BCBM102T

DISCRETE MATHEMATICS

DR. C.V. RAMAN UNIVERSITY

Kargi Road, Kota, Bilaspur, (C. G.),

Ph. 07753-253801, 07753-253872

E-mail: info@cvru.ac.in, Website: www.cvru.ac.in

BCBM102T

DISCRETE MATHEMATICS

Credit- 2

Subject Expert Team

Course Edited By:

Unit Written By:

Warning: All rights reserved, No part of this publication may be reproduced or transmitted or utilized or stored in any form or by any means now known or hereinafter invented, electronic, digital or mechanical, including photocopying, scanning, recording or by any information storage or retrieval system, without prior written permission from the publisher.

Edition: 2026

Publish by: DR. C.V. RAMAN UNIVERSITY

Kargi Road, Kota, Bilaspur, (C. G.),

Ph. 07753-253801, 07753-253872

STRUCTURE OF PROGRAM

Title of the Program	**:**	**Bachelor of Computer Application**
Program Type	**:**	**Undergraduate**
Program Credits	**:**	**120 Credit /160 Credit**
Program Duration	**:**	**3 Years / 4 Years**
Eligibility Criteria	**:**	**12th Pass**

PROGRAMME OBJECTIVE

- To demonstrate learner's fundamental computer science concepts such as programming, algorithms, and data structures.
- To design, develop, and manage software applications for a variety of platforms, including online and mobile.
- To handle databases as well as computer networks and cybersecurity principles.

PROGRAMME OUTCOME

- An ability to write, test, and debug code in several programming languages.
- An ability to create and develop software applications for several platforms, including web, mobile, and desktop.
- A capacity to design, develop, and maintain databases while assuring efficient data storage and retrieval.
- An ability to understand and use network technology & information security concepts to safeguard data and systems.
- An ability to display strong analytical and problem-solving skills, including the use of computational tools for addressing complicated issues.
- An ability to use software engineering approaches to create, test, and maintain software systems.
- An ability to work in teams while demonstrating great teamwork and communication abilities in a professional setting.
- An ability to behave ethically and professionally in their IT employment.
- Ability to adapt to evolving technology and rapid changes in the IT business.
- The ability to conduct well-planned additional study or research in computer science and related subjects.

PROGRAMME SPECIFIC OUTCOME

- Ability to create, build, and maintain dynamic websites using modern web technologies such as HTML, CSS, JavaScript, and frameworks such as Angular.
- Skill in relational and non-relational database design and management, as well as practical knowledge of SQL, NoSQL, and data retrieval techniques.
- Skill to apply project management principles to the design, execution, and delivery of software projects using tools such as Agile techniques and version control systems.
- Skill in developing and deploying mobile applications on many platforms, including Android and iOS, using languages such as Java, Swift, or Kotlin.
- Experience installing network systems and applying security measures to maintain data integrity and privacy within an organization.

STRUCTURE OF COURSE

Course Title	**:**	**Discrete Mathematics**
Course Type	**:**	**Minor Sub**
Course Code	**:**	**BCBM102T**
Course Credits	**:**	**06**
Semester	**:**	**01**

COURSE LEARNING OBJECTIVES

- To learn and apply mathematical reasoning, logic, set theory, combinatorics, graph theory, and the fundamental concept of grouping

COURSE LEARNING OUTCOMES

- To solve issues using mathematical reasoning, logic, and proof methods, to work with graphs, groups, and combinatorial structures, and to grasp key concepts in computer science and mathematics.
- Construct and execute set operations using proper notation, as well as create Venn diagrams for problem solutions.
- Apply counting principles, permutations, combinations, and discrete probabilities.
- Understand the properties and applications of sets, functions, and relations in different mathematical situations.
- Apply graph theory to model and solve path, circuit, and connectivity-related problems.
- Reviewing group, ring, coset, homomorphism, and their properties.

Course Content

BLOCK I	**SET THEORY**	**01-88**
Unit 1	**Set and Subset**	**03**
Unit 2	**Operation on Set**	**19**
Unit 3	**Principle of Inclusion-Exclusion**	**33**
Unit 4	**Derangement**	**55**
Unit 5	**Proposition**	**69**
BLOCK II	**PERMUTATIONS, COMBINATIONS & PROBABILITY**	**89-148**
Unit 6	**Permutation and Combination**	**91**
Unit 7	**Probability**	**111**
Unit 8	**Probability Distribution**	**131**
BLOCK III	**RELATIONS & FUNCTION**	**149-240**
Unit 9	**Cartesian Product and Relations**	**151**
Unit 10	**Operations and properties of relations**	**169**
Unit 11	**Important types of relations**	**181**
Unit 12	**Lattices**	**201**
Unit 13	**Function**	**217**
BLOCK IV	**GRAPH THEORY**	**241-320**
Unit 14	**Fundamental of graph theory**	**243**
Unit 15	**Isomorphism and subgraphs**	**271**
Unit 16	**Graph operations and walks**	**287**
Unit 17	**Connectivity, Eulerian graphs and planarity**	**303**
BLOCK V	**GROUP**	**321-384**
Unit 18	**Group**	**323**
Unit 19	**Subgroup**	**337**
Unit 20	**Ring**	**357**
Unit 21	**Subring**	**371**

BLOCK I	SET THEORY	01-88
Unit 1	Set and Subset	03
Unit 2	Operation on Set	19
Unit 3	Principle of Inclusion-Exclusion	33
Unit 4	Derangement	55
Unit 5	Proposition	69



This block lays the foundational concepts of Set Theory, which form the basis of modern mathematics and discrete structures. It begins with an exploration of sets and subsets, introducing learners to the fundamental language of mathematical grouping. Further, the block discusses operations on sets, enabling students to understand union, intersection, complement, and related algebraic manipulations. The Principle of Inclusion–Exclusion strengthens analytical reasoning by helping calculate cardinality in complex overlapping sets. This is followed by the concept of derangements, which deals with permutations without fixed points, enriching problem-solving skills. The block concludes with propositions, linking set concepts to basic logic, thereby preparing learners for deeper topics ahead.

UNIT 1
SET AND SUBSET

NOTE

UNIT STRUCTURE

1.1 Introduction

1.2 Objective

1.3 Set and Subset

1.3.1 Definition of Set

1.3.1.1. More examples of Set

1.3.2 Representation of a Set

1.3.2 1. Examples of set-in roster and set-builder:

1.3.3 Definition of Subset

1.3.3 1. More examples of subset

1.3.4 Types of Set

1.3.4.1. Finite Set

1.3.4.2. Infinite Set

1.3.4.3. Empty set

1.3.4.4. Singleton Set

1.3.4.5. Equal Set

1.3.4.6. Equivalent Set

1.3.4.7. Cardinal number of a finite Set

1.3.4.8. Universal Set

1.3.4.9. Power Set

1.3.5. Types of Subsets

1.3.5.1. Proper Subset

1.3.5.2. Improper Subset

1.4 Countable and Uncountable Set

1.4.1 Structure of Countable and Uncountable Set

1.4.2 Countable Set (denumerable)

1.4.3 Uncountable Set (non-denumerable)

1.5 Summary

1.6 Glossary

1.7 Answer to check your progress

1.8 Suggested readings - OER

1.9 Reference

1.10 Exercise

NOTE

1.1 INTRODUCTION

In the previous unit, you explored the basics of mathematical structures and their significance in problem-solving. You learned about fundamental mathematical objects and how they are used in various applications. Sets, an essential concept in mathematics, form the foundation for many advanced topics. This unit will expand your understanding of sets and subsets.

Building on this knowledge, this unit introduces the definition, representation, and classification of sets. You will learn how to express sets using different notations and identify various types of sets, such as finite, infinite, empty, and singleton sets. Additionally, subsets, their types, and power sets will be explored in detail.

This unit bridges the basic understanding of sets with more advanced concepts like countable and uncountable sets. By studying denumerable and non-denumerable sets, you will gain insight into infinite sets and their mathematical properties. These concepts are fundamental to logic, probability, and computer science.

1.2 OBJECTIVE

After studying this unit, you will be able to:

- Describe sets and subsets, comprehend what they mean, and differentiate between various kinds of sets and subsets.
- Examine other approaches to representing sets, including Venn diagrams, roster form, and set-builder notation.
- Determine and categories the various kinds of sets (finite, infinite, universal, null/empty, etc.) and subsets (proper, improper, etc.) along with their characteristics.
- Describe and examine the main distinctions between countable (denumerable) and uncountable (non-denumerable) collections.
- Examine the meaning, instances, and mathematical structure of countable and uncountable sets in a variety of mathematical contexts.

1.3 SET AND SUBSET

1.3.1 Definition of Set:

The term "set" refers to a well-defined group(collections) of objects. A set's elements or members are the items that make up that set. The set's components are displayed. in curly brackets "{}" and separated by commas ".". collections such as team and group. Months of a year, such as January, June, and July, as well as the letters of the alphabet, are all well-defined; therefore, it is a set. A writer may be extremely

skilled for one person but not for another; hence, this collection is not well defined and does not represent a set.

Alphabet capital letters A, B, C, X, Y, and Z, among others, are commonly used to denote sets. Whereas an element of a set, represented by the little letters a, b, c….etc. If a belongs to set A, we write $a \in$ A. Symbol $\in$ indicates that (epsilon) belongs to the Greek Alphabet. If a does not belong to set A, we write $a \notin$ A.

Example1: Consider Nazia's stationary set A.

A = { pen, pencil, rubber, sharpener, paper} is a set.

Some standard sets in math's are:

- Set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$
- Set of whole numbers, $W = \{0, 1, 2, 3, \dots\}$
- Set of integer numbers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- Set of rational numbers, $\mathbb{Q} = \{p/q \mid q \; is \; an \; integer \; and \; q \neq 0\}$
- Set of irrational numbers, $\mathbb{Q}' = \{x \mid x \; is \; not \; rational\}$
- Set of real numbers, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

1.3.1.1. More Examples of Set

1. Set has the capitals of five Indian states.

B = {Lucknow, Patna, Bhopal, Itanagar, Shillong}

2. Set of Alphabets in the English Language:

$A = \{a, b, c, \dots, z\}$

3. Set of Odd Numbers Less Than 10:

$O = \{1,3,5,7,9\}$

4. Set of Colors in a Rainbow:

$C = \{red, orange, yellow, green, blue, indigo, violet\}$

5. Set of Days in a Week:

$D = \{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday\}$

1.3.2 Representation of a Set:

Sets can be represented in two ways:

1. **Roster or Tabular Form**, (Listing Method, Tabular Method, or Enumeration Method).
2. **Set-builder Form,** Rule Method, or Set of Selection Method.

   RULES:

   Roster Form Rules:

   - Elements surrounded in curly brackets { }
   - Commas divide the elements of a set.

NOTE

Set Builder Form

- Elements are surrounded in curly brackets { }
- Commas divide the elements of a set.
- Set elements denoted by symbols such as x, y, z, etc.
- After the symbol, use the colon $':'$.
- Then characteristic property of that symbol.

