.

Linear programming is the widely used in mathematical modelling technique to determine the optimum allocation of scarce resources among competing demands. Resources typically include raw materials, manpower, machinery, time, money and space. The technique is very powerful and found especially useful because of its application to many different types of real business problems in areas like finance, production, sales and distribution, personnel, marketing and many more areas of management.

As its name implies, the linear programming model consists of linear objectives and linear constraints, which means that the variables in a model have a proportionate relationship. For example, an increase in manpower resource will result in an increase in work output.

11.2 GENERAL FORM OF A LINEAR PROGRAMMING PROBLEM

A linear programming problem consists of the determination of the values of n decision variables $x_1, x_2, ..., x_n$, such that the linear objective function $z = c_1 x_1 + c_2 x_2 + ... + c_n x_n$ attains an optimum value subject to the following linear constraints:

$$a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n \leq (\text{or} = \text{or} \geq) b_1$$

$$a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n \leq (\text{or} = \text{or} \geq) b_2$$

$$..$$

$$..$$

$$a_{m1} x_1 + a_{m2} x_2 + ... + a_{mn} x_n \leq (\text{or} = \text{or} \geq) b_m$$

and the non-negative restrictions $x_1, x_2, ..., x_n \geq 0$.

Here the constants $c_1, c_2, ... c_n$ are the coefficients of the respective decision variables $x_1, x_2, ... x_n$ in the objective function.

We note that $\dfrac{\partial z}{\partial x_j} = c_j$ for $j = 1, 2, ... n$. Thus c_j represents the contribution to the objective function z per unit increase in x_j. For example, if z denotes a profit function, then c_j will represent the contribution to profit of a unit increase in x_j, $j = 1, 2, ... n$. A similar type of interpretation can be given if z represents a cost function.

Each constant, $b_1, b_2, ..., b_m$ that appear on the right hand side of the constraints, represents a restriction or a requirement depending upon whether the inequality is expressed as $\leq$ or $\geq$. The coefficients of decision variables in the constraints a_{ij}'s, are called technical or structural coefficients. We note that $\dfrac{\partial b_i}{\partial x_j} = a_{ij}$, for $i = 1, 2, ... m$ and $j = 1, 2, ... n$. Thus a_{ij} represents the quantity of b_i required to increase x_j by one unit. The restrictions $x_j \geq 0$ $(j = 1, 2, ...n)$, simply imply that the decision variable must always be non-negative.

Using the notations of summations, the above linear programming problem can be written as:

Optimise $z = \sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \leq (\text{or} = \text{or} \geq) b_i \ (i = 1, 2, \ldots\ldots m)$

and $x_j \geq 0 \ (j = 1, 2, \ldots\ldots n)$

Further, if we denote by

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \text{ the above problem,}$$

using matrix notations, can be written as:

Optimise $z = C'X$ subject to $AX \leq (\text{or} = \text{or} \geq) B$ and $X \geq O$, where O is $n \times 1$ null vector.

Feasible Solution

A set of values $x_1, x_2, \ldots\ldots x_n$, that satisfies the given set of constraints and non-negativity conditions is called a *feasible solution*.

Feasible Region

It is the set of all possible feasible solutions.

Optimal Solution

Out of the set of feasible solutions, the one that optimises the objective function is called an optimal solution.

11.2.1 Formulation of the Problem

Generally, the problems that we may have to solve using linear programming techniques are not handed down to us in a ready to solve format. The given information is to be transformed into the form of equations and inequalities so that linear programming techniques are applicable.

The main steps in the formulation of a given problem into a linear programming model are:

(i)   Identify the unknowns or the decision variables to be determined and assign symbols to them.

(ii) Identify the objective function of the problem and express it as a linear function of the decision variables.

(iii) Identify the restrictions or requirements and express them in the form of linear relations.

(iv) Write down the non-negativity conditions for the variables.

It should, however, be noted that the formulation of linear programming model is no science but an art, which can be learned only through practice. The procedure for the

formulation of a problem into a linear programming model is explained through the following example:

Example: A chemical manufacturing company produces two products, A and B. Each product passes through three processes. The processing time (in hrs.) per unit for each of the two products in each process is given below:

	A	B
Process 1	2	5
Process 2	1	2
Process 3	4	2

The total hours available for each process in a week are 30, 24 and 20 respectively. Product A gives a profit of ₹ 18 per unit and product B gives a profit of ₹ 15 per unit. Find the quantities of A and B to be produced in the next week that will give maximum profit.

Solution:

The formulation of the above problem into a linear programming problem can be done as follows:

Let x_1, x_2 be the respective quantities, to be produced, of the two products A and B. In terms of notations, we are given:

$$c_1 = 18, c_2 = 15, b_1 = 30, b_2 = 24, b_3 = 20$$

Also $a_{11} = 2, a_{12} = 5, a_{21} = 1, a_{22} = 2, a_{31} = 4$ and $a_{32} = 2$.

Thus, the problem can be written as:

Maximise $\quad z = 18x_1 + 15x_2$

Subject to $\quad 2x_1 + 5x_2 \leq 30 \qquad \qquad$ ... (1)

$\qquad \qquad \quad x_1 + 2x_2 \leq 24 \qquad \qquad$ ... (2)

$\qquad \qquad \quad 4x_1 + 2x_2 \leq 20 \qquad \qquad$ ... (3)

and $\quad x_1, x_2 \geq 0$

11.2.2 Essentials of Linear Programming Model

For a given problem situation, there are certain essential conditions that need to be solved by using linear programming.

- *Limited Resources:* Limited number of labour, material equipment and finance.

- *Objective:* Refers to the aim to optimize (maximize the profits or minimize the costs.)

- *Linearity:* Increase in labour input will have a proportionate increase in output.

- *Homogeneity:* The products, workers' efficiency and machines are assumed to be identical.

- *Divisibility:* It is assumed that resources and products can be divided into fractions. (In case of fractions are not possible, like production of one-third of a computer, a modification of linear programming called integer programming can be used.)

11.2.3 Properties of Linear Programming Model

The following properties form the linear programming model:

- Relationship among decision variables must be linear in nature.

- A model must have an objective function.

- Resource constraints are essential.

- A model must have a non-negativity constraint.

11.3 GRAPHICAL METHOD OF SOLUTION

To find the feasible region we regard the three constraints as equations. These equations are plotted as straight lines. We note that each of these lines divides the plane into two non-overlapping regions. Since the constraints are of $\leq$ type, only the points lying on or below the line satisfy the particular constraint involved. The common part of all such regions lying in the first quadrant, where $x_j \geq 0$, $j = 1, 2$, will form the feasible region. This region is shown as shaded area OPQR in Figure 11.1.
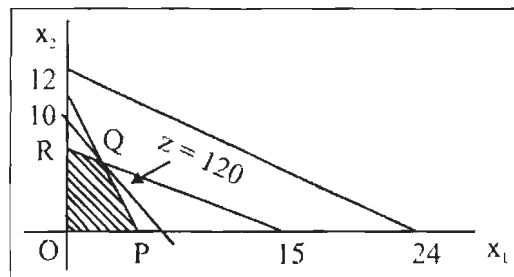
Figure 11.1

It should be noted here that the feasible region also includes all the points on its kinked boundary. Thus, we can say that the feasible region is a closed set. The points like O, P, Q and R that occur at the intersection of the boundaries of the feasible region are termed as the *corner (or extreme) points*.

Another point worth mentioning is that the constraint $x_1 + 2x_2 \leq 24$, which does not in any way affect the feasible region, is called a redundant constraint and hence can be dropped. We note that the optimal solution obtained by using the remaining constraints, which are more restrictive, will also satisfy this constraint.

The next step, in the solution of the linear programming problem, is to plot the objective function, which is profit function in the above example. We note that the graph of $z = 18x_1 + 15x_2$, for a given value of z, will be a straight line. Various points on the line give different combinations of x_1 and x_2 for a fixed value of z. Thus, the straight line can be interpreted as the iso-z or iso-profit line. Taking z as a parameter, we can draw a family of iso-profit lines. Out of these lines, we select the one with maximum value of z while still remaining in the feasible region.

As an example, let us consider three iso-z lines corresponding to $z = 90$, 120 and 140. The respective equations are

$$90 = 18x_1 + 15x_2$$

$$120 = 18x_1 + 15x_2$$

and $$140 = 18x_1 + 15x_2$$

The line $120 = 18x_1 + 15x_2$, which passes through only one point Q of the feasible region, is shown in Figure 11.1. We note that the line $140 = 18x_1 + 15x_2$ will lie to the right of this line, i.e., outside the feasible region and, therefore, ignored. On the other hand, the line $90 = 18x_1 + 15x_2$ will lie to the left of this line, i.e., pass through the feasible region. However, the associated profit is 90 (< 120) and, therefore, ignored. Thus the profits are maximised by the iso-z line corresponding to z = 120 and the optimal feasible solution or simply the optimal solution is given by the coordinates of point Q through which this iso-profit line passes. These coordinates are $x_1 = \dfrac{5}{2}$ and $x_2 = 5$. Thus the optimal quantities of A and B are $\dfrac{5}{2}$ and 5 respectively and the maximised profit is ₹ 120. We should note that, at the optimal solution, the process time on Process 1 and 3 is fully utilized but not for Process 2. Obviously, for maximum profits, we need only 2.5 + 10 = 12.5 hours of processing time on Process 2, which is less than 24 hours, the available processing time.

Alternative Method

In the example, given above, the optimal solution was obtained at the corner point corresponding to the intersection of the constraint (1) and (2). Another possible situation could be where the slope of an iso-z line is equal to the slope of one of the constraints as shown in Figure 11.2. Here the maximum iso-profit line passes through an infinite number of points in the feasible region and any one of them can be taken as an optimal solution. We note here that these solutions also contain two extreme points S and T. This observation indicates that *an optimal solution to a linear programming problem will always occur at least at one of the corner points of a feasible region.* Thus, to find an optimal solution, we need to confine our attention only to the corner points of a feasible region. An alternative method to find the optimal solution to example given above is given below:
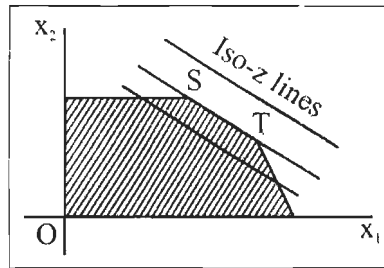
Figure 11.2

First locate all the corner points of a feasible region. These points are O, P, Q and R in Figure 11.1. Find the coordinates of these points and the associated value of z.

Corner Point	Value of z
$O(0, 0)$	$18 \times 0 + 15 \times 0 = 0$
$P(5, 0)$	$18 \times 5 + 15 \times 0 = 90$
$Q\left(\dfrac{5}{2}, 5\right)$	$18 \times \dfrac{5}{2} + 15 \times 5 = 120$
$R(0, 6)$	$18 \times 0 + 15 \times 6 = 90$

Since z is maximum at point Q, the optimal solution is $\left(\dfrac{5}{2}, 5\right)$.

Example: To maintain his health, a person must fulfill certain minimum daily requirements for several kinds of nutrients. Assume that there are only three kinds of nutrients: calcium, proteins, and calories and the person's diet consists of only two food items I and II, whose price and nutrient contents are shown in the following table:

	Food I (per lb)	Food II (per lb)	Minimum daily requirement of the nutrient
Price (₹)	0.60	1.00	
Calcium	10	4	20
Proteins	5	5	20
Calories	2	6	12

What combination of the two food items will satisfy the daily requirements and entail the least cost?

Solution:

Let x_1, x_2 denote the quantities of food I and II respectively. The linear programming formulation of the above problem can be written as

Minimise $\quad z = 0.06x_1 + x_2$

subject to $\quad 10x_1 + 4x_2 \geq 20$

$\qquad\qquad 5x_1 + 5x_2 \geq 20$

$\qquad\qquad 2x_1 + 6x_2 \geq 12$

and $\qquad x_1, x_2 \geq 0$
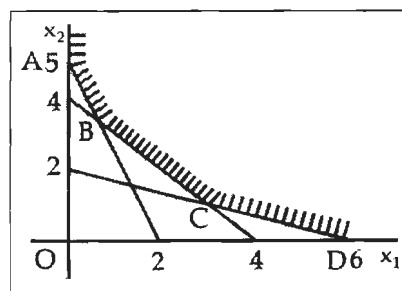
To solve this problem, we find the corners of the feasible region. From the above figure, we note that feasible region (shown as shaded) has four corners A, B, C and D. The coordinates of these points and the corresponding values of the objective function are given in the following table:

Corner Point	Value of the Objective Function
A (0, 5)	$z = 0.6 \times 0 + 1 \times 5 = 5.00$
B $\left(\dfrac{2}{3}, \dfrac{10}{3}\right)$	$z = 0.6 \times \dfrac{2}{3} + 1 \times \dfrac{10}{3} = 3.73$
C (3, 1)	$z = 0.6 \times 3 + 1 \times 1 = 2.80$
D (6, 0)	$z = 0.6 \times 6 + 1 \times 0 = 3.60$

Thus, z is minimum when $x_1 = 3$ and $x_2 = 1$.