1.3.2 1. Examples of set in Roster and Set-Builder

- **Statement:** The set of all natural numbers between10 and 14.
  Roster form {11,12,13}
  Set-builder form $\{x: x \in N, 10 < x < 14\}$.
- **Statement:** The set of all distinct letters used in the word 'Friend'.
  Roster form {F, r, i, e, n, d}
  Set-builder form{x: x is the distinct letters used in the word 'Friend'}
- **Statement:** "These set of all vowels in the word EQUATION" in Roster form.
  Roster form {A, E, I, O, U}
  Set-builder form{x: x is the distinct vowels in the word 'EQUATION'}
- **Statement** The set of all natural numbers between 10 and 14.
  Roster form {11,12,13}
  Set-builder form $\{x: x \in N, 10 < x < 14\}$.
- **Statement** The set of all distinct letters used in the word 'Friend.'
  Roster form {F, r, i, e, n, d}
  Set-builder form{x: x is the distinct letters used in the word 'Friend'}
- **Statement** "These set of all vowels in the word EQUATION" in Roster form.
  Roster form {A, E, I, O, U}
  Set-builder form{x: x is the distinct vowels in the word 'EQUATION'}

1.3.3 Definition of Subset

If a set exists within another set, it is referred to as a subset of the containing set. Let A and B be two sets. If every element of A is an element of B, then A is referred to as a subset of B. Every set is a subset of itself, and the empty set is a subset of all sets. The total number of subsets of a given set is 2^n, where n is the number of members.

For example:

a) If A = {2,4,5} has three elements, the total number of subsets is $2^n = 2^3 = 8$ The number of subsets will $\{\phi, \{2\}, \{4\}, \{5\}, \{2,4\}, \{2,5\}, \{4,5\}, \{2,4,5\}\}$.

b) If $x \in A$ and $x \in B$, the notation A ⊆ B and means " A is a subset of B." If A is not a subset of B, we express it as A ⊈ B.

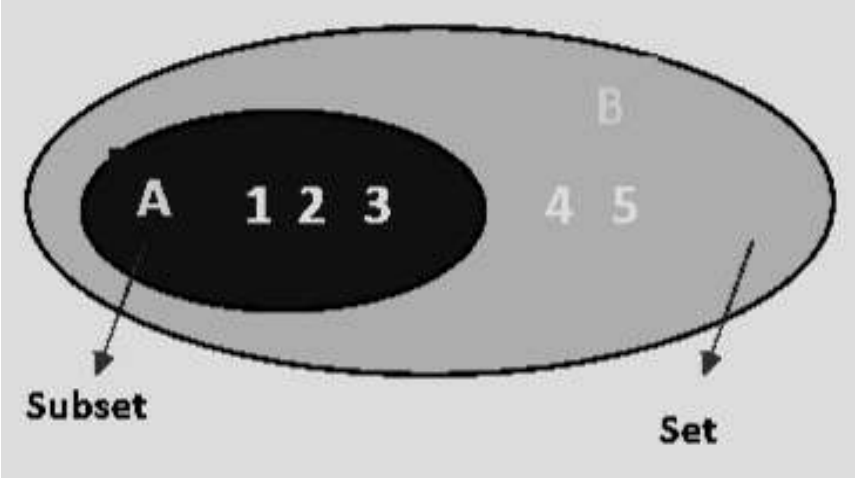


Figure No:1.1: Venn Diagram of Set and Subset

Remark:

(i) Notation ′ ⊆ ′ represents 'is subset of.'

(ii) Notation ′ ⊈ ′ represents 'is not subset of'

1.3.3 1. More Examples of Subset

1. If $A = \{2,3,5\}$ and $B = \{2,3,4,5\}$ then $A \subsetneq B$, since every element of A is also, an element of B. But $B \not\subset A$ since $4 \in B$ while $4 \notin A$.
2. If $A = \{1,4,6,8,10\}$ and $B = \{3,2,4,\}$ Then B is not a subset of A, since $3 \in B$ and

   $3 \notin A$.
3. If $A = \{6,1,4\}$ and $B = \{4,1,6\}$ then $A \subsetneq B$ and $B \subsetneq A$.
4. If $A = \{1,2,3,4,5\}$ A subset of A is $B = \{2,4\}$ because every element of B belongs to A.
5. If $X = \{a,b,c,d,e\}$ A subset of X is $Y = \{a,c,e\}$, as all elements of Y are contained in X.

1.3.4 Types of Set

1.3.4.1. Finite Set

A finite set is a set with a certain, either countable number of members.

Examples:

a) The set $\{2,3,4,5\}$ contains only 4 elements.

b) The set $B = \{\text{Lucknow, Bhopal, Patna, Shillong, Itanagar}\}$ of the capitals of 5 states of India.

1.3.4.2 Infinite Set

An infinite set is a set having an unlimited number of elements.

NOTE

Examples:

a) $A = \{1, 2, 3, 4, 5, \ldots \ldots \ldots . . \}$

b) Set of all points in a plane.

1.3.4.3 Empty set or Null Set or Void Set

An empty set is a set that has no elements. It is denoted by ϕ or $\{\}$. The number of elements in an empty set is finite.

Examples:

a) $A = \{x: x \text{ is a natural number less than } 1\}$

b) $R = \{x \in N: 6 < x < 7\}$

1.3.4.4 Singleton Set:

A singleton set is a set that consists of only one element.

Examples:

a) $\{4\}, \{7\}, \{-7\}$ are singleton sets.

b) $B = \{x: x \text{ is an integer and } 5 < x < 7\}$

1.3.4.5 Equal Set

Sets A and B are considered equal if their elements are the same, i.e., every element of A is also an element of B. Otherwise, two sets are unequal and we write $A \neq B$.

Examples:

a) Let $A = \{a, b, c, d, e\}$ and $B = \{c, d, b, a, e\}$ then $A = B$, because every element of A is in B and vice-versa. Therefore, A and B are said to be equal sets

b) The sets $A = \{1,4,5\}$ and $B = \{1, 1, 1, 4, 5, 5\}$ are equal sets because all elements of A are in B and vice-versa.

c) The sets $A = \{4, 2, 5, 6\}$ and $B = \{a, b, c, d\}$ are not equal sets.

1.3.4.6 Equivalent Set:

Sets A and B are considered identical if their cardinal numbers are the same. This means that the number of elements in A equals the number of elements in set B.

In other words, n (A) = n (B).

Examples:

a) Let $A = \{a, f, c, d\}$ and $B = \{1, 2, 6, 4\}$, then n (A) = 4 and n (B) = 4.

Therefore, A and B are said to be equivalent sets.

b) The sets A = {1,4,5} and B = {1,1,4,5,5} are not equivalent set because number of elements of A is not equal to number of elements of in B and vice-versa.

1.3.4.7 Cardinal number of a finite Set

The number of different elements in any set A is referred to as its cardinal number. It is designated as $n(A)$.

Examples:

a) $A = \{x: x \in N, x < 6\}$, i.e., $A = \{1,2,3,4,5\}$ Therefore, $n(A) = 5$

b) A = set of letters in the word 'ALGEBRA.'

i.e., $A = \{A, L, G, E, B, R\}$. Therefore, $n(A) = 6$

1.3.4.8 Universal Set

In any mathematical discussion, consider all sets to be subsets of a specified huge fixed set; this fixed set is referred to as the universal set. A set that contains all sets in each context is known as the universal set. It is denoted with " U ".

Examples:

a) Let $A = \{1,2,4,6\}$, $B = \{1,3,5\}$ and $C = \{0,7\}$, then $U = \{0,1,2,3,4,5,6,7\}$ is a universal set.

b) Let $A = \{1,2,3\}$, $B = \{2,4,5,6\}$ and $C = \{1,3,5,7\}$.

Then $U = \{1,2,3,4,5,6,7\}$ is a universal set.

1.3.4.9 Power Set

Let A represent the set. The collection of all subsets of a set A is known as its power set, and it is denoted by P(A). If A has n items, then P(A) contains 2n elements.

Example:

a) Let $A = \{3,4,2\}$ Then, $P(A) = \{\phi, \{3\}, \{4\}, \{2\}, \{3,4\}, \{3,2\}, \{4,2\}, \{3,4,2\}\}$

$\Rightarrow 2^3 = 8$ subsets form.

b) Let $B = \{a, \{b\}\}$ Then, $P(B) = \{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\} \Rightarrow 2^2 = 4$ subsets form.

1.3.5. Types of Subsets

1.3.5.1. Proper Subset

Let A and B be any two sets, then A is said to be a proper subset of B, if and only if $A \neq B$ and

i) A is a subset of B, i.e. $A \subseteq B$.

ii) There exist, at least one element in B which does not belong to A. It is denoted by $A \subset B$ and read as A is a proper subset of B.

NOTE

Examples:

a) Let A = {x: x is an even natural number} and B = {x: x is a natural number}
Then, A = {2, 4, 5, 6, 8, … … .} and B = {1, 2, 3, 4, 5, 6, 8, … … .} ∴ A ⊂ B

b) Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2, 5\}$. Here set Y ⊂ X since all elements in Y are contained in X and 'X ⊃ Y'.

1.3.5.2. Improper Subset

An **improper subset** is a subset that is the same as the original set, meaning it contains all the elements of the set. Let A and B be two sets, then A is called an improper subset of B, if and only if A = B**.** A is a subset of B i.e., A ⊆ B and A = B

Example:

a) Let, X = {1, 2, 3, 4} and Y = {1, 2, 3, 4}. Here set Y is an improper subset of set X as all the elements of set Y is in set X. Hence, we can write Y ⊆ X.

b) Let $B = \{red, blue, green\}$. The improper subset of B is $\{red, blue, green\}$

1.4 COUNTABLE AND UNCOUNTABLE SET

1.4.1 Structure of Countable and Uncountable Set

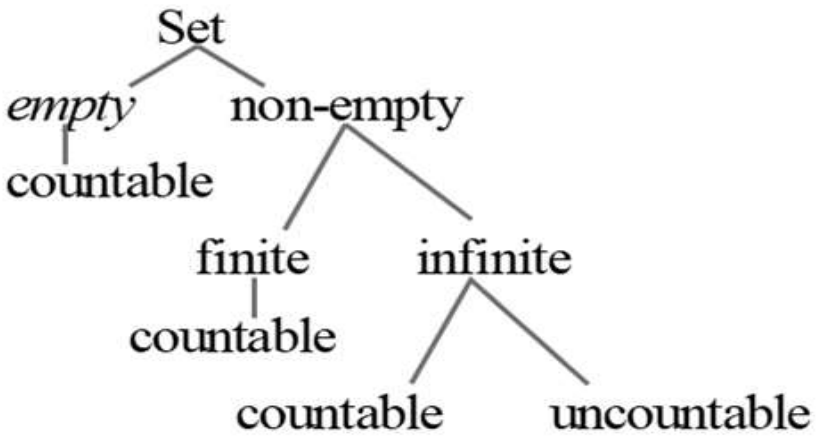


Figure No.1.2: Structure of countable and uncountable set

1.4.2 Countable Set (Denumerable)

If the components of a set can be arranged in a one-to-one correspondence with the natural numbers N, the set is said to be countable. Countable sets, also known as denumerable sets, are those whose elements can be mapped to natural numbers, such as integers or rational numbers.

Finite Set: A set of finite numbers consists of a set that is either empty or contains a particular number of entries as necessary.

Example1: The set $\{1, 2, 3, 4\}$ is a finite set.

Example2: The set $\{a, b, c, d\}$ is a finite set.

INFINITE SET: An infinite set is one with an infinite number of elements. However, they are less than countable.

Examples:

(i) Set of natural numbers $N = \{1,2,3 \dots \dots\}$

(i) Set of Integer numbers $Z = \{\dots \dots \dots -3, -2, -1,0,1,2,3 \dots \dots\}$

MORE EXAMPLES

1) The set $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3 \dots \dots 100\}$ is a finite set is countable.

i.e.,

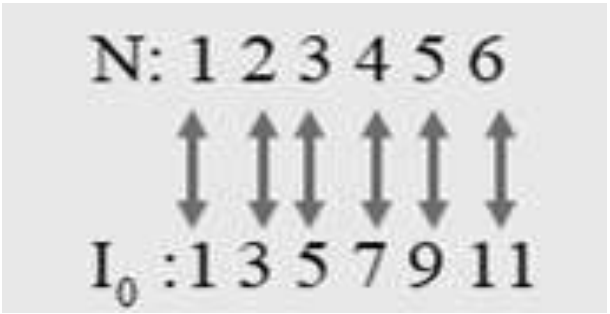


Figure No.1.3: Finite set is countable

2) Let Set of natural numbers $A = \{0,2,4,6,8,10 \dots \dots \dots \dots\}$ be an infinite set and Set of Integer Numbers $B = \{1,3,5,7,9,11 \dots \dots \dots \dots \dots\}$ be an infinite set.

Even above are infinite series but they are one-to-one mapping so they are countable.

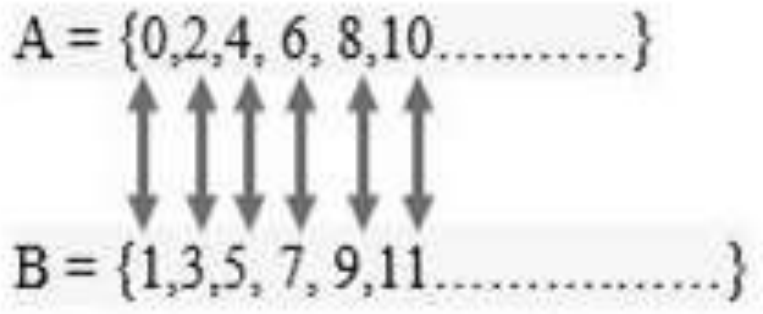


Figure No.1.4: Infinite series are countable

1.4.3 Uncountable Set (Non-Denumerable)

If a set is not countable, it is referred to as countable. If there cannot be one-to-one mapping between the elements of a set and natural numbers.

NOTE

Example: So Set of Natural and integer numbers come under countable set but set of real numbers are uncountable.

1) Set of natural numbers $\mathbb{N} = \{1,2,3,4 \dots \dots\}$

2) Set of Integer numbers $\mathbb{Z} = \{\dots \dots . -3, -2, -1,0,1,2,3 \dots \dots\}$

   We can count above infinite series.

   Next,

3) Set of Real numbers $\mathbb{Z} = \{\dots \dots \dots -3, -2, -1,0,1,2,3 \dots \dots\}$

   We cannot count above infinite series because there are infinite numbers exit between two numbers in $\mathbb{Z}. i. e.,(1.1, 1.2 \dots \dots)$ between 1 to 2.

So set of real numbers are uncountable.

Activity 1.1

Exploring Countable and Uncountable Sets

Objective:

To understand the concept of countable and uncountable sets and differentiate between them through examples and problem-solving**.**

Instructions:

1. **Classify Sets:**
   - Given the following sets, classify them as countable (denumerable) or uncountable (non-denumerable):
     a) Set of all natural numbers N
     b) Set of all real numbers R
     c) Set of all prime numbers
     d) Set of all points on a line segment
     e) Set of all whole numbers

2. **Find a One-to-One Correspondence:**
   - Show that the set of even numbers E = {2, 4, 6, 8, ...} is countable by establishing a one-to-one correspondence with natural numbers.

3. **Identify Uncountable Sets:**
   - Explain why the set of real numbers between 0 and 1 is uncountable.
   - Discuss Cantor's diagonal argument in simple terms.

4. **Real-Life Applications:**
   - Give two real-life examples where countable and uncountable sets are used (e.g., computing, data science, probability).

5. Group Discussion:

- In a group, discuss:
  - What is the main difference between countable and uncountable sets?
  - Why are rational numbers countable, but irrational numbers uncountable?

Expected Outcome: Students will develop a strong conceptual understanding of countable and uncountable sets, learn to differentiate between them, and explore their significance in mathematics.

☑ Check Your Progress 1.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define Power set by giving a suitable example.

……
……
……

2. Every proper subset of a set contains all the elements of the original set. (True or False).

3. The power set of a set $S = \{a\}$ contains:

A) $\emptyset, \{a\}$
B) $\{\emptyset\}$
C) $\{\emptyset, \{a\}\}$
D) $\{a, \{a\}\}$

4. A set that contains no elements is called a_____ set.

1.5 SUMMARY

This unit introduced the fundamental concepts of sets and subsets, which form the foundation of mathematics and logical reasoning. Students learned the definition of a set and explored various examples to understand how collections of objects, numbers, or symbols can be grouped meaningfully. Different methods of representing sets, including roster form and set-builder form, were studied to improve mathematical notation and expression. The unit also explained the concept of subsets and demonstrated how one set can be contained within another through several practical examples.

The unit further explored different types of sets such as finite sets, infinite sets, empty sets, singleton sets, equal sets, equivalent sets, universal sets, and power sets. Students learned how to identify these

NOTE

sets and calculate the cardinal number of finite sets. The concepts of proper and improper subsets were also discussed to strengthen understanding of relationships between sets. In addition, the unit introduced countable and uncountable sets, helping students distinguish between denumerable and non-denumerable collections and understand the structure of infinite sets.

By the end of this unit, students should be able to define and represent sets correctly, identify different types of sets and subsets, determine subset relationships, and understand the difference between countable and uncountable sets. These concepts are important in mathematics, computer science, statistics, logic, and data organization. Knowledge of sets is widely used in database systems, programming, probability theory, and problem-solving techniques in everyday life.

Students are encouraged to continue exploring advanced topics related to set theory, including set operations, Venn diagrams, relations, and functions. Practicing examples and solving real-world problems involving sets can further strengthen understanding and analytical thinking skills.

1.6 GLOSSARY

- **Set:** A collection of distinct objects, considered as an entity, such as numbers, letters, or symbols. Example: $\{1, 2, 3\}$.
- **Element:** An individual object within a set. For example, in the set $\{a, b, c\}$, "a" is an element.
- **Subset:** A set where every element is also a member of another set. For example, $\{1, 2\}$ is a subset of $\{1, 2, 3\}$.
- **Proper Subset:** A subset that does not contain all the elements of its parent set. Example: $\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.
- **Types of Sets:** Includes finite sets (limited elements), infinite sets (unlimited elements), null/empty sets (no elements), and universal sets (contain all objects under consideration).
- **Countable Set:** A set whose elements can be mapped to natural numbers in a one-to-one correspondence. Example: Natural numbers $(1, 2, 3, \ldots)$.
- **Uncountable Set:** A set whose elements cannot be enumerated or mapped to natural numbers. Example: Real numbers between 0 and 1.
- **Cardinality:** The measure of the number of elements in a set. For finite sets, it is the count of elements, and for infinite sets, it's expressed using concepts like $\aleph_0$ (aleph-null) or the continuum.
- **Power Set:** The set of all possible subsets of a given set. For a set with n elements, the power set has 2^n subsets.

- **Set Representation:** Methods to define sets, such as:
  - Roster form: Explicit listing of elements (e.g., $\{1, 2, 3\}$).
  - Set-builder notation: Describes elements using a rule (e.g., $\{x \mid x > 0\}$).
  - Venn diagrams: Graphical representation of sets and their relationships.

1.7 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 1.1

1. **Answer:** The power set of a set is the set of all possible subsets, including the empty set and the set itself. If a set has n elements, then its power set will have 2^n subsets.

   Example: Let $A = \{1,2\}$

   Then the power set of A, denoted by $P(A)$, is: $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

   Total number of subsets = $2^2 = 4$

   - Subsets:
     - Empty set: $\emptyset$
     - Single-element subsets: $\{1\}, \{2\}$
     - Full set: $\{1,2\}$
2. **Answer:** False
3. **The power set of a set $S = \{a\}$ contains:**

   Answer: A) $\emptyset, \{a\}$

   Explanation: Power set of $\{a\}$ is: $P(S) = \{\emptyset, \{a\}\}$ So the correct option is: $\emptyset, \{a\}$
4. **Answer:** Empty
5. **Explanation:** A set with no elements is called the empty set, denoted by $\emptyset$ or {}.

1.8 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1241	

NOTE

1.9 REFERENCE

- https://byjus.com/maths/what-is-a-set/
- https://byjus.com/maths/sets/
- https://www.varsitytutors.com/hotmath/hotmath_help/topics/number-systems
- https://www.geeksforgeeks.org/types-of-sets/
- https://www.geeksforgeeks.org/representation-of-a-set/
- https://people.math.carleton.ca/~kcheung/math/books/giam-ON/html/sec-venn.html
- https://byjus.com/maths/subsets/#:~:text=Set%20A%20is%20said%20to,also%20present%20in%20set%20B.

1.10 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following is an example of a set?

a) A group of beautiful flowers

b) The collection of vowels in the English alphabet

c) The tallest buildings in the world

d) The richest people on Earth

2. What is the representation of the set of even numbers less than 10 in roster form?

a) $\{2,4,6,8\}\, x < 10\}$

b) $\{x \mid x$, is an even number $x < 10\}$

c) $\{2n \mid n \in \mathbb{N}, n < 5\}$

d) All of the above

3. If $A = \{1, 2, 3, 4\}$, which of the following is a proper subset of A?

a) {1,2,3,4}

b) {1,2}

c) {1,2,3,4,5}

d) $\emptyset$

4. Which of the following is a countable set?

a) The set of natural numbers

b) The set of real numbers

c) The set of points on a line

d) The set of irrational numbers

5. What is the cardinality of the power set of $S = \{a, b, c\}$?

a) 3

b) 6

c) 8

d) 9

Answers with Explanations:

1. b) The collection of vowels in the English alphabet

Explanation: A set must be well-defined. The vowels in the English alphabet (a, e, i, o, u) form a well-defined collection, whereas "beautiful" or "tallest" is subjective and not well-defined.

2. d) All of the above

Explanation: The set of even numbers less than 10 can be represented in multiple ways: as a list (roster form), using a property (set-builder notation), or with a formula involving n. All options are correct representations.

3. b) $\{1, 2\}$

Explanation: A proper subset contains some, but not all, elements of the original set. {1,2,3,4}, is an improper subset, {1,2,3,4,5} is not a subset, and ∅ set is an improper subset but not proper in this context.

4. a) The set of natural numbers

Explanation: A countable set can be listed in a one-to-one correspondence with natural numbers. Natural numbers themselves are countable, whereas real numbers, points on a line, and irrational numbers are uncountable.