Unbounded Feasible Region

A feasible region in which the objective function can be increased indefinitely, is called an unbounded feasible region. The solution to such a linear programming model is said to be unbounded. Consider the linear programming problem:

Maximise $z = x_1 + x_2 s$

subject to $-2x_1 + 3x_2 \leq 2$

                $-2x_1 + x_2 \leq 1$

and $x_1, x_2 \geq 0$
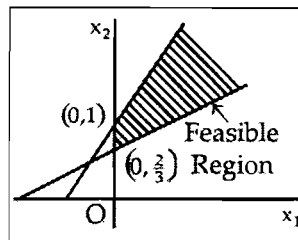
Figure 11.3

The feasible region, shown as shaded in Figure 11.3, is unbounded from above. Since the objective function can be increased indefinitely, the solution of the problem is said to unbounded.

Infeasible Solution

When the feasible region of a linear programming problem (LPP) is a null set, we say that its solution is infeasible or there is no feasible solution.

Feasible Region and the Convex Set

It can be shown that the set of points defined by the objective function or the feasible region of a linear programming problem is closed and convex.

Using mathematical techniques, it can also be shown that if the sets corresponding to an objective function and the feasible region are closed and convex, then the optimal value of the objective function, if it exists, will be found at one of the corner points of the feasible region. It should be noted here that the simplex method of solving a linear programming problem, to be discussed now, is based on this result.

11.4 SIMPLEX METHOD FOR MAXIMISATION OF LPP IN STANDARD FORM

The graphical approach and the approach of evaluating objective function are at the every corner point of the feasible region, for the solution of a linear programming problem it becomes cumbersome when the numbers of constraints become large. Simplex method provides a technique to find the corners of a feasible region and the associated value of the objective function. This technique will be explained in two steps:

(a) Finding the corner points, and

(b) Moving from one corner to another.

11.4.1 Finding the Corner Points

Consider the following standard maximisation problem:

Maximise $\quad z = c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$

subject to $\quad a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n \le b_1$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n \le b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n \le b_m$$

and $\quad x_j \ge 0$ for $j = 1, 2, \ldots\ldots n$.

Notes:

A linear programming problem that satisfies the following conditions is termed as a standard maximisation problem.

(i) All the variables are non-negative.

(ii) Each constraint is written as a linear expression that is less than or equal to a positive constant.

Since a two variable case can be conveniently handled, both by geometry as well as algebra, we shall explain the technique of finding corner points by using only two variables. This technique, as we shall see later, can be easily generalised to the n-variables case.

Slack Variables

Let us assume that the constraints and non-negativity conditions for a two variable case are:

$$x_1 + x_2 \le 15$$
$$2x_1 + x_2 \le 20$$
$$\text{and} \quad x_1, x_2 \ge 0$$

We convert each of the above constraint inequality into an equality by introducing slack variables. Thus, we can write:

$$x_1 + x_2 + s_1 = 15$$
$$\text{and} \quad 2x_1 + x_2 + s_2 = 20,$$

where $s_i \ge 0 \; (i = 1, 2)$ are called slack variables.

We may note here that a slack variable s_i represents the unused component of b_i. For example, if the number of hours available on machine I is 15, then s_1 will represent the number of unused hours etc. It may be pointed out that machine I requires one hour for processing each unit of commodity X_1 and X_2. This is indicated by the respective coefficients of x_1 and x_2 in the first constraint.

Now consider the following equations:
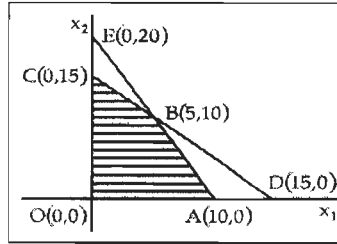
Figure 11.4

$$x_1 + x_2 = 15 \qquad \qquad \text{... (1)}$$

$$2x_1 + x_2 = 20 \qquad \qquad \text{... (2)}$$

$$x_1 = 0 \qquad \qquad \text{... (3)}$$

$$x_2 = 0 \qquad \qquad \text{... (4)}$$

We note that the corners of the feasible region can be obtained as the points of intersection of the above equations taken two at a time, as given below:

(a) The intersection of the equations (1) and (2) is at $x_1 = 5$ and $x_2 = 10$.

(b) The intersection of the equations (1) and (3) is at $x_1 = 0$ and $x_2 = 15$.

(c) The intersection of the equations (1) and (4) is at $x_1 = 15$ and $x_2 = 0$.

(d) The intersection of the equations (2) and (3) is at $x_1 = 0$ and $x_2 = 20$.

(e) The intersection of the equations (2) and (4) is at $x_1 = 10$ and $x_2 = 0$.

(f) The intersection of the equations (3) and (4) is at $x_1 = 0$ and $x_2 = 0$.

We note that with each point of intersection there is an associated value of each of the slack variables, s_1 and s_2. These values are obtained by substituting the coordinates of the corresponding point in the following equations:

$$x_1 + x_2 + s_1 = 15 \qquad \qquad \text{... (5)}$$

$$2x_1 + x_2 + s_2 = 20 \qquad \qquad \text{... (6)}$$

The values of s_1 and s_2, for each point, are given in the following table:

Point of Intersection	Values of s_1 and s_2
(i) O (0, 0)	$s_1 = 15, s_2 = 20$
(ii) A (10, 0)	$s_1 = 5, s_2 = 0$
(iii) B (5, 10)	$s_1 = 0, s_2 = 0$
(iv) C (0, 15)	$s_1 = 0, s_2 = 5$
(v) D (15, 0)	$s_1 = 0, s_2 = -10$
(vi) E (0, 20)	$s_1 = -5, s_2 = 0$

Note that for points of intersection D and E the value of one of the slack variables is negative. Since, by assumption s_i's are supposed to be non-negative, these points cannot be taken as corner points of the feasible region.

Thus the corner points of the feasible region are O, A, B and C. On writing the values of x_1, x_2, s_1 and s_2, associated with each point as an ordered 4-tuple, we have O(0, 0, 15, 20), A(10, 0, 5, 0), B(5, 10, 0, 0) and C(0, 15, 0, 5).

We note that the number of zeros in each 4-tuple is two, which is equal to the number of decision variable in the problem. It may be mentioned here that the number of zeros greater than two is also possible. Thus, we can say that *a point having at least n zeros in its ordered (n + m)-tuple, where m is the number of constraints, will correspond to a corner point of a feasible region consisting of n decision variables.*

Alternatively, the coordinates of the corner points O, A, B and C, as ordered 4-tuples, are some solutions of a system of two simultaneous equations, (5) and (6), in four unknowns x_1, x_2, s_1 and s_2. We recall here that a system of 2 equations in 4 unknowns has an infinite number of solutions. Thus out of these solutions, those having at least 2 zeros in their ordered 4-tuple, subject to the non-negative restrictions on the variables, will correspond to the corner points of the feasible region. This result can be generalised as follows:

Given the m constraints $a_{i1}x_1 + a_{i2}x_2 + \ldots\ldots + a_{in}x_n \le b_i$ $(i = 1, 2, \ldots\ldots m)$ and the non-negativity restrictions $x_j \ge 0$ $(j = 1, 2, \ldots\ldots n)$, we can, by introducing a slack variable $s_i \ge 0$ in the ith constraint $(i = 1, 2, \ldots\ldots m)$, write a system of m equations in $(m + n)$ unknowns. Since this system has less number of equations than the number of unknowns, there are infinite numbers of solutions. Out of these solutions, those having at least n zeros in their $(m + n)$-tuples, subject to the non-negative restrictions on the variables, will correspond to the corner points of the feasible region.

Example: For the system of equations:

$$2x_1 + 3x_2 + x_3 + x_4 = 30$$

$$x_1 + 2x_2 + 0.x_3 + x_4 = 20,$$

find two solutions having at least two zeros in their respective 4-tuples.

Solution:

This is a system of 2 equations in 4 variables. To obtain a solution with two zero, we put $x_1 = x_2 = 0$.

Thus the equations become

$$x_3 + x_4 = 30$$

$$x_4 = 20$$

Thus the solution becomes (0, 0, 10, 20)

Further, if we put $x_3 = x_4 = 0$, the two equations become

$$2x_1 + 3x_2 = 30$$

$$x_1 + 2x_2 = 20$$

Solving these equations simultaneously, the solution is (0, 10, 0, 0)

Alternate method:

The above equations can be rewritten in matrix form as follows:

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$$

The augmented matrix is given by $\begin{bmatrix} 2 & 3 & 1 & 1 & | & 30 \\ 1 & 2 & 0 & 1 & | & 20 \end{bmatrix}$

Performing $R_1 = R_1 - R_2$, we get $\begin{bmatrix} 1 & 1 & 1 & 0 & | & 10 \\ 1 & 2 & 0 & 1 & | & 20 \end{bmatrix}$ ... (1)

The equations represented by this matrix are

$$x_1 + x_2 + x_3 + 0.x_4 = 10$$

$$x_1 + 2x_2 + 0.x_3 + x_4 = 20$$

Thus if we take $x_1 = x_2 = 0$, we have $x_3 = 10$ and $x_4 = 20$

The required solution is (0, 0, 10, 20).

We note here that the columns corresponding to variables having non-zero values in the solution make an identity matrix. Similarly, if we want to have non-zero values of x_1 and x_2 in the solution, we have to make such transformation(s) in the augmented matrix so that its first and second columns make an identity matrix. To get this, we perform the transformation $R_2 = R_2 - R_1$, in the augmented matrix of the equation (1) and obtain $\begin{bmatrix} 1 & 1 & 1 & 0 & | & 10 \\ 0 & 1 & -1 & 1 & | & 10 \end{bmatrix}$.

Now perform $R_1 = R_1 - R_2$ to get $\begin{bmatrix} 1 & 0 & 2 & -1 & | & 0 \\ 0 & 1 & -1 & 1 & | & 10 \end{bmatrix}$

The system of equations given by the above augmented matrix is

$$x_1 + 2x_3 - x_4 = 0$$

$$x_2 - x_3 + x_4 = 10$$

Thus, if we take $x_3 = x_4 = 0$, we have $x_1 = 0$ and $x_2 = 10$.

Remarks:

1. The variables corresponding to the linearly independent columns, that form an identity matrix, are termed as the *basic variables* and the remaining are termed as the *non-basic variables*. The basic variables, taken together, are said to form a *basis*. For example, in the solution (0, 0, 10, 20), x_3 and x_4 are basic variables and, therefore, they are said to be in the basis. Similarly, in the solution (0, 10, 0, 0), x_1 and x_2 are in the basis.

2. If a feasible solution to a linear programming model has basic variables, it is termed as a *basic feasible solution*.

3. The above discussion implies that a corner point of a feasible region can be located simply by locating a basic feasible solution.

11.4.2 Moving from One Corner to Another

After the location of a corner point, the search for the optimal solution is done by moving from one corner of the feasible region to another, i.e., by shifting from one Basic Feasible Solution (BFS) to another. We note that the shift from one BFS to another can be done by converting a non-basic variable into a basic variable. This process of shifting is called *pivoting*. The column corresponding to the entering variable is called as *pivot column* and the row corresponding to the departing variable

is called as *pivot row*. The element at the intersection of pivot column and pivot row is called the *pivot element*. As will be explained in the following example, in standard maximisation problems, the variable with the highest positive coefficient in the objective function is to be chosen as the entering variable and the outgoing variable is chosen to be that variable such that the values of all the decision and the slack variables remain non-negative.

11.4.3 The Simplex Tableau

A simplex tableau is a table that displays the augmented matrix of the coefficients of a system of equations in a particular format. Consider the following standard maximisation linear programming problem:

Maximise $\quad z = c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$

subject to $\quad a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n \le b_1$

$\qquad\qquad a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n \le b_2$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\qquad\qquad a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n \le b_m$

and $\qquad\qquad x_j \ge 0 \text{ for } j = 1, 2, \ldots.. n$

Adding slack variables to various inequalities, we can write the constraints and the objective function as the following matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 & 0 \\ -c_1 & -c_2 & \cdots & -c_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ s_1 \\ s_2 \\ \vdots \\ s_m \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ 0 \end{bmatrix}$$

Note that the coefficients of slack variables in the objective function are taken as zeros. The augmented matrix of the above matrix equation, written in the following format, is termed as the *simplex tableau*.

Basis	x_1	x_2	...	x_n	s_1	s_2	...	s_m	z	VVB*
s_1	a_{11}	a_{12}	...	a_{1n}	1	0	...	0	0	b_1
s_2	a_{21}	a_{22}	...	a_{2n}	0	1	...	0	0	b_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
s_m	a_{m1}	a_{m2}	...	a_{mn}	0	0	...	1	0	b_m
z	$-c_1$	$-c_2$	...	$-c_n$	0	0	...	0	1	0

*Values of Variables in Basis

To shift from one BFS to another, we have to apply elementary row operations to the above tableau so that pivot element is replaced by 1 and all the other elements in pivot

column are replaced by zeros. This process is also termed as *pivoting*. Various steps for pivoting are explained through the following example.