5. c) 8

Explanation: The cardinality of the power set of a set with n elements is 2^n. Since S has 3 elements, the power set has $2^3 = 8$ subsets.

B. Short answer type question

1. Define a set with one suitable example.
2. Differentiate between roster form and set-builder form of representing a set.
3. What is a subset? Give an example.
4. Explain the difference between a finite set and an infinite set.
5. Define empty set and singleton set with examples.

NOTE

Part 2

C. Long answer type question (200 words)

1. What is a set? Provide a brief definition with one example.
2. Write the set of vowels in the English alphabet using both roster form and set-builder notation.
3. What is a subset? Provide one example of a subset of the set $A = \{1,2,3,4\}$
4. Differentiate between countable and uncountable sets with one example of each.
5. Define types of sets with suitable examples.

Part 3

D. Long answer type question (300 words)

1. Explain properties of sets also give importance of sets in mathematics and real-world applications.
2. Discuss various methods of representing a set, including roster form, set-builder notation, and Venn diagrams.
3. Explain the difference between proper and improper subsets. Discuss the concept of a power set and demonstrate how to calculate the power set of a set with three elements.
4. Explain structure of countable and uncountable sets. Provide examples of countable sets (e.g., natural numbers, integers) and uncountable sets (e.g., real numbers).
5. Explain how the concept of countability relates to the idea of mapping elements and how it influences advanced mathematical theories.

UNIT 2
OPERATION ON SET

NOTE

UNIT STRUCTURE

2.1 Introduction

2.2 Objective

2.3 Operation on Set

- **2.3.1 Venn diagram**
- **2.3.2 Types of Operation on set**
  - **2.3.2.1 Union of Sets**
  - **2.3.2.2 Intersection of sets**
  - **2.3.2.3 Complement of a Set**
  - **2.3.2.4 Difference of Sets**
  - **2.3.2.5 Symmetric difference.**
  - **2.3.2.6 Disjoint of Sets**
  - **2.3.2.7 Cartesian Product of sets**

2.4 Summary

2.5 Glossary

2.6 Answer to check your progress

2.7 Suggested readings - OER

2.8 Reference

2.9 Exercise

2.1 INTRODUCTION

In the previous unit, you explored the basics of sets and subsets, understanding their types and representations. These concepts are fundamental in mathematics and are widely used in various fields. Now, we extend this knowledge to operations on sets, which help in combining, comparing, and analyzing sets. Venn diagrams provide a visual representation of these operations, making them easier to understand.

Building on this knowledge, this unit introduces different set operations such as union, intersection, complement, and difference. These operations are essential for solving problems in probability, logic, and database systems. The Cartesian product extends set operations by forming ordered pairs, which are widely used in relations and functions. Understanding these operations will enhance problem-solving skills in various mathematical domains.

This unit bridges the gap between fundamental set theory and its applications in real-world scenarios. It introduces how set operations are used in data analysis, computer science, and logic. By learning

NOTE

Venn diagrams and operations like symmetric difference and disjoint sets, you will gain a deeper understanding of relationships between different sets. This knowledge is crucial for advanced topics in discrete mathematics and beyond.

2.2 OBJECTIVE

After studying this unit, you will be able to:

- Compute the union of sets and its significance in combining elements from multiple sets.
- Identify and determine the intersection of sets to find common elements between them.
- Explore the concept of the complement of a set and its relationship with the universal set.
- Calculate the difference of sets and apply it to solve problems involving subtraction of elements.
- Distinguish between disjoint sets and non-disjoint sets and understand their properties.

2.3 OPERATION ON SET

2.3.1 Venn Diagram

The Venn diagram is a pictorial representation of data sets.

To draw the Venn-diagram following points are to be considered:

British mathematician John-Venn (1834–1883) put an idea into effect. In a Venn diagram, the universal set U is represented by a single point inside a rectangle and its subsets by a circle within the rectangle.

2.3.2 Operation on Set

1. Union of Sets
2. Intersection of sets
3. Complement of a Set
4. Difference of Sets
5. Symmetric difference.
6. Disjoint of Sets
7. Cartesian Product of Sets

2.3.2.1 Union of Sets

Union are denoted by the symbol " ∪ ".

Let A and B represent any two sets. The union of A and B is the collection of all elements that appear as either A or B, or both. $A \cup B$ means ' A union B.'

Thus, $A \cup B = \{x: x \in A \text{ or } x \in B\}$.

NOTE

In Venn diagram Shaded portion represents $A \cup B$.

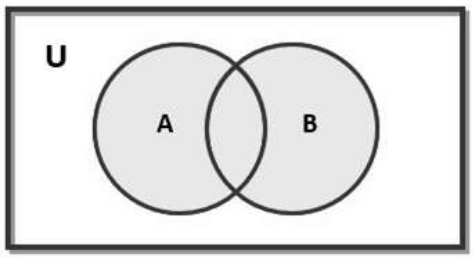


Figure No 2.1: Venn diagram of Union of sets

Examples:

a) Let $A = \{7,3\}$ and $B = \{3,4,5\}$ then $A \cup B = \{3,4,5,7\}$.

b) If $A = \{1,2,3,4\}$, $B = \{3,4,5\}$ and $C = \{4,7\}$ then $A \cup B \cup C = \{1,2,3,4,5,7\}$.

NOTE: $A \cup \phi = A = \phi \cup A$

i.e., Union of any set with an empty set is always the set itself.

2.3.2.2 Intersection of Sets

Intersection are denoted by the symbol "$\cap$".

Let A and B represent any two sets. The intersection of A and B is the set of all elements from both A and B. It is denoted by $A \cap B$ and read as 'A intersection B.'

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

In Venn diagram portion represents $A \cap B$.

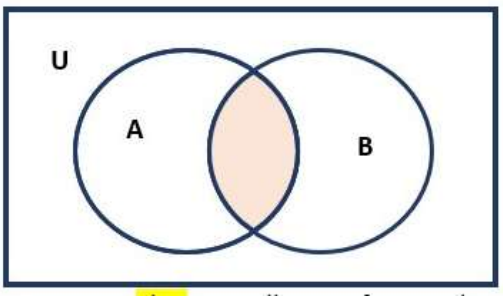


Figure No.2.2: Venn diagram of Intersection of sets

Examples:

a) Let $A = \{2,1,4,5\}$ and $B = \{1,3,4,6\}$ then $A \cap B = \{1,4\}$

b) Let $A = \{2,3,4,5\}, B = \{1,3,4,6\}$ and $C = \{4\}$ then $A \cap B \cap C = \{A\}$

NOTES:

1. $A \cap \phi = \phi \cap A = \phi$

i.e., intersection of any set with the empty set is always the empty set.

NOTE

2.3.2.3 Complement of A Set

Complement of a set is denoted by A' or A^C and read as 'A's complement.' Let U be the universal set and A be any subset of U. The complement of A to U is the set of all elements of U that are not in A. Thus, $A = \{x: x \in U \text{ and } x \notin A\}$ or $A' = U - A$

The shaded portion in **Venn diagram** represents the **complement of set A.**

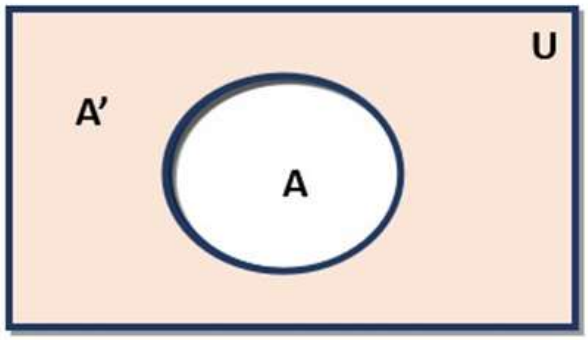


Figure No.2.3: Venn diagram of Complement of set

Example:

a) Let $U = \{1,2,3,4,5,6\}$ and $A = \{3,4\}$ then,

$$A' = U - A$$
$$= \{1,2,3,4,5,6\} - \{3,4\}$$
$$= \{1, 2, 5, 6\}$$

b) Let $U = \{1,2,3,4,5 \ldots \ldots 10\}$and $A = \{1,3,5,7,9\}$ then

$$A' = U - A$$
$$= \{1,2,3,4,5\ldots\ldots 10\} - \{1,3,5,7,9\}$$
$$= \{2, 4, 6, 8,10\}$$

2.3.2.4 Difference of Sets

Difference of sets is denoted as " − " or "/" .

Consider any two sets, A and B. The set of all the elements in A that are not part of B is the difference between sets A and B in this order.

It is written as A minus B and is represented by the symbols $A - B$ or A/B.

$$A - B = \{x: x \in A \text{ and } x \notin B\}$$
$$B - A = \{x: x \in B \text{ and } x \notin A\}$$

Similarly,

It is possible to depict the difference between sets A and B. shaded portion in **Venn-diagram.**

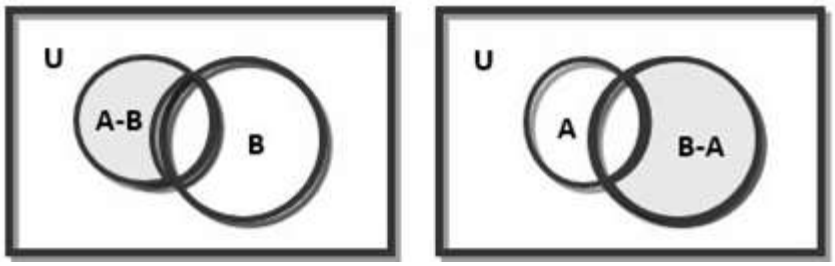


Figure No.2.4: Venn diagram of Difference of sets

Example:

a) Let $P = \{1, 2, 3, 4, 5\}$ and $Q = \{1, 5, 7, 9\}$.

Consequently, $P - Q = \{2, 3, 4\}$ and $Q - P = \{7, 9\}$

b) Let $\mathrm{P} = \{1, 2, 3, 4\}$ and $\mathrm{Q} = \{2, 4, 6, 8\}$

Then, $\mathrm{P} - \mathrm{Q} = \{1,3\}$ and $\mathrm{Q} - \mathrm{P} = \{6, 8\}$

2.3.2.5 Symmetric Difference of Two Sets:

The symbol '△' is for the symmetric difference.

The set of elements that do not belong to both A and B is known as the symmetric difference $(A - B) \cup (B - A)$ for any two sets A and B. It can be read as A symmetric difference B and is represented by the symbols $A \triangle B$ or $A \oplus B$.

$A \triangle B = (A - B) \cup (B - A) = \{x : x \in (A \cup B)$ but $x \notin (A \cap B)\}$, then

The Venn diagram's shaded area depicts the symmetric difference between sets A and B.

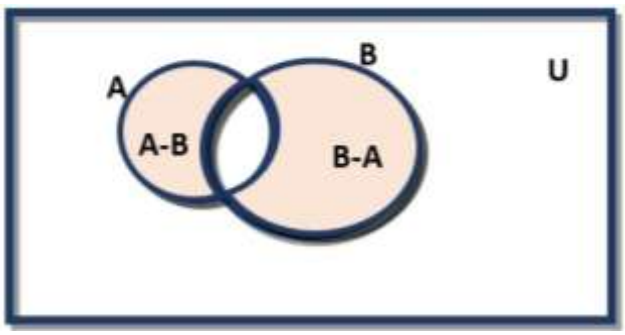


Figure No.2.5: Venn diagram of Symmetric difference of sets

Examples:

a) Let $\mathrm{P} = \{1, 2, 3, 4\}$ and $\mathrm{Q} = \{3, 4, 5, 6\}$

Now, $\mathrm{P} - \mathrm{Q} = \{1,2\}$, $\mathrm{Q} - \mathrm{P} = \{5, 6\}$

Thus, $\mathrm{P} \triangle \mathrm{Q} = (\mathrm{P} - \mathrm{Q}) \cup (\mathrm{Q} - \mathrm{P}) = \{1,2,5,6\}$

b) Let $\mathrm{P} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ and $\mathrm{Q} = \{\mathrm{a}, \mathrm{b}, \mathrm{p}, \mathrm{q}\}$

Thus, $\mathrm{P} \triangle \mathrm{Q} = (\mathrm{P} - \mathrm{Q}) \cup (\mathrm{Q} - \mathrm{P}) = \{\mathrm{c}, \mathrm{d}, \mathrm{p}, \mathrm{q}\}$

NOTE

2.3.2.6 Disjoint of Sets:

If two sets, A and B, do not share an element, then they are said to be disjoint sets ($A \cap B = \phi$). $A \cap B = \phi$ is therefore interpreted as "A and B are disjoint set."

Disjoint of two sets A and B are represented in **Venn-diagram.**

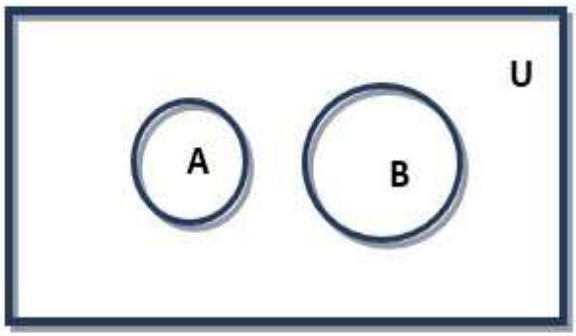


Figure No.2.6: Venn diagram of Disjoint of sets

Example:

a) Let A = $\{2,4,6\}$ and B = $\{1,3,5\}$ then A ∩ B = ϕ.

b) Let A = $\{1,2,3\}$ and B = $\{6,7,8,9\}$ then A and B are disjoint sets.

As no element is common i.e., A ∩ B = ϕ.

2.3.2.7 Cartesian Product of Sets:

The **Cartesian product** of two sets A and B, denoted as $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

Mathematically, $A \times B = \{(a, b) \mid a \in A, b \in B\}$

If either A or B is empty, their Cartesian product is also an empty set.

Example :

a) Let $A = \{1,2\}$ and $B = \{x, y\}$ $A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$

b) Let $C = \{a, b, c\}$ and $D = \{1\}$. $C \times D = \{(a, 1), (b, 1), (c, 1)\}$

Key Notes:

- Cartesian products are not commutative, meaning $A \times B \neq B \times A$ unless $A = B$.
- The size of $A \times B$ is the product of the sizes of A and B: $| A \times B | = | A | \cdot | B |$.

Activity 2.1

Exploring Set Operations Using Venn Diagrams

Objective:

To understand and visualize different set operations using Venn diagrams and real-life examples.

NOTE

Instructions:

1. Draw Venn Diagrams:

- Take two sets, A and B, and represent them using overlapping circles in a Venn diagram.
- Label the regions corresponding to different set operations (union, intersection, difference, complement, etc.).

2. Perform the Following Operations on Given Sets:

- Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$
- Find:

  a) $A \cup B$ (Union)

  b) $A \cap B$ (Intersection)

  c) $A - B$ (Difference)

  d) $B - A$ (Difference)

  e) A' and B' (Complements considering Universal Set $= \{1,2,3,4,5,6,7,8,9,10\}$)

  f) Symmetric Difference $A \Delta B$

  g) Cartesian Product $A \times B$

3. Real-Life Application:

- Think of two real-life sets (e.g., students who play cricket vs. students who play football).
- Apply the above operations and interpret the results.

4. Discussion Questions:

- What do you observe in each operation?
- How do Venn diagrams help in understanding set operations?
- Can you think of any real-world problems where set operations are useful?

Outcome:

This activity will help students visualize and understand set operations using both numerical and real-world examples.

☑ Check Your Progress 2.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is the union and intersection of two sets?

……

NOTE

.....

.....

2. Disjoint sets have no elements in common. (True or False).

3. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, what is A–B?

(A) {1} (B) {4}

(C) {1,4} (D) {2,3}

4. The complement of a set contains elements in the _____ but not in the set.

2.4 SUMMARY

This unit introduced the fundamental concepts of operations on sets and their importance in mathematics and logical reasoning. Students learned how sets can be represented using Set Theory and visualized through Venn diagrams, which help in understanding relationships between different groups of elements. The unit also explained various operations on sets, including union, intersection, complement, difference, symmetric difference, disjoint sets, and Cartesian products. These operations provide the foundation for solving mathematical problems involving classification, comparison, and relationships between data.

The key learning outcome of this unit is the ability to perform and interpret different set operations accurately. Students should now understand how the union of sets combines elements, how intersections identify common elements, and how complements and differences help determine excluded or unique elements. The concept of symmetric difference explains elements that belong to either set but not both, while disjoint sets demonstrate sets with no common elements. Students also explored Cartesian products, which are useful for forming ordered pairs and relations between sets. Venn diagrams further strengthened conceptual understanding by providing a simple visual representation of these operations.

The concepts studied in this unit have wide practical applications in areas such as computer science, database management, probability, statistics, business analysis, and decision-making. Set operations are commonly used in searching and filtering data, designing logical circuits, organizing information, and solving real-life problems involving groups and categories. Understanding Cartesian products also supports further studies in relations, functions, and coordinate geometry.

Students are encouraged to continue exploring advanced topics related to sets, relations, functions, and logic to build stronger mathematical

NOTE

reasoning skills. Practicing more problems with Venn diagrams and set operations can improve analytical thinking and problem-solving abilities. Additional learning can be supported through mathematics textbooks, online tutorials, and interactive exercises that provide real-world examples of set applications.

2.5 GLOSSARY

- **Union of Sets**:

  The operation that combines all elements from two or more sets, removing duplicates. Denoted as A∪B.

- **Intersection of Sets**:

  The operation that identifies the common elements shared between two or more sets. Denoted as A∩B.

- **Complement of a Set**:

  The set of all elements in the universal set U that are not in the given set A. Represented as A^c.

- **Difference of Sets**:

  The set of elements present in one set A but not in another set B. Denoted as $A - B$ or $A \setminus B$.

- **Symmetric Difference**:

  The set of elements that are in either of the two sets A or B, but not in their intersection. Denoted as $A \triangle B$.

- **Disjoint Sets**

  Two sets are disjoint if their intersection is empty, i.e., $A \cap B = \emptyset$. They share no common elements.

- **Cartesian Product of Sets**:

  The set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Denoted as $A \times B$.

- **Universal Set**:

  The set that contains all elements under consideration for a particular discussion or problem. Represented as U.

- **Empty Set**:

  A set that contains no elements, denoted as ∅ or {}. Also known as the null set.

- **Subset**:

  A set A is a subset of another set B if every element of A is also an element of B. Denoted as $A \subseteq B$.

2.6 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 2.1

NOTE

1. **Answer:** The union of two sets is the set containing all elements that are in either set (or both).

   The intersection of two sets is the set containing only those elements that are common to both sets.

   For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then:

   - $A \cup B = \{1, 2, 3, 4\}$ (Union)
   - $A \cap B = \{2, 3\}$ (Intersection)

2. **Answer:** True.

   Disjoint sets are sets that have no common elements.

   For example, $A = \{1, 2\}$ and $B = \{3, 4\}$ are disjoint because $A \cap B = \emptyset$.

3. **If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, what is $A - B$?**

   Answer: $A - B$ means elements in A that are **not** in B.

   $$A - B = \{1\}$$

   Correct option: (A) $\{1\}$

4. ***Answer:*** The complement of a set contains elements in the universal set but not in the given set. So, the blank is filled as: "universal set"

   Example: If $U = \{1,2,3,4,5\}$ and $A = \{2,4\}$, then $A' = \{1,3,5\}$

2.7 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1242	

2.8 REFERENCE

- https://byjus.com/maths/set-operations/
- https://byjus.com/maths/venn-diagrams/
- https://probability4datascience.com/slides/Slide_2_01.pdf
- https://byjus.com/maths/union-of-sets/
- https://byjus.com/maths/intersection-of-sets/
- https://byjus.com/maths/complement-of-set/

2.9 EXERCISE

NOTE

Part 1

A. Multiple choice question

1. **In a Venn diagram, what does the shaded region of $A \cup B$ represent?**

   a) Only elements in A

   b) Only elements in B

   c) Elements in A or B or both

   d) Elements common in A and B

2. **The intersection of two sets $A \cap B$ represents:**

   a) Elements only in A

   b) Elements only in B

   c) Elements common in both A and B

   d) Elements in A or B

3. **What is the symmetric difference of two sets $A \triangle B$?**

   a) Elements common in A and B

   b) Elements that are in either A or B but not in both

   c) Union of A and B

   d) Empty set

4. **If two sets A and B are disjoint, then their intersection $A \cap B$ is:**

   a) A

   b) B

   c) Universal set

   d) Empty set

5. **The Cartesian product of two sets $A \times B$**

   a) The set of elements common in A and B

   b) The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

   c) The union of A and B

   d) The intersection of A and B

Answers with Explanations:

1. **c) Elements in A or B or both**

   Explanation: In a Venn diagram, the union of two sets A∪B includes all elements present in A or B or both.

NOTE

2. c) Elements common in both A and B

Explanation: The intersection of two sets consists of elements that belong to both sets.

3. b) Elements that are in either A or B but not in both

Explanation: The symmetric difference of two sets includes elements that are in either set but not in their intersection: $A \triangle B = (A - B) \cup (B - A)$

4. d) Empty set

Explanation: Two sets are disjoint if they have no common elements, meaning their intersection is empty.

5. b) The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

Explanation: The Cartesian product of two sets is the set of all possible ordered pairs formed by taking an element from A and an element from B:

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

B. Short answer type question

1. Draw a Venn diagram representing two overlapping sets A and B and shade the region representing $A \cap B$.
2. If $A = \{1,2,3\}$ and $B = \{3,4,5\}$, find $A \cup B$ using Venn diagram?
3. If the universal set $U = \{1,2,3,4,5,6,7,8,9,10\}$ and $A = \{2,4,6,8,10\}$, find the complement of A (denoted as A').
4. If $P = \{a, b, c, d\}$ and $Q = \{c, d, e, f\}$, find $P \triangle Q$ symmetric difference of P and Q.
5. If $X = \{1,2\}$ and $Y = \{a, b\}$, list the elements of $X \times Y$ Cartesian product of X and Y.