Example: A dealer wishes to purchase certain number of fans and sewing machines. He has only ₹ 5,760 to invest and has a space for at the most 20 items. A fan costs him ₹ 360 and a sewing machine ₹ 240. His expectation is that he can sell a fan at a profit of ₹ 20 and a sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he buys, how should he invest his money in order to maximise his profit. Formulate this problem as a linear programming problem and solve it by simplex method.

Solution:

Let x_1 be the number of fans and x_2 be the number of sewing machines. The given problem, formulated as LPP, can be written as

Maximise $\quad z = 22x_1 + 18x_2$

subject to $\quad 360x_1 + 240x_2 \leq 5,760$

$\quad\quad\quad\quad x_1 + x_2 \leq 20$

and $\quad\quad x_1 \geq 0, x_2 \geq 0$

On adding slack variables, s_1 and s_2, the above problem can be rewritten as

Maximise $\quad z = 22x_1 + 18x_2 + 0.s_1 + 0.s_2$

subject to $\quad 360x_1 + 240x_2 + s_1 = 5,760$

$\quad\quad\quad\quad x_1 + x_2 + s_2 = 20$

and $\quad\quad x_1, x_2, s_1, s_2 \geq 0$

To facilitate the construction of a simplex tableau, we can write the constraints and the objective function as

$$360x_1 + 240x_2 + s_1 = 5,760$$

$$x_1 + x_2 + s_2 = 20$$

$$-22x_1 - 18x_2 - 0.s_1 - 0.s_2 + z = 0$$

From this, we can easily write the first simplex tableau, as given below:

Tableau I

Basis	x_1	x_2	s_1	s_2	z	VVB
s_1	(360)	240	1	0	0	5760
s_2	1	1	0	1	0	20
z	−22	−18	0	0	1	0

From the objective function, we note that a unit increase of x_1 will contribute ₹ 22 to profit while a unit increase of x_2 will contribute only ₹ 18. Thus the variable to enter the basis should be x_1. In terms of above tableau, the variable having the *largest negative coefficient* in the last row is selected as the entering variable. We note that the largest negative element in last row of the tableau is −22, therefore, x_1 is taken as the entering variable. To determine the outgoing variable such that the values of all the decision variables as well as the slack variables remain non-negative, we take the

minimum of the displacement ratios $\frac{5760}{360} = 16$ and $\frac{20}{1}$. Since the displacement ratio corresponding to first row is minimum, s_1 will be the outgoing variable. Thus the first column and the first row of the above tableau become the pivot column and the pivot row respectively and the element belonging to both of them i.e., 360, becomes the pivot element. The pivot element is encircled. The next step is to transform the pivot column into a unit vector with 1 at the position of pivot element. Tableau II, after making relevant transformations, is obtained as given below:

Tableau II

Basis	x_1	x_2	s_1	s_2	z	VVB
x_1	1	$\frac{2}{3}$	$\frac{1}{360}$	0	0	16
s_2	0	$\left(\frac{1}{3}\right)$	$-\frac{1}{360}$	1	0	4
z	0	$-\frac{10}{3}$	$\frac{11}{180}$	0	1	352

Note that the next entering variable is x_2 and the outgoing variable is s_2. Therefore, the pivot element is $\frac{1}{3}$. After making the necessary transformations, the next tableau is obtained as follows:

Tableau III

Basis	x_1	x_2	s_1	s_2	z	VVB
x_1	1	0	$\frac{1}{120}$	-2	0	8
x_2	0	1	$-\frac{1}{120}$	3	0	12
z	0	0	$\frac{1}{30}$	10	1	392

Finally, since no entry in the last row is negative, the objective function cannot be increased further and hence reached its maximum. The last column of the above table gives the optimum values of various variables. This gives $x_1 = 8, x_2 = 12$ and $z = 392$.

Example: A farmer has 1,000 acres of land on which he can grow corn, wheat or soya-beans. Each acre of corn costs ₹ 100 for preparation and requires 7 man-days of work, and yields a profit of ₹ 30. An acre of wheat costs ₹ 120 to prepare, requires 10 man-days of work and yields a profit of ₹ 40. An acre of soya-bean costs ₹ 70 to prepare, requires 8 man-days of work and yields a profit of ₹ 20. If the farmer has ₹ 1,00,000 for preparation and can count on 8,000 man-days of work, how many acres should be allocated to each crop to maximise profits?

Solution:

Let x_1, x_2, x_3 acres of land be allocated to corn, wheat and soya-bean respectively.

Then we have to

maximise $\quad z = 30x_1 + 40x_2 + 20x_3$

subject to $\quad 100x_1 + 120x_2 + 70x_3 \leq 1,00,000$

$\qquad\qquad 7x_1 + 10x_2 + 8x_3 \leq 8,000$

$$x_1 + x_2 + x_3 \le 1,000$$

and $\qquad x_1, x_2, x_3 \ge 0.$

Dividing the first constraint by 10 on both sides, we can write

$$10x_1 + 12x_2 + 7x_3 \le 10,000$$

On introducing slack variables, we can write the constraints and the objective function as

$$10x_1 + 12x_2 + 7x_3 + s_1 = 10,000$$

$$7x_1 + 10x_2 + 8x_3 + s_2 = 8,000$$

$$x_1 + x_2 + x_3 + s_3 = 1,000$$

$$-30x_1 - 40x_2 - 20x_3 - 0.s_1 - 0.s_2 - 0.s_3 + z = 0$$

Tableau I

Basis	x_1	x_2	x_3	s_1	s_2	s_3	z	VVB
s_1	10	12	7	1	0	0	0	10,000
s_2	7	(10)	8	0	1	0	0	8,000
s_3	1	1	1	0	0	1	0	1,000
z	−30	−40	−20	0	0	0	1	0

Tableau II

Basis	x_1	x_2	x_3	s_1	s_2	s_3	z	VVB
s_1	$\left(\dfrac{16}{10}\right)$	0	$-\dfrac{26}{10}$	1	$-\dfrac{12}{10}$	0	0	400
x_2	$\dfrac{7}{10}$	1	$\dfrac{8}{10}$	0	$\dfrac{1}{10}$	0	0	800
s_3	$\dfrac{3}{10}$	0	$\dfrac{2}{10}$	0	$-\dfrac{1}{10}$	1	0	200
z	−2	0	12	0	4	0	1	32,000

Tableau III

Basis	x_1	x_2	x_3	s_1	s_2	s_3	z	VVB
x_1	1	0	$-\dfrac{26}{16}$	$\dfrac{10}{16}$	$-\dfrac{12}{16}$	0	0	250
x_2	0	1	$\dfrac{31}{16}$	$-\dfrac{7}{16}$	$\dfrac{10}{16}$	0	0	625
s_3	0	0	$\dfrac{11}{16}$	$-\dfrac{3}{16}$	$\dfrac{2}{16}$	1	0	125
z	0	0	$\dfrac{35}{4}$	$\dfrac{5}{4}$	$\dfrac{5}{2}$	0	1	32,500

Since no entry in the last row of the above tableau is negative, the objective function has reached maximum. The required solution is $x_1 = 250$, $x_2 = 625$, $x_3 = 0$ and the value of the objective function is $z = 32,500$.

To get a better insight about the working of simplex method, a simplex tableau can be written in the following format.

Basis	c_i	c_1	c_2	...	c_n	0	0	...	0	VVB
		x_1	x_2	...	x_n	s_1	s_2	...	s_n	
s_1	0	a_{11}	a_{12}	...	a_{1n}	1	0	...	0	b_1
s_2	0	a_{21}	a_{22}	...	a_{2n}	0	1	...	0	b_2
⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
s_m	0	a_{m1}	a_{m2}	...	a_{mn}	0	0	...	1	b_m
z_j		0	0	...	0	0	0	...	0	0
$c_j - z_j$		c_1	c_2	...	c_n	0	0	...	0	

The entries $c_1, c_2, ..., c_n, 0, 0, ...$ in the top row of the tableau represent the contribution to profit from the unit increase of the variable.

The entries in the first column of the above table indicate the basic variables i.e., the variables in the basis of a linear programming problem. The second column gives their respective c_j values.

The elements in the third, fourth, ... columns can be interpreted as marginal rates of substitution. For example, a value a_{ij} is the quantity of the ith basic variable sacrificed so as to increase the quantity of jth variable by one unit.

Since reducing a basic variable by one unit implies the loss in profit equal to the associated value in c_j column, the entries of c_j column can be regarded as the cost per unit. Thus, the inner product of c_j column with any column under a variable can be regarded as the cost of increasing that variable by one unit.

For example, the cost of increasing variable x_1 by one unit is

$$z_1 = 0 \times a_{11} + 0 \times a_{21} + ... + 0 \times a_{m1} = 0 \text{ etc.}$$

These costs for various variables are written in the z_j row of the above tableau.

The net contribution to profit by each variable are written in the last, $c_j - z_j$, row of the tableau. This row is also known as the index row.

Following this approach, an alternative solution of example given above is given below:

Tableau I

Basis	c_j	30	40	20	0	0	0	VVB
		x_1	x_2	x_3	s_1	s_2	s_3	
s_1	0	10	12	7	1	0	0	10,000
s_2	0	7	⑩	8	0	1	0	8,000
s_3	0	1	1	1	0	0	1	1,000
z_j		0	0	0	0	0	0	0
$c_j - z_j$		30	40	20	0	0	0	

The index or $c_j - z_j$ row indicates that the net contribution to profit of a unit increase in x_1 is 30, for x_2 is 40 and that for x_3 is 20. Thus we take x_2 as the entering variable.

Further the minimum of $\dfrac{10000}{12}$, $\dfrac{8000}{10}$ and $\dfrac{1000}{1}$ is $\dfrac{8000}{10}$. Therefore s_2 will be the outgoing variable. The new tableau, after pivoting is given below.

Tableau II

Basis	c_i	30 x_1	40 x_2	20 x_3	0 s_1	0 s_2	0 s_3	VVB
s_1	0	$\left(\dfrac{16}{10}\right)$	0	$-\dfrac{26}{10}$	1	$-\dfrac{12}{10}$	0	400
x_2	40	$\dfrac{7}{10}$	1	$\dfrac{8}{10}$	0	$\dfrac{1}{10}$	0	800
s_3	0	$\dfrac{3}{10}$	0	$\dfrac{2}{10}$	0	$-\dfrac{1}{10}$	1	200
z_i		28	40	32	0	4	0	32000
$c_j - z_j$		2	0	-12	0	-4	0	

It is obvious that the entering variable now is x_1. Further, the outgoing variable is s_1. After pivoting, we get the following tableau.

Tableau III

Basis	c_j	30 x_1	40 x_2	20 x_3	0 s_1	0 s_2	0 s_3	VVB
x_1	30	1	0	$-\dfrac{26}{16}$	$\dfrac{10}{16}$	$-\dfrac{12}{16}$	0	250
x_2	40	0	1	$\dfrac{31}{16}$	$-\dfrac{7}{16}$	$\dfrac{10}{16}$	0	625
s_3	0	0	0	$\dfrac{11}{16}$	$-\dfrac{3}{16}$	$\dfrac{2}{16}$	1	125
z_i		30	40	$\dfrac{115}{4}$	$\dfrac{5}{4}$	$\dfrac{5}{2}$	0	32500
$c_j - z_j$		0	0	$-\dfrac{35}{4}$	$-\dfrac{5}{4}$	$-\dfrac{5}{2}$	0	

Since no entry in the $c_j - z_j$ row is positive, the objective function has reached maximum. As before, the solution is $x_1 = 250$, $x_2 = 625$ and $x_3 = 0$. The value of z is 32500.

Remarks:

The value in the z_j row under a slack variable in the final tableau denotes the shadow price or accounting price or opportunity cost of the associated resource. It gives the change in objective function as a result of one unit increase in input. For example, the shadow price of the first resource with slack variable s_1, in the above example, is 5/4 which implies that by increasing one unit of first resource we can increase the profit by ₹ 5/4.

Example: Given the simplex tableau for a maximisation problem of linear programming:

c_j	Product Mix	x_1	x_2	s_1	s_2	Quantity
4	x_2	1	1	1	0	6
0	s_2	1	0	−1	1	2
	c_j	3	4	0	0	

(i) Test whether the above tableau gives optimum solution. If not, improve it and find the optimum solution.

(ii) If s_1 and s_2 are slack variables at machine I and machine II, what is the maximum price you will like to pay for one hour of each machine?

Solution:

(i) Rewriting the above tableau in standard format we have

Tableau I

Basis	c_j	3	4	0	0	VVB
		x_1	x_2	s_1	s_2	
x_2	4	1	1	1	0	6
s_2	0	1	0	−1	1	2
z_j		4	4	4	0	24
$c_j - z_j$		−1	0	−4	0	

Since all the elements of $c_j - z_j$ row are either negative or zero, the solution is optimal. The optimal solution is $x_1 = 0$, $x_2 = 6$ and the optimal value of the objective function is 24.

(ii) The elements in z_j row under s_1 and s_2 are shadow prices for one hour of machine I and II respectively. The price for one hour of machine I is 4 and that of machine II is zero.

Example: Given below is a table obtained after few iteration using simplex method, to solve a linear programming problem to maximise the total contribution from product 1 and 2.

Basic Variables	x_1	x_2	s_1	s_2	s_3	Quantity
x_2	0	1	3/5	−2/5	0	300
x_1	1	0	−2/5	3/5	0	300
s_3	0	0	−1/5	−1/5	1	400
c_j	8.5	10.5	0	0	0	

Give short answers to the following giving reasons:

(i) Is the above solution optimal?

(ii) Is the above solution feasible?

(iii) Does the problem have alternative solution? If so, find the other solution?

(iv) What are the shadow prices of the resources?

Solution:

Rewriting the above table in standard format, we have

Basis	c_j	8.5	10.5	0	0	0	VVB
		x_1	x_2	s_1	s_2	s_3	
x_2	10.5	0	1	3/5	−2/5	0	300
x_1	8.5	1	0	−2/5	3/5	0	300
s_3	0	0	0	−1/5	−1/5	1	400
z_j		8.5	10.5	2.9	0.9	0	5700
$c_j - z_j$		0	0	−2.9	−0.9	0	

(i) Since none of the elements in $c_j - z_j$ row is positive, therefore the solution is optimal.

(ii) Since the solution satisfies all the constraints and non-negativity conditions, it is feasible.

(iii) Since none of the non-basic variables (s_1 and s_2 in the above example) have zero in the $c_j - z_j$ row, there is no alternative solution.

(iv) Shadow prices of first second and third resources are 2.9, 0.9 and 0 respectively.

LPP having an Unbounded Solution

Example:

Maximise $z = 3x_1 + 4x_2$

subject to $-x_1 + x_2 \leq 5$

$-3x_1 + x_2 \leq 10$ and $x_1, x_2 \geq 0$.

Solution:

On introducing slack variables, the constraints and the objective function can be written as

$$-x_1 + x_2 + s_1 = 5$$

$$-3x_1 + x_2 + s_2 = 10$$

$$-3x_1 - 4x_2 + z = 0$$

Tableau I

Basis	x_1	x_2	s_1	s_2	z	VVB
s_1	−1	①	1	0	0	15
s_2	−3	1	0	1	0	10
z	−3	−4	0	0	1	0

Tableau II

Basis	x_1	x_2	s_1	s_2	z	VVB
x_2	−1	1	1	0	0	5
s_2	−2	0	−1	1	0	5
z	−7	0	4	0	1	20

The only negative entry in the last row is -7. Thus x_1 should be the entering variable.

But since all the entries in the pivot column are negative, the solution is unbounded. Note that *only positive entries of the pivot column are considered to take a decision about the outgoing variable.*

LPP having Multiple Solutions

A linear programming problem having more than one optimal solution is said to have multiple solutions. This situation occurs when the objective function is parallel to one of the constraints (i.e. are linearly dependent). This is explained through the following example.

Example:

$$\text{Maximise} \quad z = 5x + 3y$$
$$\text{subject to} \quad x + y \le 14$$
$$10x + 6y \le 100$$
$$\text{and} \quad x \ge 0, y \ge 0.$$

Solution:

On introducing slack variables, the initial tableau can be written as

Tableau I

Basis	x	y	s_1	s_2	z	VVB
s_1	1	1	1	0	0	14
s_2	⑩	6	0	1	0	100
z	-5	-3	0	0	1	0

Tableau II

Basis	x	y	s_1	s_2	z	VVB
s_1	0	$\dfrac{4}{10}$	1	$-\dfrac{1}{10}$	0	4
x	1	$\dfrac{6}{10}$	0	$\dfrac{1}{10}$	0	10
z	0	0	0	$\dfrac{1}{2}$	1	50

Since there is no negative entry in the last row of the above tableau, the objective function has reached an optimum. The optimum solution is $x = 10$, $y = 0$ and $z = 50$. However, we note that *the entry in the last row corresponding to y column is zero but y is a non basic variable.* This situation indicates that there are *multiple solutions* to the problem. We note that the objective function is parallel to one of the given constraints. If we introduce y into basis, s_1 becomes the outgoing variable. Thus, the new tableau can be written as

Basis	x	y	s_1	s_2	z	VVB
y	0	1	$\dfrac{10}{4}$	$-\dfrac{1}{4}$	0	10
x	1	0	$-\dfrac{6}{4}$	$\dfrac{1}{4}$	0	4
z	0	0	0	$\dfrac{1}{2}$	1	50

Thus, we get another optimum solution with $x = 4$, $y = 10$ and $z = 50$.

11.5 MINIMISATION PROBLEM IN STANDARD FORM

A linear programming minimisation problem which satisfies the following conditions is said to be in standard form.

(i) All the decision variables are non-negative.

(ii) All the constraints are given with type inequalities.

(iii) The coefficients of the decision variables in the objective function must be non-negative.

Although a separate simplex method for the solution of a minimisation problem can also be given, we shall tackle this problem through the solution of its dual.

11.5.1 The Dual Problem

We shall state without proof that corresponding to every maximisation linear programming problem, there always exists an associated minimisation linear programming problem and vice-versa. The original problem is known as the *primal problem* and the associated one as its *dual*. It can be shown that optimal value of the objective function of a primal and its dual are always identical. Further, the primal problem can be converted into its dual and vice-versa. Thus, it follows that a linear programming problem can also be solved through the solution of its dual.

Consider the following maximisation problem:

Maximise $z = c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$

subject to $a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n \le b_1$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n \le b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n \le b_m$$

and $x_j \ge 0$, for $j = 1, 2, \ldots, n$.

Assuming the m inequalities as equations, the augmented matrix of the above equation system can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\ c_1 & c_2 & \cdots & c_n & z \end{bmatrix} \quad \therefore \quad A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & c_1 \\ a_{12} & a_{22} & \cdots & a_{m2} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} & c_n \\ b_1 & b_2 & \cdots & b_m & z \end{bmatrix}$$

Using this matrix, we can write down the following minimisation problem which is called the dual of the above maximisation problem.

Minimise $z = b_1 y_1 + b_2 y_2 + \ldots\ldots + b_m y_m$

subject to constraints

$$a_{11} y_1 + a_{21} y_2 + \ldots\ldots + a_{m1} y_m \ge c_1$$

$$a_{21} y_1 + a_{22} y_2 + \ldots\ldots + a_{m2} y_m \ge c_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n$$

and $\qquad y_i \geq 0$, for $i = 1, 2, \ldots, m$.

Note that $y_1, y_2, \ldots y_m$ are the decision variables of the dual problem.

Example: Write the dual of the following problems:

(i) Minimise $z = 4x_1 + 5x_2$

    subject to $2x_1 + 3x_2 \geq 4$

              $x_1 + 4x_2 \geq 3$

              $3x_1 + x_2 \geq -5$

              $x_1, x_2 \geq 0$

(ii) Minimise $z = 2x_1 - x_2 + 3x_3$

    subject to $x_1 + x_2 + 2x_3 \leq 10$

              $3x_1 + 5x_2 \leq 12$

              $4x_2 + x_3 \leq 9$

              $x_1, x_2 \geq 0$

Solution:

(i) Taking the inequalities as equations, we can write

$$2x_1 + 3x_2 = 4$$
$$x_1 + 4x_2 = 3$$
$$3x_1 + x_2 = -5$$
$$4x_1 + 5x_2 = z$$

Thus the augmented matrix can be written as $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 4 & 3 \\ 3 & 1 & -5 \\ 4 & 5 & z \end{bmatrix}$

Taking transpose, we get $A' = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 1 & 5 \\ 4 & 3 & -5 & z \end{bmatrix}$

Thus the dual problem can be written as

    Maximise $z = 4y_1 + 3y_2 - 5y_3$

    subject to $2y_1 + y_2 + 3y_3 \leq 4$

                  $3y_1 + 4y_2 + y_3 \leq 5$

    and $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

(ii) Taking inequalities as equations, we can write the augmented matrix of the equation system and its transpose as

$$A = \begin{bmatrix} 1 & 1 & 2 & 10 \\ 3 & 5 & 0 & 12 \\ 0 & 4 & 1 & 9 \\ 2 & -1 & 3 & z \end{bmatrix} \text{ and } A' = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 1 & 5 & 4 & -1 \\ 2 & 0 & 1 & 3 \\ 10 & 12 & 9 & z \end{bmatrix} \text{ respectively.}$$

Thus, the dual of the given problem can be written as

    Minimise $z = 10y_1 + 12y_2 + 9y_3$

subject to $\quad y_1 + 3y_2 \geq 2$

$$y_1 + 5y_2 + 4y_3 \geq -1$$

$$2y_1 + y_3 \geq 3$$

and $\quad y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.

11.5.2 Solution of a Minimisation Problem in Standard Form

A minimisation problem, in standard form, is often solved through its dual. The optimum value of the objective function of a primal problem is same as the optimum value of the objective function of its dual. The optimum values of decision variables of the minimisation problem are directly read from the final tableau of the dual problem. These values are given by the elements in the last row under the slack variables' columns.

Example: Vitamins A, B and C are found in foods F_1 and F_2. One unit of F_1 contains 1 mg. of A, 100 mg. of B and 10 mg. of C. One unit of F_2 contains 1 mg. of A and 10 mg. of B and 100 mg. of C. The minimum daily requirements of A, B and C are 1 mg., 50 mg. and 10 mg. respectively. The cost per unit of F_1 and F_2 are ₹ 1 and ₹ 1.50 respectively. Formulate the above as a linear programming problem minimising the cost per day.

Solution:

Let x_1 and x_2 denote the quantities of foods F_1 and F_2 respectively.

Thus the problem can be written as

Minimise $\quad z = x_1 + 1.5x_2$

subject to $\quad x_1 + x_2 \geq 1$

$$100x_1 + 10x_2 \geq 50$$

$$10x_1 + 100x_2 \geq 10 \text{ and } x_1, x_2 \geq 0.$$

Assuming the constraints as equalities, we can write the above equation system, including the objective function, in the form of the following augmented matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 100 & 10 & 50 \\ 10 & 100 & 10 \\ 1 & 1.5 & z \end{bmatrix} \quad \therefore A' = \begin{bmatrix} 1 & 100 & 10 & 1 \\ 1 & 10 & 100 & 1.5 \\ 1 & 50 & 10 & z \end{bmatrix}$$

Let y_1, y_2 and y_3 be the decision variables of the dual. Thus the dual problem can be written as

Maximise $\quad z = y_1 + 50y_2 + 10y_3$

subject to $\quad y_1 + 100y_2 + 10y_3 \leq 1$

$$y_1 + 10y_2 + 100y_3 \leq 1.5 \text{ and } y_1, y_2, y_3 \geq 0.$$

Adding slack variables s_1 and s_2 to the first and the second constraints respectively, Tableau I can be written as

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Tableau I

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
s_1	1	(100)	10	1	0	0	1
s_2	1	10	100	0	1	0	1.5
z	−1	−50	−10	0	0	1	0

Tableau II

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
y_2	$\dfrac{1}{100}$	1	$\dfrac{1}{10}$	$\dfrac{1}{100}$	0	0	$\dfrac{1}{100}$
s_2	$\dfrac{9}{10}$	0	(99)	$-\dfrac{1}{10}$	1	0	$\dfrac{14}{10}$
z	$-\dfrac{1}{2}$	0	−5	$\dfrac{1}{2}$	0	1	$\dfrac{1}{2}$

Tableau III

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
y_2	$\left(\dfrac{1}{110}\right)$	1	0	$\dfrac{1}{99}$	$-\dfrac{1}{990}$	0	$\dfrac{85}{9900}$
y_3	$\dfrac{1}{110}$	0	1	$-\dfrac{1}{990}$	$\dfrac{1}{99}$	0	$\dfrac{14}{990}$
z	$-\dfrac{5}{11}$	0	0	$\dfrac{49}{99}$	$\dfrac{5}{99}$	1	$\dfrac{113}{198}$

Tableau IV

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
y_1	1	110	0	$\dfrac{110}{99}$	$-\dfrac{11}{99}$	0	$\dfrac{85}{90}$
y_3	0	−1	1	$-\dfrac{11}{990}$	$\dfrac{11}{990}$	0	$\dfrac{1}{180}$
z	0	50	0	1	0	1	1

Since no entry in the last row is negative, the optimum is reached. The optimal solution of the dual problem is $y_1 = \dfrac{85}{90}$, $y_2 = 0$ and $y_3 = \dfrac{1}{180}$ with maximum value of the objective function $= 1$. Further, the optimal solution to the primal problem is $x_1 = 1$ and $x_2 = 0$ with minimum value of the objective function $= 1$.