Part 2

C. Long answer type question (200 words)

1. Let $A = \{1,3,5,7,9,11\}$ and $B = \{3,6,9,12\}$ Find $A \cup B$ (Union) and $A \cap B$ (Intersection).

   (c) Find $A \triangle B$ (Symmetric Difference).

2. Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ and two subsets be $A = \{2,4,6,8,10\}$ and $B = \{1,3,5,7,9,11\}$. Find the complement of A (i.e., A') and complement of B (i.e., B').

3. In a class of 50 students,
   - 30 students like Mathematics (M)
   - 25 students like Science (S)

NOTE

- 15 students like both Mathematics and Science

Using set notation and a Venn diagram, answer the following:

(a) How many students like only Mathematics?

(b) How many students like only Science?

(c) How many students like either Mathematics or Science or both?

(d) How many students do not like either subject?

4. Let $X = \{a, b, c, d, e\}$ and $Y = \{c, d, e, f, g\}$. Find $X \triangle Y$ (Symmetric Difference).

   Also, if $P = \{1,2,3\}$ and $Q = \{4,5,6\}$, show that P and Q are disjoint sets.

5. Let $A = \{1,2,3\}$. and $B = \{x, y\}$.

   (a) Find the Cartesian product $A \times B$ and list all ordered pairs.

   (b) Define a relation R from A to B as $R = \{(1, x), (2, y)\}$. is R a function? Justify your answer.

Part 3

D. Long answer type question (300 words)

1. In a school, students participate in either Cricket (C) or Football (F) or both. A survey conducted among 80 students revealed:
   - 45 students play Cricket.
   - 38 students play Football.
   - 20 students play both Cricket and Football.

   Using set operations and a Venn diagram, answer the following:

   (a) Find the number of students who play only Cricket.

   (b) Find the number of students who play only Football.

   (c) Find the number of students who play either Cricket or Football or both.

   (d) Find the number of students who do not play any sport.

   (e) Prove that the total number of students is given by the formula:

   $$| C \cup F | = | C | + | F | - | C \cap F |$$

2. Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$. Define two subsets:
   - $A = \{2,4,6,8,10,12,14\}$ (Even numbers)
   - $B = \{3,6,9,12,15\}$ (Multiples of 3)

NOTE

Using set operations, answer the following:

(a) Find the complement of A, i.e., A′.

(b) Find the complement of B, i.e., B′.

(c) Verify De Morgan's Laws:

$$(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$$

3. Two sets are defined as follows:
   - $X = \{a, b, c, d, e\}$
   - $Y = \{c, d, e, f, g\}$

   (a) Find $X \triangle Y$ (the symmetric difference of X and Y).

   (b) If $P = \{1,2,3,4\}$ and $Q = \{5,6,7,8\}$, prove that P and Q are disjoint sets.

   (c) Show that for any two sets A and B, the symmetric difference satisfies the identity:

$$A \triangle B = (A - B) \cup (B - A)$$

4. A company conducted a survey on 100 customers to understand their preferences for two products: Product X and Product Y. The findings are:
   - 60 customers purchased Product X.
   - 50 customers purchased Product Y.
   - 30 customers purchased both products.

   Using set theory, answer the following:

   (a) Represent the data using a Venn diagram.

   (b) Find the number of customers who purchased only Product X.

   (c) Find the number of customers who purchased only Product Y.

   (d) Find the number of customers who purchased at least one product.

   (e) Find the number of customers who purchased neither product.

UNIT 3
PRINCIPLE OF INCLUSION-EXCLUSION

NOTE

UNIT STRUCTURE

3.1 Introduction

3.2 Objective

3.3 Principle of Inclusion-Exclusion

- **3.3.1 Definition of Principle of Inclusion-Exclusion**
- **3.3.2 Basic Theorems on Principle of Inclusion-Exclusion**
- **3.3.3. Using Vann Diagram Principle of Inclusion-Exclusion (PIE)**
- **3.3.4. Proof of basic theorems on Principle of Inclusion-Exclusion**
- **3.3.5. Examples based on Principle of Inclusion-Exclusion (PIE)**

3.4 Summary

3.5 Glossary

3.6 Answer to check your progress

3.7 Suggested readings - OER

3.8 Reference

3.9 Exercise

3.1 INTRODUCTION

In the previous unit, you explored the basics of set operations, including union, intersection, and complement. These operations helped in understanding relationships between different sets. Now, we extend this knowledge to a more advanced concept called the Principle of Inclusion-Exclusion (PIE). This principle is crucial in solving counting problems where multiple sets overlap.

Building on this knowledge, the Principle of Inclusion-Exclusion (PIE) provides a systematic way to count elements in the union of multiple sets while avoiding overcounting. This principle is widely used in combinatorics, probability, and real-life applications like data analysis. Using Venn diagrams, we can visualize and prove the key theorems of PIE, making complex counting problems easier.

This unit bridges set theory and advanced counting techniques by introducing the Principle of Inclusion-Exclusion. It explains its definition, fundamental theorems, and proofs using Venn diagrams. By applying PIE to real-world examples, we learn how to correctly count

NOTE

elements in overlapping sets, solving problems in probability, computer science, and combinatorial mathematics.

3.2 OBJECTIVE

After studying this unit, you will be able to:

- Analysethe concept and significance of the Principle of Inclusion-Exclusion (PIE).
- Apply the Principle of Inclusion-Exclusion to solve counting problems involving overlapping sets.
- Derive and explain the basic theorems related to the Principle of Inclusion-Exclusion.
- Use the PIE method to calculate the number of elements in the union of multiple sets.
- Solve real-world problems involving conditions like divisibility, coloring, and arrangement using PIE.

3.3 PRINCIPAL OF INCLUSION-EXCLUSION

3.3.1 Definition of Principle of Inclusion-Exclusion (PIE)

The Principle of Inclusion-Exclusion (PIE) is a fundamental counting technique used in combinatorial mathematics to calculate the cardinality (number of elements) of the union of multiple sets. It efficiently handles overlapping elements by adding and subtracting the sizes of individual sets and their intersections.

The Principle of Inclusion-Exclusion states that for any finite sets $A_1, A_2, \dots, A_n$

$$| A_1 \cup A_2 \cup \cdots \cup A_n | = \sum_{i=1}^{n} | A_i | - \sum_{1 \le i < j \le n} | A_i \cap A_j |$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap Ak| - \cdots + (-1)^{n+1} | A_1 \cap A_2 \cap \cdots \cap A_n |$$

Where:

- $| A_i | =$ = Number of elements in set A_i
- $| A_i \cap A_j | =$ = Number of elements common in sets A_i and A_j

Alternating signs ensure that elements in multiple sets are not overcounted.

3.3.2. Basic Theorems on Principle of Inclusion-Exclusion

1. **For Two Sets:** $| A \cup B | = | A | + | B | - | A \cap B |$
2. **For Three Sets:**

$$| A \cup B \cup C | = | A | + | B | + | C | - | A \cap B | - | A \cap C | - | B \cap C | + | A \cap B \cap C |$$

NOTE

General Form for n Sets:

$$| A_1 \cup A_2 \cup \cdots \cup A_n |= \sum_{k=1}^{n} (-1)^{k+1} + \sum_{1 \le i_1 \ldots\ldots < i_k \le n} | A_{i_1} \cap \ldots \ldots \cap A_{i_k} |$$

Example: How many integers between 1 and 100 are divisible by $2, 3, or\ 5$?

Solution: Let:

- $A = \{Multiples\ of\ 2\}$
- $B = \{Multiples\ of\ 3\}$
- $C = \{Multiples\ of\ 5\}$

Using PIE:

$$| A \cup B \cup C |=| A | +| B | +| C | -| A \cap B | -| A \cap C | -| B \cap C | +| A \cap B \cap C |$$

Calculating each term:

$| A |= 50$ (Multiples of 2)

$| B |= 33$ (Multiples of 3)

$| C |= 20$ (Multiples of 5)

$| A \cap B |= 16$ (Multiples of 6)

$| A \cap C |= 10$ (Multiples of 10)

$| B \cap C |= 6$ (Multiples of 15)

$| A \cap B \cap C |= 3$ (Multiples of 30)

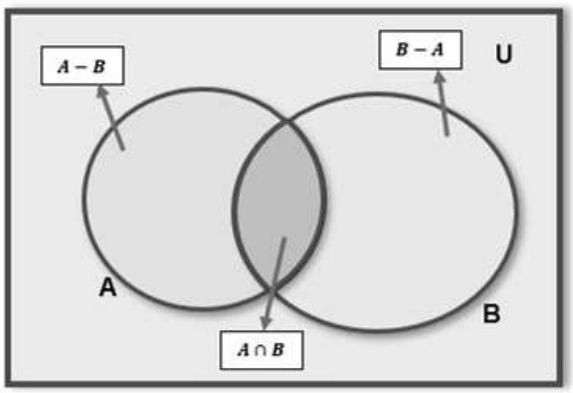
Now,

$$| A \cup B \cup C |= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

Thus, there are 74 integers between 1 and 100 that are divisible by 2, 3, or 5.

3.3.3. Using Vann Diagram Principle of Inclusion-Exclusion (Pie)

$A - B, B - A,$ and $(A \cap B)$ are disjoint if A and B are subsets of the universal set U.



NOTE

Figure No.3.1: Venn diagram of two disjoint sets

From the figure we have, $(A - B) \cup (B - A) \cup (A \cap B)$ is the same as $(A \cup B)$.

Three disjoint sets (based on addition principle)

$$|(A - B) \cup (B - A) \cup (A \cap B)| = |A - B| + |B - A| + |A \cap B|$$

[1] (Sum rule method) Suppose A and B are disjoint sets, then

$$|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$$

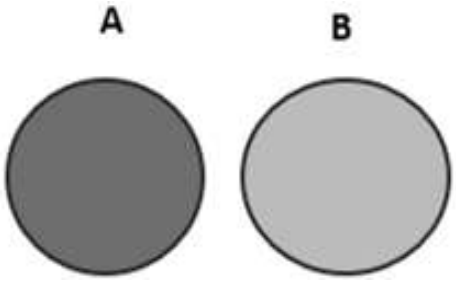


Figure No.3.2: Venn diagram Sum rule of two disjoint sets

[2] Suppose A and B are not-disjoint sets, then

$$|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$$

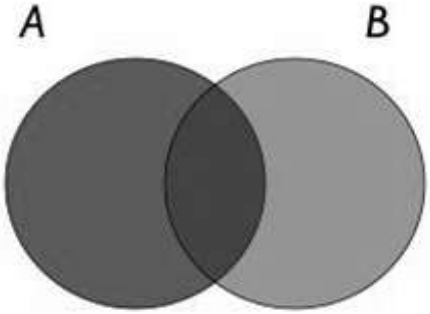


Figure No.3.3: Venn diagram Sum rule of two non-disjoint sets

[3] Suppose ***A, B and C*** are finite set, then

$$|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}| = |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}| - |\mathbf{B} \cap \mathbf{C}| - |\mathbf{A} \cap \mathbf{B}| - |\mathbf{A} \cap \mathbf{C}| + |\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}|$$

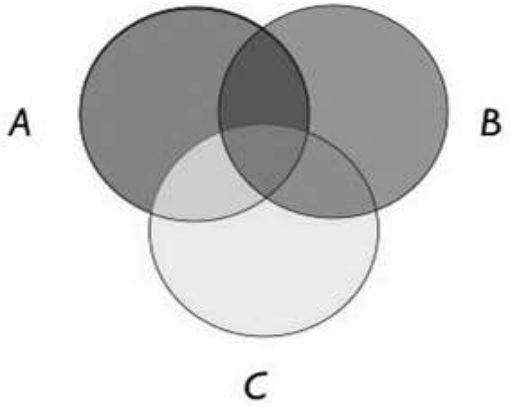


Figure No.3.4: Venn diagram Sum rule of three non-disjoint sets

NOTE

3.3.4. Proof of Basic Theorems on Principle of Inclusion-Exclusion

CASE-I: The inclusion and exclusion principle for two sets, P and Q.