Note: The simplification of the constraints must invariably be avoided whenever a problem is being solved through its dual.

Example:

Minimise $\quad z = 2x_1 + 3x_2$

subject to $\quad x_1 + 2x_2 \geq 12$

$\qquad\qquad x_1 + x_2 \geq 8$

$\qquad\qquad x_1 + 10x_2 \leq 20$

and $\qquad\quad x_1 \geq 0, \ x_2 \geq 0$

We note that the third constraint is a $\leq$ type inequality. If we multiply both sides by -1, this can be changed to a $\geq$ type inequality. Thus the above problem can be rewritten as

Minimise $\quad z = 2x_1 + 3x_2$

subject to $\quad x_1 + 2x_2 \geq 12$

$$x_1 + x_2 \geq 8$$

$$-x_1 - 10x_2 \geq -20$$

and $\quad x_1 \geq 0, \; x_2 \geq 0$

We note that this is a standard minimisation problem. The augmented matrix of the problem is

$$A = \begin{bmatrix} 1 & 2 & 12 \\ 1 & 1 & 8 \\ -1 & -10 & -20 \\ 2 & 3 & z \end{bmatrix} \quad \therefore \; A' = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & -10 & 3 \\ 12 & 8 & -20 & z \end{bmatrix}$$

Thus the dual problem can be written as

Maximise $\quad z = 12y_1 + 8y_2 - 20y_3$

subject to $\quad y_1 + y_2 - y_3 \leq 2$

$$2y_1 + y_2 - 10y_3 \leq 3$$

and $\quad y_1, y_2, y_3 \geq 0$

Using slack variables s_1 and s_2, we can write the simplex tableau as

Tableau I

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
s_1	1	1	−1	1	0	0	2
s_2	②	1	−10	0	1	0	3
z	−12	−8	20	0	0	1	0

Tableau II

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
s_1	0	$\frac{1}{2}$	④	1	$-\frac{1}{2}$	0	$\frac{1}{2}$
y_1	1	$\frac{1}{2}$	−5	0	$\frac{1}{2}$	0	$\frac{3}{2}$
z	0	−2	−40	0	6	1	18

Note that we consider only positive entries in the pivot column for the selection of a pivot row.

Tableau III

Basis	y_1	y_2	y_3	s_1	s_2	z	VVB
y_3	0	$\frac{1}{8}$	1	$\frac{1}{4}$	$-\frac{1}{8}$	0	$\frac{1}{8}$
y_1	1	$\frac{9}{8}$	0	$\frac{5}{4}$	$-\frac{1}{8}$	0	$\frac{17}{8}$
z	0	3	·0	10	1	23	23

Since no entry in the last row is negative, optimum is reached. From the above tableau, the solution to the primal problem is: $x_1 = 10$, $x_2 = 1$ and min. $z = 23$.

11.6 ECONOMIC INTERPRETATION OF THE DUAL

Let us assume that the primal problem is a standard profit maximisation problem. In terms of notations used in § 11.4, we assume that the firm wants to determine the level of x_j's so that its profit z is maximised. We may recall that c_j's denote the profit per unit of x_j and b_i's the quantities of scarce inputs (capacities). Further, a_{ij} denotes the quantity of ith input required to produce a unit of jth commodity.

The objective function of the dual is given by

Minimise $z = b_1 y_1 + b_2 y_2 + \ldots\ldots + b_m y_m$.

Since z of the primal represents total profit of the firm expressed in rupees, therefore, z of the dual will also represent rupees. We note that the coefficient of y_i's in the objective function of the dual denote quantities of inputs. Therefore, y_i's should represent the some kind of rates like rupees per unit of input which indeed are the *shadow prices of the inputs*.

A shadow price gives change in the value of the objective function per unit change of the associated constraint. For example, the shadow price of an input in a profit maximisation problem gives the increase in total profit as a result of unit increase in the use of that input. Shadow prices, thus provides the marginal imputed value or worth of each input to the firm. If a particular input is not fully employed, its shadow price is zero because increasing this input would leave profits unchanged. A firm should attempt to expand its use of an input as long as the input's marginal value or shadow price exceeds the cost of hiring its additional unit.

Further, the constraints of the dual are given by:

$$a_{11} y_1 + a_{21} y_2 + \ldots\ldots + a_{m1} y_m \geq c_1$$
$$a_{21} y_1 + a_{22} y_2 + \ldots\ldots + a_{m2} y_m \geq c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{1n} y_1 + a_{2n} y_2 + \ldots\ldots + a_{mn} y_m \geq c_n$$

We know that the coefficients a_{ij} denote the quantity of the ith input used in producing a unit of jth product. Thus the left-hand side of the first constraint represents the total cost of producing a unit of x_1. The term on the right hand side of the first constraint c_1 represents the contribution to profit by a unit of x_1. Thus, the first constraint implies that the cost of producing a unit of x_1 be at least as large as c_1 in optimum solution.

This indicates that if the cost of production of x_1 is greater than c_1 (i.e., strict inequality holds) in optimum solution, the allocation of resources to production of x_1 is certainly non-optimal implying there by that x_1 should not be produced at all and the related resources should be used elsewhere.

As opposed to this, if cost of production of x_1 equals c_1 (i.e., equality holds in the first constraint) in optimum solution, x_1 is actually produced.

A similar interpretation can be given to other constraints.

Remarks:

1. The shadow prices are useful in decentralized decision making to ensure optimal use of scarce resources. A firm may have a number of plants using some common facilities. Obviously, if one plant manager uses more of a facility, less of it will be available to others. So imputing of appropriate shadow prices will help. Shadow prices are also of great help in optimising resource use in economic planning, taking decision about locational choice of a project etc.

2. Duals serve a significant practical purpose as well. In some situations, a primal may involve a very large number of constraints. Hence, solving it in its original form might be very cumbersome. However, if one constructs its dual, the number of constraints may come down drastically and the problem can be easily solved.

Example: A firm manufactures two product A and B on which profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M_1 and M_2.

The product A requires one minute of processing time on machine M_1 and two minutes on M_2 while B requires one minute each on M_1 and M_2. Machine M_1 is available for use for not more than 7 hours and 30 minutes, while M_2 is available for 10 hours during any working day.

(i) Formulate the above as a linear programming problem to find the number of units of products A and B to be manufactured to get the maximum profit.

(ii) Find the optimal values of the decision variables and the objective function.

(iii) Write the dual of the above problem and its solution.

(iv) Interpret the decision variables of the dual.

Solution:

Let x_1 and x_2 denote the respective number of units of product A and B to be manufactured.

(i) The linear programming formulation of the given problem is:

Maximise $\quad z = 3x_1 + 4x_2$

subject to $\quad x_1 + x_2 \le 450$

$\qquad\qquad 2x_1 + x_2 \le 600$

and $\qquad x_1, x_2 \ge 0$

(ii) Using slack variables, we can write:

Tableau I

Basis	x_1	x_2	s_1	s_2	z	VVB
s_1	1	①	1	0	0	450
s_2	2	1	0	1	0	600
z	−3	−4	0	0	1	0

Tableau II

Basis	x_1	x_2	s_1	s_2	z	VVB
x_2	1	1	1	0	0	450
s_2	1	0	−1	1	0	150
z	1	0	4	0	1	1800

Since no entry in the last row is negative, we have reached an optimal solution with $x_1 = 0, x_2 = 450$ and max. profit = ₹ 1,800.

(iii) The augmented matrix of the primal problem is:

$$A = \begin{bmatrix} 1 & 1 & 450 \\ 2 & 1 & 600 \\ 3 & 4 & z \end{bmatrix} \quad \therefore \quad A' = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \\ 450 & 600 & z \end{bmatrix}$$

Let the decision variable of the dual be denoted by y_1 and y_2.

The dual problem can be written as:

Minimise $z = 450y_1 + 600y_2$

subject to $y_1 + 2y_2 \geq 3$

$y_1 + y_2 \geq 4$

and $y_1, y_2 \geq 0$.

From tableau II, the solution to the dual can be written as $y_1 = 4$, $y_2 = 0$ with min. $z = ₹ 1,800$.

(iv) The values of y_1 and y_2 can be interpreted as the opportunity costs or the shadow prices of using the two machine's time in manufacturing. We may note that the shadow price of M_1's time is ₹ 4. This implies that an additional minute used on M_1 will increase the optimal profit by ₹ 4. We note that this resource is fully utilised i.e. $s_1 = 0$ in the optimal solution. Further, the shadow price of M_2's time is zero. This indicates that optimal profit cannot be increased by using an additional minute on M_2. We note that s_2 is not zero i.e. M_2's time is not fully utilised in the optimal solution.

Remarks:

1. The shadow price of a resource is positive for a constraint that is satisfied as equality in the optimal solution i.e., no slack exists. Similarly a zero shadow price of a resource indicates that the constraint is not satisfied as equality in the optimal solution i.e., slack exists.

2. The cost of production of a unit of x_1 is 4 $\left(=y_1 + 2y_2 \right) > 3$ (the contribution to profit by a unit of x_1). Therefore, production of x_1 is zero in the above example.

Example: A company produces 2 types of leather belts — Type A and B. The contribution to profit per belt is ₹ 4 for type A and ₹ 3 for type B. The time requirement of one belt of type A and type B are in the ratio of 2 : 3. Time available is sufficient to produce 500 belts of type A. The leather is sufficient for only 400 belts. Belt A requires a fancy buckle and only 200 fancy buckles are available.

(i) Formulate the above as a linear programming problem.

(ii) Solve it by simplex method and comment upon the optimum solution.

(iii) Write it's dual and read its optimal solution from the tableau of the optimal solution of the primal.

Solution:

Let x_1 belts be of type A and x_2 belts be of type B. Then the objective function is $z = 4x_1 + 3x_2$.

If time required to produce, a belt is 2 units then total time available for the production of belts is $500 \times 2 = 1000$ units.

$\therefore$ We have $2x_1 + 3x_2 \leq 1000$

Further $x_1 + x_2 \leq 400$ and $x_1 \leq 200$ $x_1, x_2 \geq 0$

(i) The linear programming problem is:

maximise $z = 4x_1 + 3x_2$

subject to $2x_1 + 3x_2 \le 1000$

$x_1 + x_2 \le 400$

$x_1 \le 200$

and $x_1, x_2 \ge 0$

(ii) Using slack variables, we can write Tableau 1 as

· Tableau I

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
s_1	2	3	1	0	0	0	1000
s_2	1	1	0	1	0	0	400
s_3	①	0	0	0	1	0	200
z	−4	−3	0	0	0	1	0

Tableau II

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
s_1	0	3	1	0	−2	0	600
s_2	0	①	0	1	−1	0	200
x_1	1	0	0	0	1	0	200
z	0	−3	0	0	4	1	800

Since the displacement ratios $\dfrac{600}{3}$ and $\dfrac{200}{1}$ are equal, the linear program is said to be *degenerate* and we can select either s_1 or s_2 as the outgoing variable. Selecting s_2 as the outgoing variable, we get the following tableau.

Tableau III

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
s_1	0	0	1	−3	1	0	0
x_2	0	1	0	1	−1	0	200
x_1	1	0	0	0	1	0	200
z	0	0	0	3	1	1	1400

Since no entry in the last row is negative, we have reached an optimum solution with $x_1 = x_2 = 200$ and the maximum profit = ₹ 1400. The solution indicates that the company should produce 200 belts of each type. Since the value of one of the variables in the basis of tableau III i.e., s_1 is zero, this indicates that the columns of the coefficient matrix are not linearly independent. We note that

$$200\begin{bmatrix}0\\0\\1\end{bmatrix} + 200\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\200\\200\end{bmatrix}$$ i.e., the last column is a linear combination of first and

second columns. This indicates the existence of more than one possible solutions.

(iii) Let y_1, y_2 and y_3 be the decision variables of the dual.

∴   The dual problem can be written as

Minimise $z = 1000y_1 + 400y_2 + 200y_3$

subject to $2y_1 + y_2 + y_3 \geq 4$

$3y_1 + y_2 \geq 3$

and $\qquad y_1, y_2, y_3 \geq 0$

From Tableau III, we have $y_1 = 0$, $y_2 = 3$ and $y_3 = 1$

Remarks:

1. Since there are more than one possible solution to the given problem, accordingly there will be more than one possible solutions to the dual.

2. When all the columns of the coefficient matrix of a linear programming problem are not linearly independent then a zero shadow price of a resource need not imply that the slack exists in the optimum solution. This is obvious from tableau III where shadow price of first resource is zero but $s_1 = 0$ i.e., the constraints is satisfied as equality in the optimal solution.

11.7 LINEAR PROGRAMMING PROBLEM NOT IN STANDARD FORM

If a linear programming problem is neither of a standard maximisation type nor of a standard minimisation type, then it has to be modified so that the simplex method is applicable.

11.7.1 Use of Artificial Variables

The use of artificial variables for the modification of a non-standard linear programming problem for the maximisation of an objective function is explained through the following example:

Example:

Maximise $\quad z = 3x_1 + 4x_2$

subject to $\quad x_1 + 2x_2 \geq 6$

$3x_1 - x_2 \geq -5$

$5x_1 - x_2 \leq 7$ and $x_1, x_2 \geq 0$.

Solution:

As a first step, we have to ensure that the constants appearing on the right hand side of the constraints are positive. We note that the second constraint has -5 on its right hand side. Therefore, we multiply both sides of the constraint by -1 to make its right hand side positive. Thus the constraints can now be written as

$x_1 + 2x_2 \geq 6$

$-3x_1 + x_2 \leq 5$

$5x_1 - x_2 \leq 7$

Add a slack variable to the left-hand side of each constraint with $\leq$ type inequality. Similarly subtract a surplus variable from the left-hand side of each constraint with $\geq$ type inequality. Thus we have:

$x_1 + 2x_2 - s_1 = 6$

$-3x_1 + x_2 + s_2 = 5$

$5x_1 - x_2 + s_3 = 7$

Here s_1 is a surplus variable while s_2 and s_3 are slack variables.

We recall that when the maximisation problem is in standard form, we always have an identity matrix consisting of the columns, excluding the last row, corresponding to the slack variables in the initial tableau. Thus, we begin with an initial feasible solution in which all the decision variables are equal to zero.

Contrary to this, in the above problem, the matrix of the columns corresponding to s_1, s_2 and s_3 is not an identity matrix. Therefore, we do not have an initial feasible solution with all the decision variables equal to zero and hence the simplex method is not directly applicable. Thus to use simplex method, the problem needs to be modified.

The main role of modification is to provide an initial identity matrix consisting of columns corresponding to those variables which are not decision variables of the problem. We note that, for the above example, this can be done if we add an artificial variable to the first constraint. In order that this variable doesn't affect the result of the problem, we attach a very large negative number as its coefficient in the objective function. We note that the coefficients of variables in the objective function of a maximisation problem often represent marginal profit or marginal revenue etc.; negative coefficient implies marginal loss etc., and hence, variables with negative coefficients can never be included in the optimal solution. After the inclusion of an artificial variable in the first constraint, the above problem can be rewritten as:

Maximise $z = 3x_1 + 4x_2 + 0.s_1 + 0.s_2 + 0.s_3 - M.A_1$

(where M is very large positive number and A_1 denotes an artificial variable)

$$\text{subject to} \qquad x_1 + 2x_2 - s_1 + A_1 = 6$$

$$-3x_1 + x_2 + s_2 = 5$$

$$5x_1 - x_2 + s_3 = 7$$

$$\text{and} \qquad x_1, x_2, s_1, s_2, s_3, A_1 \geq 0$$

The initial simplex tableau can be written as:

Tableau I

Basis	x_1	x_2	s_1	s_2	s_3	A_1	z	VVB
A_1	1	2	−1	0	0	1	0	6
s_2	−3	1	0	1	0	0	0	5
s_3	5	−1	0	0	1	0	0	7
z	−3	−4	0	0	0	M	1	0

Note that the basic feasible solution implied by the above tableau is $x_1 = x_2 = s_1 = 0$, $A_1 = 6$, $s_2 = 5$ and $s_3 = 7$.

The value of z corresponding to this solution $= -6M$, which is different from the entry in the lower right corner of the above tableau, written as 0. In order to replace this 0 by $-6M$, we make a transformation $R_4 = R_4 - MR_1$. Thus we get:

Tableau II

Basis	x_1	x_2	s_1	s_2	s_3	A_1	z	VVB
A_1	1	②	−1	0	0	1	0	6
s_2	−3	1	0	1	0	0	0	5
s_3	5	−1	0	0	1	0	0	7
z	$-3-M$	$-4-2M$	M	0	0	0	1	$-6M$

After this tableau we can apply the usual simplex method.

Tableau III

Basis	x_1	x_2	s_1	s_2	s_3	A_1	z	VVB
x_2	$\dfrac{1}{2}$	1	$-\dfrac{1}{2}$	0	0	$\dfrac{1}{2}$	0	3
s_2	$-\dfrac{7}{2}$	0	$\left(\dfrac{1}{2}\right)$	1	0	$-\dfrac{1}{2}$	0	2
s_3	$\dfrac{11}{2}$	0	$-\dfrac{1}{2}$	0	1	$\dfrac{1}{2}$	0	10
z	-1	0	-2	0	0	$2+M$	1	12

Tableau IV

Basis	x_1	x_2	s_1	s_2	s_3	A_1	z	VVB
x_2	-3	1	0	1	0	0	0	5
s_1	-7	0	1	2	0	-1	0	4
s_3	(2)	0	0	1	1	0	0	12
z	-15	0	0	4	0	M	1	20

Tableau V

Basis	x_1	x_2	s_1	s_2	s_3	A_1	z	VVB
x_2	0	1	0	$\dfrac{5}{2}$	$\dfrac{3}{2}$	0	0	23
s_1	0	0	1	$\dfrac{11}{2}$	$\dfrac{7}{2}$	-1	0	46
x_1	1	0	0	$\dfrac{1}{2}$	$\dfrac{1}{2}$	0	0	6
z	0	0	0	$\dfrac{23}{2}$	$\dfrac{15}{2}$	M	1	110

Since there is no negative entry in the last row, optimum is reached. Hence, the optimum solution is $x_1 = 6, x_2 = 23$ and max. $z = 110$.

Example:

Minimise $z = 4x_1 + 2x_2$

subject to $3x_1 + x_2 \geq 10$

$x_1 + x_2 = 6$

$2x_1 + x_2 \leq 10$

and $x_1, x_2 \geq 0.$

Solution:

Since minimisation of z is same as the maximisation of $-z$, the above problem can be restated as:

Maximise $P = -z = -4x_1 - 2x_2$

subject to $3x_1 + x_2 \geq 10$

$x_1 + x_2 = 6$

$2x_1 + x_2 \leq 10$

and $x_1, x_2 \geq 0.$

Subtracting a surplus variable s_1 and adding an artificial variable A_1 to the first constraint, we get $3x_1 + x_2 - s_1 + A_1 = 10$. Adding only an artificial variable A_2 to the second constraint, we get $x_1 + x_2 + A_2 = 6$. Further, adding a slack variable s_3 to the third constraint, we get $2x_1 + x_2 + s_3 = 10$. We note that $x_1, x_2, s_1, s_3, A_1, A_2 \geq 0$. In view of the slack, surplus and artificial variables, the objective function becomes:

$$P = -4x_1 - 2x_2 + 0.s_1 - M.A_1 - M.A_2 + 0.s_3$$

Thus we have

Tableau I

Basis	x_1	x_2	s_1	A_1	A_2	s_3	P	VVB
A_1	3	1	−1	1	0	0	0	10
A_2	1	1	0	0	1	0	0	6
s_3	2	1	0	0	0	1	0	10
P	4	2	0	M	M	0	1	0

Here $A_1 = 10$, $A_2 = 6$, $s_3 = 10$, $x_1 = 0, x_2 = 0, s_1 = 0$ is the basic feasible solution and the associated value of the objective function is $-16M$. In order to bring this value in the lower right corner of the tableau, we multiply the sum of the first and second row by $-M$ and add to the last row.

Tableau II

Basis	x_1	x_2	s_1	A_1	A_2	s_3	P	VVB
A_1	③	1	−1	1	0	0	0	10
A_2	1	1	0	0	1	0	0	6
s_3	2	1	0	0	0	1	0	10
P	$4-4M$	$2-2M$	M	0	0	0	1	$-16M$

Tableau III

Basis	x_1	x_2	s_1	A_1	A_2	s_3	P	VVB
x_1	1	$\dfrac{1}{3}$	$-\dfrac{1}{3}$	$\dfrac{1}{3}$	0	0	0	$\dfrac{10}{3}$
A_2	0	$\left(\dfrac{2}{3}\right)$	$\dfrac{1}{3}$	$-\dfrac{1}{3}$	1	0	0	$\dfrac{8}{3}$
s_3	0	$\dfrac{1}{3}$	$\dfrac{2}{3}$	$-\dfrac{2}{3}$	0	1	0	$\dfrac{10}{3}$
P	0	$\left(\dfrac{2}{3}-\dfrac{2}{3}M\right)$	$\left(\dfrac{4}{3}-\dfrac{1}{3}M\right)$	$\left(-\dfrac{4}{3}+\dfrac{4}{3}M\right)$	0	0	1	$-\dfrac{40}{3}-\dfrac{8}{3}M$

Tableau IV

Basis	x_1	x_2	s_1	A_1	A_2	s_3	P	VVB
x_1	1	0	$-\dfrac{1}{2}$	$\dfrac{1}{2}$	$-\dfrac{1}{2}$	0	0	2
x_2	0	1	$\dfrac{1}{2}$	$-\dfrac{1}{2}$	$\dfrac{3}{2}$	0	0	4
s_3	0	0	$\dfrac{1}{2}$	$-\dfrac{1}{2}$	$-\dfrac{1}{2}$	1	0	2
P	0	0	1	$(M-1)$	$(M-1)$	0	1	−16

Since no entry, excluding the last, in the last row is negative, the optimal solution is reached. Thus $x_1 = 2$ and $x_2 = 4$ and the minimised $z = 16$.

11.7.2 Two-phase Method of Solution

In the two-phase method, all the constraints are written as $\le$ type inequalities. If a constraint is given as an equation it is replaced by two inequalities. For example, the constraint $x_1 + x_2 = 5$ will be replaced by the inequalities $x_1 + x_2 \le 5$ and $x_1 + x_2 \ge 5$. Finally, these constraints are written as equations by using slack variables.

Phase I

This phase is applicable if the tableau, at any stage, consists of one or more negative entries in its last column, excluding the element in the last row. The row having the largest negative entry in last column of the tableau is taken as the pivot row. Pivot column is the column, excluding the last, having the largest negative entry in the pivot row. We note that *if all the entries in the pivot row are non-negative, the problem has no solution.*

Phase II

This phase is applicable if the tableau, at any stage, consists of all non-negative entries in its last column. The pivoting is done by the method used for standard maximisation problems.

We shall attempt, examples by using two-phase method.

$$\text{Maximise} \quad z = 3x_1 + 4x_2$$

$$\text{subject to} \quad x_1 + 2x_2 \ge 6$$

$$3x_1 - x_2 \ge -5$$

$$5x_1 - x_2 \le 7$$

$$\text{and} \quad x_1, x_2 \ge 0$$

Writing all the constraints as $\le$ type inequalities and adding positive slack variables, we get,

$$-x_1 - 2x_2 + s_1 = -6$$

$$-3x_1 + x_2 + s_2 = 5$$

$$5x_1 - x_2 + s_3 = 7$$

Thus we have,

Tableau I

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
s_1	−1	(−2)	1	0	0	0	−6
s_2	−3	1	0	1	0	0	5
s_3	5	−1	0	0	1	0	7
z	−3	−4	0	0	0	1	0

We note that the first entry in the last column is negative, therefore, phase I is applicable and pivot element is −2.

Tableau II

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
x_2	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	0	3
s_2	$-\frac{7}{2}$	0	$\left(\frac{1}{2}\right)$	1	0	0	2
s_3	$\frac{11}{2}$	0	$-\frac{1}{2}$	0	1	0	10
z	-1	0	-2	0	0	1	12

Phase II is now applicable.

Tableau III

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
x_2	-3	1	0	1	0	0	5
s_1	-7	0	1	2	0	0	4
s_3	(2)	0	0	1	1	0	12
z	-15	0	0	4	0	1	20

Tableau IV

Basis	x_1	x_2	s_1	s_2	s_3	z	VVB
x_2	0	1	0	$\frac{5}{2}$	$\frac{3}{2}$	0	23
s_1	0	0	1	$\frac{11}{2}$	$\frac{7}{2}$	0	46
x_1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	6
z	0	0	0	$\frac{23}{2}$	$\frac{15}{2}$	1	110

Thus the optimal solution is $x_1 = 6$, $x_2 = 23$ and max. $z = 110$.

Example is a minimisation problem. This can, alternatively be written as

Maximise $P = -4x_1 - 2x_2$

subject to $-3x_1 - x_2 \le -10$

$x_1 + x_2 \le 6$

$-x_1 - x_2 \le -6$

$2x_1 + x_2 \le 10$

Adding the slack variables s_1, s_2, s_3, s_4, the initial tableau can be written as:

Tableau I

Basis	x_1	x_2	s_1	s_2	s_3	s_4	P	VVB
s_1	(-3)	-1	1	0	0	0	0	-10
s_2	1	1	0	1	0	0	0	6
s_3	-1	-1	0	0	1	0	0	-6
s_4	2	1	0	0	0	1	0	10
P	4	2	0	0	0	0	1	0

Since two entries in the last column are negative, Phase I is applicable.