Theorem 1: To prove for two finite sets P and Q

$$|P \cup Q| = |P| + |Q| - |P \cap Q|$$

Proof: We know by the above Venn diagram.

$|(P - Q) \cup (Q - P) \cup (P \cap Q)| = |P \cup Q|$ ..... (i)

$$|(P - Q) \cup (Q - P) \cup (P \cap Q)| = |P - Q| + |Q - P| + |P \cap Q|$$

$$= |P| - |P \cap Q| + |Q - P| + |P \cap Q| \quad [\because |P - Q| = |P| - |P \cap Q|]$$

$$= |P| - |P \cap Q| + |Q| - |P \cap Q| + |P \cap Q|$$

$$= |P| + |Q| - |P \cap Q| \quad \text{..... (ii)}$$

Hence, by (i) and (ii) $|P \cup Q| = |P| + |Q| - |P \cap Q|$

To determine the components of a union, we include the individual elements of P and Q. However, we must delete the items that are like both P and Q to avoid counting them twice.

CASE-II: Three sets P, Q, and R using the inclusion and exclusion principle.

Theorem 2: To prove for two finite sets P, Q and R

$$|\mathbf{P \cup Q \cup R}| = |\mathbf{P}| + |\mathbf{Q}| + |\mathbf{R}| - |\mathbf{Q \cap R}| - |\mathbf{P \cap Q}| - |\mathbf{P \cap R}| + |\mathbf{P \cap Q \cap R}|$$

Proof: Using formula $|P \cup Q \cup R| = |A| + |Q \cup R| - |P \cap (Q \cup R)|$

$$= |P| + |Q| + |R| - |Q \cap R| - |(P \cap Q) \cup (P \cap R)|$$

$$= |P| + |Q| + |R| - |Q \cap R| - (|P \cap Q| + |P \cap R| - |P \cap Q \cap R|)$$

$$= |P| + |Q| + |R| - |Q \cap R| - |P \cap Q| - |P \cap R| + |P \cap Q \cap R|$$

3.3.5. Examples Based On Principle of Inclusion-Exclusion (Pie)

Example 1: Find the number of integers between 1 and 100 that are divisible by 2 or 3.

Solution: Let: $A = \{Multiples\ of\ 2\}$ and $B = \{Multiples\ of\ 3\}$

By principle of inclusion-exclusion (PIE),

$|A \cup B| = |A| + |B| - |A \cap B|$

NOTE

$|A| = 50$ (Multiples of 2)

$|B| = 33$ (Multiples of 3)

$|A \cap B| = 16$ (Multiples of 6, LCM of 2 and 3)

Using PIE,

$|A \cup B| = 50 + 33 - 16 = 67$

Thus, there are 67 integers divisible by 2 or 3 between 1 and 100.

Example 2: How many integers between 1 and 200 are divisible by 4, 5, or 6?

Solution: Let: A ={Multiples of 4} = 50, B = {Multiples of 5} = 40 and

C = {Multiples of 6}= 33

By principle of inclusion-exclusion (PIE),

$$|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}| = |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}| - |\mathbf{B} \cap \mathbf{C}| - |\mathbf{A} \cap \mathbf{B}| - |\mathbf{A} \cap \mathbf{C}| + |\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}|$$

$|A \cap B| = 10$ (Multiples of 20)

$|A \cap C| = 16$ (Multiples of 12)

$|B \cap C| = 13$ (Multiples of 30)

$|A \cap B \cap C| = 3$ (Multiples of 60)

Using PIE,

$|A \cup B \cup C| = 50 + 40 + 33 - 10 - 16 - 13 + 3 = 87$

Thus, there are 87 integers divisible by 4, 5, or 6 between 1 and 200.

Example 3: How many integers between 1 and 200 are divisible by 4 or 5?

Solution: Let:

$$A = \{Multiples\ of\ 4\} = \frac{200}{4} = 50$$

$$B = \{Multiples\ of\ 5\} = \frac{200}{5} = 40$$

$$|A \cap B| = \{Multiples\ of\ 20\} = \frac{200}{20} = 10$$

By PIE,

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B| = 50 + 40 - 10 = 80$

Thus, there are 80 integers between 1 and 200 that are divisible by 4 or 5.

NOTE

Example 4: How many numbers between 1 and 500 are divisible by 7 or 11?

Solution: Let: $A = \{\text{Multiples of } 7\} = 71$ and $B = \{\text{Multiples of } 11\} = 45$

By principle of inclusion-exclusion (PIE),

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = 71$

$|B| = 45$

$|A \cap B| = 6$

By PIE,

$|A \cup B| = 71 + 45 - 6 = 110$

Thus, there are 110 numbers divisible by 7 or 11 between 1 and 500**.**

Example 5: How many integers between 1 and 500 are divisible by 3, 5, or 7?

Solution: Let:

$$A = \{Multiples\ of\ 3\} = \left[\frac{500}{3}\right] = 166$$

$$B = \{Multiples\ of\ 5\} = \left[\frac{500}{5}\right] = 100$$

$$C = \{Multiples\ of\ 7\} = \left[\frac{500}{7}\right] = 71$$

Step 1: Calculate Pairwise Intersections

$$|A \cap B| = \{Multiples\ of\ LCM(3, 5) = 15\} = \left[\frac{500}{15}\right] = 33$$

$$|A \cap C| = \{Multiples\ of\ LCM(3, 7) = 21\} = \left[\frac{500}{21}\right] = 23$$

$$|B \cap C| = \{Multiples\ of\ LCM(5, 7) = 35\} = \left[\frac{500}{35}\right] = 14$$

Step 2: Calculate Triple Intersection

$$|A \cap B \cap C| = \{Multiples\ of\ LCM(3, 5, 7) = 105\} = \left[\frac{500}{105}\right] = 4$$

Step 3: Apply the PIE Formula

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$|A \cup B \cup C| = 166 + 100 + 71 - 33 - 23 - 14 + 4 = 271$

Thus, there are 271 integers between 1 and 500 that are divisible by 3, 5, or 7.

NOTE

Example 6: How many integers between 1 and 1000 are divisible by 3, 5, or 7?

Solution: Let:

A={Multiples of 3}=[1000/3]=333

B={Multiples of 5}=[1000/5]=200

C={Multiples of 7}=[1000/7]=142

Step 1: Find pairwise intersections:

|A∩B|=[1000/15]=66

|A∩C|=[1000/21]=47

|B∩C|=[1000/35]=28

Step 2: Find the intersection of all three sets:

|A∩B∩C|=[1000/105]=9

Step 3: Apply the Principle of Inclusion-Exclusion (PIE):

|A∪B∪C|=|A|+|B|+|C|-|A∩B|-|A∩C|-|B∩C|+|A∩B∩C|

|A∪B∪C|=333+200+142-66-47-28+9=543

Thus, there are 543 integers between 1 and 1000 that are divisible by 3, 5, or 7.

Example 7: How many numbers between 1 and 1000 are divisible by 4, 6, or 8?

Solution: Let:

A={Multiples of 4}=[1000/4]=250

B={Multiples of 6}=[1000/6]=166

C={Multiples of 8}=[1000/8]=125

Step 1: Calculate Pairwise Intersections

|A∩B|={Multiples of LCM(4, 6) = 12}=[1000/12]=83

|A∩C|={Multiples of LCM(4, 8) = 8}=[1000/8]=125

|B∩C|={Multiples of LCM(6, 8) = 24}=[1000/24]=41

Step 2: Calculate Triple Intersection

|A∩B∩C|={Multiples of LCM(4, 6, 8) = 24}=41

Step 3: Apply the PIE Formula

|A∪B∪C|=|A|+|B|+|C|-|A∩B|-|A∩C|-|B∩C|+|A∩B∩C|

|A∪B∪C|=250+166+125-83-125-41+41=333

NOTE

Thus, there are 333 numbers between 1 and 1000 that are divisible by 4, 6, or 8.

Example 8: How many integers between 1 and 1000 are divisible by 4, 5, or 6?

Solution: Let:

A={Multiples of 4}=[1000/4]=250

B={Multiples of 5}=[1000/5]=200

C={Multiples of 6}=[1000/6]=166

Step 1: Calculate Pairwise Intersections

|A∩B|={Multiples of LCM(4, 5) = 20}=[1000/20]=50

|A∩C|={Multiples of LCM(4, 6) = 12}=[1000/12]=83

|B∩C|={Multiples of LCM(5, 6) = 30}=[1000/30]=33

Step 2: Calculate Triple Intersection

|A∩B∩C|={Multiples of LCM(4, 5, 6) = 60}=[1000/60]=16

Step 3: Apply the Principle of Inclusion-Exclusion

|A∪B∪C|=|A|+|B|+|C|-|A∩B|-|A∩C|-|B∩C|+|A∩B∩C|

|A∪B∪C|=250+200+166-50-83-33+16=466

Thus, there are 466 integers between 1 and 1000 that are divisible by 4, 5, or 6.

Example-8 How many elements does P∩Q have if P and Q are two sets with 40 elements each, P having 22 elements and Q having 28?

Solution: Given n (P ∪ Q)= 40,n (P)= 22,n (Q) = 28

By formula n (P U Q) = n (P) + n (Q) – n (P ∩ Q)

40= 22 + 28 – n (P ∩ Q)

40 =50 – n (P ∩ Q)

∴ n (P ∩ Q)= 50 – 40 =10

Example-9 Determine n(A ∩ B) given two finite sets A and B such that

n (A) = 20,n (B) = 28,and n (A ∪ B) = 36.

Solution: By formula n(A □ B) =n(A) + n(B) – n (A ∩ B).

n (A)= 20,n (B)= 28,and n (A ∪ B)= 36.

n(A ∩ B)=20 + 28 - 36=48 - 36=12

Example-10 In a class of 40 students, 15 like to play cricket and football and 20 like to play cricket. How many like to play football only but not cricket ?