Tableau II

Basis	x_1	x_2	s_1	s_2	s_3	s_4	P	VVB
x_1	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	$\frac{10}{3}$
s_2	0	$\frac{2}{3}$	$\frac{1}{3}$	1	0	0	0	$\frac{8}{3}$
s_3	0	$\left(-\frac{2}{3}\right)$	$-\frac{1}{3}$	0	1	0	0	$-\frac{8}{3}$
s_4	0	$\frac{1}{3}$	$\frac{2}{3}$	0	0	1	0	$\frac{10}{3}$
P	0	$\frac{2}{3}$	$\frac{4}{3}$	0	0	0	1	$-\frac{40}{3}$

Phase I is again applicable to Tableau II

Tableau III

Basis	x_1	x_2	s_1	s_2	s_3	s_4	P	VVB
x_1	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	2
s_2	0	0	0	1	1	0	0	0
x_2	0	1	$\frac{1}{2}$	0	$-\frac{3}{2}$	0	0	4
s_4	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	2
P	0	0	1	0	1	0	1	-16

Thus the optimum solution is $x_1 = 2$, $x_2 = 4$. Also the maximised $P = -16$ or equivalently minimised $z = 16$.

Example: The standard weight of a special purpose brick is 5 kg and it contains two basic ingredients B_1 and B_2. B_1 costs ₹ 5 per kg and B_2 costs ₹ 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of B_1 and minimum 2 kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find the minimum cost of the brick satisfying the above conditions.

Solution:

Let x_1 kg of B_1 be mixed with x_2 kg of B_2. Then the above problem can be formulated as:

Minimise $z = 5x_1 + 8x_2$

subject to $x_1 + x_2 = 5$

$x_1 \leq 4$, $x_2 \geq 2$ and $x_1, x_2 \geq 0$

To apply two-phase method, we rewrite the above problem as:

Maximise $P = -5x_1 - 8x_2$

subject to $x_1 + x_2 \leq 5$

$x_1 + x_2 \geq 5$ or $-x_1 - x_2 \leq -5$

$x_1 \leq 4$, $-x_2 \leq -2$

Adding slack variables, the constraints can be written as:

$$x_1 + x_2 + s_1 = 5$$

$$-x_1 - x_2 + s_2 = -5$$

$$x_1 + s_3 = 4$$

$$-x_2 + s_4 = -2$$

Tableau I

Basis	x_1	x_2	s_1	s_2	s_3	s_4	P	VVB
s_1	1	1	1	0	0	0	0	5
s_2	-1	(-1)	0	1	0	0	0	-5
s_3	1	0	0	0	1	0	0	4
s_4	0	-1	0	0	0	1	0	-2
P	5	8	0	0	0	0	1	0

Tableau II

Basis	x_1	x_2	s_1	s_2	s_3	s_4	P	VVB
s_1	0	0	1	1	0	0	0	10
x_2	1	1	0	-1	0	0	0	5
s_3	1	0	0	0	1	0	0	4
s_4	(1)	0	0	0	0	1	0	3
P	-3	0	0	8	0	0	1	-40

Tableau III

Basis	x_1	x_2	s_1	s_2	s_3	s_4	P	VVB
s_1	0	0	1	1	0	0	0	10
x_2	0	1	0	-1	0	-1	0	2
s_3	0	0	0	0	1	-1	0	1
x_1	1	0	0	0	0	1	0	3
P	0	0	0	8	0	0	1	-31

Thus the optimal solution is $x_1 = 3$, $x_2 = 2$ and the minimised value of: objective function is 31.

Check Your Progress

Fill in the blanks:

1. The coefficients of decision variables in the constraints are called technical or _____

2. Out of the set of feasible solutions, the one that optimises the objective function is called an _____ solution.

3. A feasible region in which the objective function can be increased indefinitely, is called an _____ feasible region.

4. _____ provides a technique to find the corners of a feasible region and the associated value of the objective function.

11.8 LET US SUM UP

- Formulation of linear programming is the representation of problem situation in a mathematical form. It involves well defined decision variables, with an objective function and set of constraints.

- The objective of the problem is identified and converted into suitable objective function. The objective function represents the aim or goal of the system (i.e., decision variables) which has to be determined from the problem. Generally, the objective in most cases will be either to maximize resources or profits or to minimize the cost or time.

- When the availability of resources is in surplus, there will be no problem in making decisions. But in real life, organizations normally have scarce resources within which the job has to be performed in the most effective way. Therefore, problem situations are within confined limits in which the optimal solution to the problem must be found.

- Once the linear programming problem has been formulated on the basis of given objective and constraints, then, we have to solve the problem and obtain the best possible or optimal solution for the problem. There are two methods of solving linear programming problems: graphical method and simplex method.

11.9 LESSON END ACTIVITY

Two sums of money which differ by ₹ 2000 are lent out, the larger at 4% and smaller at 6% p.a. simple interest. If the difference between the amounts is ₹ 1840 at the end of 2 years, find the sum lent at each rate.

11.10 KEYWORDS

Divisibility: It is assumed that resources and products can be divided into fractions. In case of fractions are not possible, like production of one-third of a computer, a modification of linear programming called integer programming can be used.

Homogeneity: The products, workers' efficiency, and machines are assumed to be identical.

Linearity: Increase in labour input will have a proportionate increase in output.

Objective: Refers to the aim to optimize (maximize the profits or minimize the costs.)

Optimal Solution: Alternative or approach that best fits the situation, employs resources in a most effective and efficient manner, and yields the highest possible return under the circumstances. Any tinkering with an optimum decision makes it only worse.

Variables: Variables used in statistics can be divided into three types: "dependent variable", "independent variable", or other.

Linear Programming: It is a mathematical technique for maximizing or minimizing a linear function of several variables, such as output or cost.

11.11 QUESTIONS FOR DISCUSSION

1. What do you understand by a linear programming problem? Give a brief description of the problem with illustrations.

2. Explain how would you identify the cases of redundant constraint, no solution, multiple solutions and an unbounded solution from the graph of the linear

programming problems involving two variables? Give a rough sketch of each case.

3. Explain the concept of slack variables and dual of a given linear programming problem from the business point of view.

4. (i) What are the three component of a linear programming problem?

   (ii) Give a brief description of the solution of a linear programming problem using graphical method.

5. Mark the feasible region represented by the following constraints:

$$x_1 + x_2 \le 1, \ 3x_1 + x_2 \le 3, \ x_1 \ge 0, x_2 \ge 0$$

6. Given the restrictions,

$$x \ge 0, y \ge 0$$

$$2x + y \le 20$$

$$x + 2y \le 20$$

6. Indicate the feasible region on a graph, and maximise the function: $z = x + 3y$.

7. Indicate on a graph paper, the region satisfying the constraints,

$$12x + 12y \le 840$$

$$3x + 6y \le 300$$

$$8x + 4y \le 480$$

and the non-negativity conditions: $x \ge 0, y \ge 0$.

Under the above conditions, maximise the function $5x + 7y$.

8. Indicate the feasible region of the following linear programming problem on a graph paper and minimise $z = 2x + y$, subject to the constraints $5x + 10y \le 50$, $x + y \ge 1$, $y \le 4$ and $x \ge 0, y \ge 0$.

9. Maximise $z = 10x + 6y$ subject to $3x + 2y \le 12$, $2x + 5y \le 34$ and $x \ge 0, y \ge 0$, by graphical method.

10. Show the solution zone of the following inequalities on a graph:

$$5x + y \ge 10$$

$$x + y \ge 6$$

$$x + 4y \ge 12 \text{ and } x \ge 0, y \ge 0.$$

Find x and y for which $3x + 2y$ is minimum, subject to these inequalities. Use graphic method.

11. Minimise $z = 8x + 6y$ subject to $x + y \ge 10$, $5x + 3y \ge 20$, $2x + 3y \ge 15$ and $x \ge 0, y \ge 0$. Use graphical method.

12. Find, graphically, the minimum of $z = 16x + 25y$

subject to $2x + y \ge 7$, $x + y \ge 5$, $2x + 5y \ge 16$

and $x \ge 0, y \ge 0$.

13. Use simplex method to find the solution of the following linear programming problems:

(i) Maximise $u = 8x + 2y + 3z$

subject to

$8x + 10y - 9z \le 80$

$30x - 25y + 60z \le 300$

$-22x + 35y + 56z \le 280$

and $x \ge 0, y \ge 0, z \ge 0$

(ii) Maximise $z = 5x_1 + x_2 + x_3$

subject to

$x_1 + x_2 + x_3 \le 66$

$-x_1 + 3x_2 + 2x_3 \le 72$

$2x_1 + x_2 + 3x_3 \le 108$

and $x_1 \ge 0, x_1 \ge 0 \ x_1 \ge 0$

(iii) Minimise $z = 4x_1 + 2x_2 + 3x_3$

subject to

$x_1 + 2x_2 + x_3 \ge 3$

$x_1 + x_2 - x_3 \ge 6$

$x_1 + 2x_2 + x_3 \ge 3$

and $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

(iv) Minimise $u = x + y + 2z$

subject to

$4x + y - z \ge 2$

$3x - y - z \ge -24$

$-x - y + z \ge 8$

and $x \ge 0, y \ge 0, z \ge 0$.

Hint: Attempt its dual.

14. Explain the relationship between a primal and a dual in a linear programming problem. Also explain the economic interpretation of slack variables used. Obtain the dual of the following and solve it by simplex method:

Minimise $c = 3x + 4y$ subject to $5x + 10y \ge 0$, $15x + 10y \ge 1200$, $x \ge 0$ and $y \ge 0$.

15. A machine producing either product A or B can produce A by using 2 units of chemicals and 1 unit of a compound and can produce B by using 1 unit of chemicals and 2 units of compound. Only 800 units of chemicals and 1,000 units of compound are available. The profits available per unit of A and B are ₹ 30 and ₹ 40 respectively. Draw a suitable diagram to show the feasible region. Also, find the optimum allocation of units between A and B to maximise profits. Find the maximum profit.

16. A firm makes two types of furniture: chairs and tables. Profits are ₹ 20 per chair and ₹ 30 per table. Both products are processed on three machines M_1, M_2 and M_3. The time required for each product in hours and total time available in hours on each machine are as follows:

Machine	Chair	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

(i) Formulate the above as a linear programming problem to maximise profits;

(ii) Write its dual; and

(iii) Solve the primal problem by simplex method.

17. (i) Following data are available for a firm which manufactures three items A, B and C:

   (a) Express the data in the form of a linear programming problem.

   (b) Solve it by simplex method.

Time Required (hours)			
Product	Assembly	Finishing	Profit (in ₹)
A	10	2	80
B	4	5	60
C	5	4	30
Firm's Capacity	2,000	1,009	

   (ii) Show graphically, a situation when a linear programming problem has no solution.

18. Two materials A and B are required to construct tables and bookcases. For one table 12 units of A and 16 units of B are needed, while for a book case 16 units of A and 8 units of B are required. The profit is ₹ 25 on a bookcase and ₹ 20 on a table. 100 units of material A and 80 units of material B are available. How many bookcases and tables should be produced to have maximum profit?

   (i) Formulate this as a linear programming problem.

   (ii) Solve it by simplex method.

   (iii) Write the dual for it.

19. A carpenter has 90, 80 and 50 running feet respectively of teak, plywood and rosewood. The product A requires 2, 1 and 1 running feet and the product B requires 1, 2 and 1 running feet of teak, plywood and rosewood respectively. If A would sell for ₹ 48, and B would sell for ₹ 40 per unit, how much of each should he make and sell in order to obtain the maximum gross income out of his stock of wood?

   (i) Give a mathematical formulation to this linear programming problem.

   (ii) Solve it by simplex method.

20. A company produces two types of goods A and B that require gold and silver. Each unit of type A require 3 gms of silver and 1 gm of gold while each unit of type B requires 1 gm silver and 2 gms of gold. The company can produce 9 gms of silver and 8 gms of gold. If each unit of type A brings a profit of ₹ 40 and that of type B brings a profit of ₹ 50, find the number of units of each type the company should produce to maximise profit.

21. A person consumes two types of food A and B every day to obtain 8 units of protein, 12 units of carbohydrates and 9 units of fats which is his daily minimum requirement. 1 kilo of food A contains 2, 6 and 1 unit of protein, carbohydrates and fats respectively. 1 kilo of food B contains 1, 1 and 3 unit of protein, carbohydrates and fats respectively. Food A costs ₹ 8.50 per kilo, while B costs ₹ 4 per kilo. Determine how many kilos of each food should he buy to minimise his cost and still meet the minimum requirements?

22. A retailer deals in two items only — item A and item B. He has ₹ 50,000 to invest and a space to store at the most 60 pieces. An item A costs him ₹ 2,500 and B costs him ₹ 500. A net profit to him on item A is ₹ 500 and on B is ₹ 150. If he can sell all the items that he purchases, how should he invest his amount to have maximum profit?

23. A firm manufacturing two types of electrical items, A and B, can make a profit of ₹ 20 per unit of A and ₹ 30 per unit of B. Both A and B make use of two essential components, a motor and a transformer. Each unit of A requires 3 motors and 2 transformers and each unit of B requires 2 motors and 4 transformers. Total supply of components per month is restricted to 210 motors and 300 transformers. Type B is an export model requiring a voltage stabilizer, restricted to 65 units per month. How many each of A and B should the firm manufacture per month in order to maximise profit?

24. A firm produces three items A, B and C at two plants X and Y. The number of items produced and operating cost per hour are as follows:

Plant	Items produced per hour			Operating cost per hour
	A	B	C	
X	30	60	40	₹ 1,200
Y	30	20	100	₹ 800

It is desired to produce at least 400 items of type A, at least 800 items of type B and at least 880 items of type C per day. Find, by simplex method and using the concept of dual, the number of hours each plant will run on a day to have minimum costs.

25. A firm produces three types of biscuits A, B and C. It packs them in assortments of two sizes I and II. The size I contains 20 biscuits of type A, 50 of type B and 10 of type C. The size II contain contains 10 biscuits of type A, 80 of type B and 60 of type C. A buyer intends to buy at least 120 biscuits of type A, 740 of type B and 240 of type C. Determine the least number of packets he should buy. Use the simplex method and the concept of dual.

26. Give a mathematical formulation of the following linear programming problem:

The standard weight of a special purpose of brick is 5 kg and it contains two basic ingredients B_1 and B_2. B_1 costs ₹ 5 per kg and B_2 costs ₹ 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of B_1 and at least 2 kg of B_2. Determine the amount of ingredients B_1 and B_2 so that the cost of brick may become minimum. Use graphical method.

27. A diet is to contain at least 20 ounces of protein and 15 ounces of carbohydrate. There are three foods A, B and C available in the market costing ₹ 2, ₹ 1 and ₹ 3 per unit respectively. Each unit of A contains 2 ounces of protein and 4 ounces of carbohydrate; each unit of B contains 3 ounces of protein and 2 ounces of carbohydrates; each unit of C contains 4 ounces of protein and 2 ounces of carbohydrate. Find by using simplex method, how many units of each food should the diet contain so that the cost per diet is minimum?

28. A company produces three products P_1, P_2 and P_3 from two raw materials A and B, and labor L. One unit of product P_1 requires 1 unit of A, 3 units of B and 2 units of L. One unit of product P_2 requires 2 units of A and B each, and 3 units of L, while one unit of product P_3 needs 2 units of A, 6 units of B and 4 units of L. The company has a daily availability of 8 units of A, 12 units of B and 12 units of L. It is further known that the unit contribution margin for the products is ₹ 3, 2 and 5 respectively for P_1, P_2 and P_3. Formulate this as a linear programming problem, and then solve it to determine the optimal product mix. Is the solution obtained unique? Identify an alternate optimal solution, if any. Also obtain the shadow prices of the resources.

29. A firm engaged in producing 2 models, A and B, performs three operations: Assembly, Painting and Testing. The relevant data are as follows:

	Sale Price	Hours required for each unit		
		Assembly	Painting	Testing
Model A	₹ 50	1.0	0.2	0.0
Model B	₹ 80	1.5	0.2	0.1

Total number of hours-available are: Assembly 600, Painting 100, and Testing 30. Determine the weekly production schedules to maximise revenue.

30. A confectioner manufactures two types of biscuits, one Regular Type and the other Non-Regular type. The Regular type sells at a profit of ₹ 40 per box and the Non-Regular type sells at a profit of ₹ 50 per box. The biscuits are processed in three main operations: Blending, Cooking and Packaging. The average time taken, in minutes for each box, for each of the processing operations is given below:

	Blending	Cooking	Packaging
Regulars	1 min.	5 min.	3 min.
Non-Regulars	2 min.	4 min.	1 min.

The Blending equipment is available for a maximum of 12 hours, The Cooking facilities for 30 hours at the most, and Packaging equipment for 15 hours. Determine how many boxes of each type the confectioner should manufacture to maximise profits.

31. Consider a firm that produces two products X_1 and X_2 with three fixed resources R_1, R_2 and R_3. The profit contribution is ₹ 20 per unit of X_1 and ₹ 25 per unit of X_2 sold. Production of a unit of X_1 requires 10 units of R_1 and 15 units of R_2; while that of X_2 requires 10 units of R_2 and 30 units of R_3. The resources R_1, R_2 and R_3. are available in 5000, 9000 and 12000 units respectively.

    (i) Find the profit maximising product mix.

    (ii) Find the shadow prices of the three resources.

    (iii) Are there any bottlenecks in the production process?

32. A drug manufacturer desires to obtain a production schedule to maximise the net revenue from the sale of two drugs (A and B) whose net revenues per unit are ₹ 60 and ₹ 40 respectively. The constraints are given below:

    (i) Capacity to produce drugs is 1,000 units of A and B together. (A and B are measured in same units).

    (ii) Capacity to produce containers is 1,600 units of A or 800 units of B or a weighted average of A and B.

    (iii) Labour is available for 800 units of A or 1,600 units of B or a weighted average of A and B.

    What product mix of A and B should he produce so as to maximise net revenue?

33. A business firm, producing two products I and II consists of three production departments: cutting, mixing and packaging. The equipment in each department can be used for 12 hours a day, thus, we shall regard 12 hours as the daily capacity in each department. The process of production can be summarised as follows:

    (i) Product I is first cut, then packaged. Each unit of this product uses up 1/2 hour of cutting capacity and 1/3 hour of packaging capacity.

(ii) Product II is first mixed, then packaged. Each unit of this product uses up 1 hour of mixing capacity and 2/3 hour of packaging capacity.

Products I and II can be sold at prices of ₹ 80 and 60 per unit respectively and the variable cost per unit for product I is ₹ 30 and for product II is ₹ 10. What should be the optimum product mix?

34. A firm assembles and sells two different types of out-board motors, A and B, using four resources. The production process can be described as follows:

Resource	Capacity per month
Motor unit shop resource	400 Type A units or 250 Type B units or any linear combination of the two
Type A gear and drive shop resource	175 Type A units
Type B gear and drive shop resource	225 Type b units
Final assembly resource	200 Type A units or 350 Type B units or any linear combination of the two

Type A unit brings in a profit of ₹ 90/- each and Type B unit, ₹ 60/- each. What should be the optimum product mix?

35. A company sells two different products A and B. The company makes a profit of ₹ 400 and ₹ 300 per unit on products A and B respectively. The two products are produced in a common production process and are sold in two different markets. The production process has a capacity of 30,000 man-hours. It takes 300 man-hours to produce a unit of A and 100 man-hours to produce a unit of B. The company estimates that the maximum number of units of A that can be sold is 80 and that of B is 120. Subject to these limitations, the product can be sold in any combination. Find the optimum product mix.

36. A plant manufactures two products, A and B. The products use the same fabricating and machining facilities but are assembled separately. Product A requires 30 minutes for fabricating, 48 minutes for machining and 70 minutes for assembly. Product B requires 24 minutes for fabricating, 60 minutes for machining and 90 minutes for assembly. The hours available for the processes are: fabricating - 1,000 hours; machining - 2,400 hours; A assembly - 4,125 hours; and B assembly - 3,450 hours. Each unit of A contributes ₹ 50 and each unit of B ₹ 75 to profit. Find the optimum production program.

37. A company produces two types of pens, say A and B. Pen A is of superior quality and Pen B is of inferior quality. Profits on pen A and B are ₹ 5 and ₹ 3 per pen respectively. Raw materials required for each pen A is twice as that for pen B. The supply of raw material is sufficient only for 1,000 pens of type B per day. Pen A requires a special clip and only 400 clips are available per day. For pen B, only 700 clips are available per day. Find the product mix so that the company can make maximum profit.

38. A manufacturer makes three types of decorative lamps: Model A, Model B and Model C. The cost of raw material for each model of lamp is same. However, the cost of production differs. Each model A lamp requires 0.1 hour of assembly time, 0.2 hour of wiring time and 0.1 hour of packaging time. Each model B lamp requires 0.2 hour of assembly time, 0.1 hour of wiring time and 0.1 hour of packaging time. Each model C lamp requires 0.2 hour of assembly time, 0.3 hour of wiring time and 0.1 hour of packaging time. The manufacturer makes a profit of ₹ 12 on each model A lamp, ₹ 19 on each model of B lamp and ₹ 21 on each

model of C lamp. The manufacturer can schedule up to 80 hours of assembly labour, 120 hours of wiring labour and 100 hours of packaging labour. Assuming that the market potential for model A and model C is unlimited and it is 30 units for model B, determine the optimal quantities of each model. Write the dual of the given problem and give its economic interpretation.

39. A manufacturing company produces two products, each of which requires stamping, assembly and painting operations. Total productive capacity by operation if it were devoted solely to one product or other is:

Productive Capacity (units per week)

Operation	Product A	Product B
Stamping	500	700
Assembly	650	350
Painting	450	700

Pro-rata allocation of productive capacity is permissible so that combinations of production of two products are permissible. Demand for the two products is unlimited and the profit on A and B are ₹ 150 and ₹ 120 respectively.

Find the optimal solution graphically. What is the amount of slack for each constraint?

40. The ABC company sells two type of porch furniture; gliders and chairs. It makes a profit of ₹ 100 on each glider and ₹ 40 on each chair. Each glider requires 40 square feet of display space and each chair requires 25 square feet of display space. It takes 5.5 hours to assemble a glider and 40 minutes to assemble a chair. ABC has 900 square feet of display space and 30 hours of labour available for assembly. The sales manager wants at least two chairs displayed for every displayed glider. Formulate and solve the linear programming model.

41. A firm has two grades of cashew nuts: Grade I — 750 kg and Grade II — 1200 kg. These are to be mixed in two types of packets of 1 kg each—economy and special. The economy pack consists of Grade I and Grade II in the proportion of 1 : 3, while the special pack combines the two in equal proportion. The contribution of the economy and the special packs are ₹ 5 and ₹ 8 per pack respectively.

(i) Formulate this as a linear programming problem to maximise contribution and solve it by simplex method.

(ii) Write the dual of the above problem, read the optimum solution of the dual and give its economic interpretation.

42. A manufacturing company makes three products, each of which requires three operations as a part of the manufacturing process. The company can sell all of the products it can manufacture but its production capability is limited by the capacity of its operation centres. Additional data concerning the company are as follows:

Product	Manufacturing Requirements (hours/unit)			Cost (₹)	Price (₹)
	Centre 1	Centre 2	Centre 3		
A	1	3	2	11	15
B	3	4	1	12	20
C	2	2	2	10	16
Hours Available	160	120	80		

What should the product mix be? Write the dual of the given problem. Give its economic interpretation and use it for checking the optimal solution of the given problem.

43. A city hospital has the following minimal daily requirements for nurses:

Period	Clock Time (24 hours Day)	Minimal Number of Nurses required
1	6 A.M. - 10 A.M.	2
2	10 A.M. - 2 P.M.	7
3	2 P.M. - 6 P.M.	15
4	6 P.M. - 10 P.M.	8
5	10 P.M. - 2 A.M.	20
6	2 A.M. - 6 A.M.	6

The nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital superintendent wants to determine the minimal number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this as a linear programming problem.

44. A 24-hour super market has the following minimal requirement for cashiers:

Period	:	1	2	3	4	5	6
Time (24 hrs. clock)	:	3-7	7-11	11-15	15-19	19-23	23-3
Minimum number	:	7	20	14	20	10	5

Period 1 follows immediately after period 6. A cashier works 8 consecutive hours, starting at the beginning of one of the six periods. Determine a daily employee work sheet, which satisfies the requirements with least number of personnel.

Formulate the problem as a linear programming problem.

45. The Megacity Power Company (MPC) wishes to schedule its work force for the coming month. Employees work an 8-hour shift beginning at 8 A.M., 12 A.M., 4 P.M., 8 P.M. and 12 A.M. All but the late shift (12 A.M. to 8 A.M.) overlap the next shift by 4 hours. Based on the past experience, the following number of workers will be needed during the times shown:

Hours	Workers Required
12 A.M.- 8 A.M.	6
8 A.M.-12 P.M.	10
12 P.M. - 4 P.M.	16
4 P.M.- 8 P.M.	18
8 P.M.-12 P.M.	8

How many workers should be scheduled to report for each of the five shifts if MPC wishes to minimise its total work force? Formulate this as a linear programming problem.

Check Your Progress: Model Answer

1. Structural coefficients
2. Optimal
3. Unbounded
4. Simplex method

11.12 SUGGESTED READINGS

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

D C Sanchethi and V K Kapoor. *Business Mathematics.*

Sivayya and Sathya Rao, *An Introduction to Business Mathematics.*

Laboratory manual - Mathematics, secondary stage - NCERT Publication.