NOTE

Solution: Let C= Students who like cricket

F= Students who like football

C∩F= Students who like cricket and football both

C-F = Students who like cricket only

F-C= Students who like football only.

n (C)=20,n(C ∩ F)= 15,n (C U F)= 40,n (F)=?

n (C ∪ F)=n(C)+ n(F)– n (C ∩ F)

40=20 + n (F)- 1

40=5 + n (F)

40 – 5 = n (F)

∴ n (F) =35

Now, n (F ∪ C)= n(F)– n (C ∩ F)

=35 – 15 = 20

∴ Number of students who like football only but not cricket =20

Example 11: In a class of 60 students, 25 like Mathematics, 30 like Science, and 20 like English. If 10 like both Mathematics and Science, 8 like both Mathematics and English, 6 like both Science and English, and 3 like all three subjects, how many students like at least one subject?

Solution: Let: M = {Students who like Mathematics} 25

S={Students who like Science}=30

E={Students who like English}=20

By principle of inclusion-exclusion (PIE),

$$|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}| = |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}| - |\mathbf{B} \cap \mathbf{C}| - |\mathbf{A} \cap \mathbf{B}| - |\mathbf{A} \cap \mathbf{C}| \ |\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}|$$

|M∩S|=10

|M∩E|=8

|S∩E|=6

|M∩S∩E|=3

By PIE,

|M∪S∪E|=25+30+20-10-8-6+3=54

Answer: 54 students like at least one subject.

Example 12: In a group of 100 students, 40 play cricket, 50 play football and 30 play volleyball. If 20 play both cricket and football, 15

NOTE

play both cricket and volleyball, and 10 play both football and volleyball, and 5 play all three games, how many students play at least one game?

Solution:

Let: C={Cricket players}=40

F={Football players}=50

V={Volleyball players}=30

By principle of inclusion-exclusion (PIE),

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| \; |A \cap B \cap C|$$

|C∩F|=20

|C∩V|=15

|F∩V|=10

|C∩F∩V|=5

By PIE, |C∪F∪V|=40+50+30-20-15-10+5=80

Thus, there are 80 students play at least one game.

Example 13: In a group of 50 students, 20 like Mathematics, 18 like science, and 15 like English. If 8 like both Mathematics and Science, 6 like both Mathematics and English, 5 like both Science and English, and 3 like all three subjects, how many students like at least one subject?

Solution: Let:

M={Mathematics lovers}=20

S={Science lovers}=18

E={English lovers}=15

By principle of inclusion-exclusion (PIE),

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

|M∩S|=8

|M∩E|=6

|S∩E|=5

|M∩S∩E|=3

By PIE, |M∪S∪E|=20+18+15-8-6-5+3=37

Thus, there are 37 students like at least one subject.

Example 14: In a survey of 120 students, 60 students like Mathematics, 50 students like Physics, and 40 students like Chemistry. If 30 students like both Mathematics and Physics, 20 students like both Mathematics

NOTE

and Chemistry, 15 students like both Physics and Chemistry, and 10 students like all three subjects, how many students like at least one subject?

Solution: Let:

M={Students who like Mathematics}=60

P={Students who like Physics}=50

C={Students who like Chemistry}=40

Step 1: Find the number of students in pairwise intersections:

$|M \cap P|=30$

$|M \cap C|=20$

$|P \cap C|=15$

Step 2: Find the number of students in all three subjects:

$|M \cap P \cap C|=10$

Step 3: Apply the Principle of Inclusion-Exclusion (PIE):

$|M \cup P \cup C|=|M|+|P|+|C|-|M \cap P|-|M \cap C|-|P \cap C|+|M \cap P \cap C|$

$|M \cup P \cup C|=60+50+40-30-20-15+10=95$

Thus, there are 95 students like at least one subject.

Example 15: In a group of 200 students, 100 study Mathematics, 80 study Science, and 60 study English. If 40 study both Mathematics and Science, 30 study both Mathematics and English, 20 study both Science and English, and 10 study all three subjects, how many students study at least one subject?

Solution: Let:

M={Students who study Mathematics}=100

S={Students who study Science}=80

E={Students who study English}=60

Step 1: Calculate Pairwise Intersections

$|M \cap S|=40$

$|M \cap E|=30$

$|S \cap E|=20$

Step 2: Calculate Triple Intersection

$|M \cap S \cap E|=10$

Step 3: Apply the PIE Formula

$|M \cup S \cup E|=|M|+|S|+|E|-|M \cap S|-|M \cap E|-|S \cap E|+|M \cap S \cap E|$

|M∪S∪E|=100+80+60-40-30-20+10=160

NOTE

Thus, there are 160 students' study at least one subject.



Activity 3.1

Counting Overlapping Groups Using PIE

Activity Title: "Classroom Club Membership Counting"

Objective: To understand the Principle of Inclusion-Exclusion by identifying overlapping groups in a real-life scenario.

Materials Needed:

- Whiteboard or chart paper
- Markers
- Sticky notes (3 different colors)

Instructions:

1. **Step 1: Form Groups**
   - Divide the class into three groups:
     - Students who like Cricket (Group A)
     - Students who like Football (Group B)
     - Students who like Basketball (Group C)
2. **Step 2: Data Collection**
   - Ask students to write their names on sticky notes:
     - Red sticky notes for Cricket players
     - Blue sticky notes for Football players
     - Green sticky notes for Basketball players
3. **Step 3: Identify Overlaps**
   - Students who belong to two or more groups should combine their sticky notes accordingly (e.g., Red + Blue for students in both Cricket and Football).
4. **Step 4: Application of PIE**
   - Count the number of students in each category:
     - $|A|$= Number of Cricket players
     - $|B|$= Number of Football players
     - $|C|$= Number of Basketball players
     - $|A \cap B|$= Students in both Cricket and Football
     - $|A \cap C|$= Students in both Cricket and Basketball
     - $|B \cap C|$= Students in both Football and Basketball

NOTE

- $|A \cap B \cap C|$= Students in all three groups

5. **Step 5: Calculation**
   - **Use the PIE formula:**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C||$$

6. **Step 6: Discussion**
   - Discuss how PIE ensures no student is counted twice, ensuring an accurate count of students involved in at least one activity.

Learning Outcome:
Students will gain hands-on experience in applying the Principle of Inclusion-Exclusion to real-life counting problems.

☑ Check Your Progress 3.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. State the Principle of Inclusion-Exclusion (PIE) for two sets.

……

……

……

2. The Principle of Inclusion-Exclusion is used only for counting elements in disjoint sets. (True or False).

……

3. Which of the following formulas represents the Principle of Inclusion-Exclusion for two sets A and B?

(A) $|A \cup B| = |A| - |B| - |A \cap B|$

(B) $|A \cup B| = |A| + |B| - |A \cap B|$

(C) $|A \cup B| = |A| + |B|$

(D) $|A \cup B| = |A| \times |B|$

4. The Principle of Inclusion-Exclusion is commonly applied to solve problems involving _____ conditions.

3.4 SUMMARY

The unit on the Principle of Inclusion-Exclusion (PIE) introduced the fundamental methods used to count elements in overlapping sets accurately and avoid double counting. It covered the definition of the Principle of Inclusion-Exclusion, the basic theorems associated with it, and the use of Venn diagrams to visually represent relationships among

NOTE

sets. The unit also explained proofs of important theorems and demonstrated how PIE can be applied through various examples and problem-solving techniques. Students learned how to calculate unions and intersections of sets, analyze overlapping data, and apply logical counting methods in mathematics and probability. These concepts are highly useful in areas such as computer science, statistics, data analysis, survey interpretation, and logical reasoning. Understanding PIE strengthens analytical and problem-solving skills and forms a strong foundation for advanced topics in discrete mathematics and probability theory. Students are encouraged to practice additional numerical problems, explore complex set relationships, and use visual tools like Venn diagrams for deeper understanding and better application of the Principle of Inclusion-Exclusion in real-life situations and future studies.

3.5 GLOSSARY

- **Inclusion-Exclusion Principle:** A combinatorial method used to calculate the number of elements in the union of multiple sets by adding individual set sizes and subtracting overlapping regions.
- **Union:** The set containing all distinct elements from two or more sets.
- **Intersection:** The set containing common elements between two or more sets.
- **Cardinality:** The total number of elements in each set.
- Overlapping Sets: Sets that share one or more common elements.
- **Disjoint Sets:** Sets that have no elements in common, meaning their intersection is empty.
- **Subset:** A set in which all its elements are also present in another set.
- **Venn Diagram:** A graphical representation used to illustrate the relationships between sets, often useful for visualizing PIE problems.
- **LCM (Least Common Multiple):** The smallest multiple that is divisible by two or more numbers, often used in PIE problems involving multiples.
- **Counting Principle:** A mathematical technique that helps calculate the total number of possible outcomes in a set of events, often combined with PIE for accuracy.

3.6 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 3.1

1. **Answer:** The Principle of Inclusion-Exclusion for three sets A, B, and C states:

NOTE

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

It is used to find the number of elements in the union of three sets by including individual sets, excluding pairwise intersections, and including the intersection of all three.

2. Answer: False.

The Principle of Inclusion-Exclusion is especially useful for non-disjoint sets, where some elements may be common among sets.

3. Answer: Correct Option: (B)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This formula accounts for the elements counted twice in both A and B.

4. Answer: The Principle of Inclusion-Exclusion is commonly applied to solve problems involving overlapping conditions.

3.7 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1243	

3.8 REFERENCE

- https://byjus.com/jee/principle-of-inclusion-and-exclusion-and-derangement/
- https://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle
- https://tjyusun.com/mat202/sec-inclusion-exclusion
- https://www.isical.ac.in/~sush/Discrete-maths-2014/Principle%20of%20inclusion%20and%20exclusion.pdf
- https://artofproblemsolving.com/wiki/index.php/
- Principle_of_InclusionExclusion?srsltid=AfmBOor6tm4LwmNdPrS4y7LlRgzBYJjx5Lg2QkYfjirArjWuuuChSZ6b
- https://math.stackexchange.com/questions/3793030/questions-relating-to-inclusion-exclusion-principle
- https://mathspace.co/textbooks/syllabuses/Syllabus-501/topics/Topic-9608/subtopics/Subtopic-127624/

3.9 EXERCISE

NOTE

Part 1

A. Multiple choice question

1. **How many integers between 1 and 100 are divisible by 2, 3, or 5?**

   a) 67

   b) 65

   c) 70

   d) 68

2. **In a group of 100 students, 40 study Mathematics, 35 study Physics, and 30 study Chemistry. If 15 study both Mathematics and Physics, 10 study both Mathematics and Chemistry, 8 study both Physics and Chemistry, and 5 study all three subjects, how many students study at least one subject?**

   a) 75

   b) 70

   c) 80

   d) 85

3. **How many integers between 1 and 500 are divisible by 4, 5, or 6?**

   a) 270

   b) 280

   c) 275

   d) 26

4. **In a class of 200 students, 90 play Cricket, 70 play Football, and 50 play Basketball. If 30 play both Cricket and Football, 20 play both Cricket and Basketball, 15 play both Football and Basketball, and 10 play all three sports, how many students play at least one sport?**

   a) 145

   b) 140

   c) 135

   d) 150

NOTE

5. How many positive integers less than 1000 are divisible by 3, 5, or 7?

a) 477

b) 480

c) 485

d) 490

Answers with Explanations:

1. b) 65

Explanation: Let:

A={Multiples of 2}=⌊1002⌋=50

B={Multiples of 3}=⌊1003⌋=33

C={Multiples of 5}=⌊1005⌋=20

Pairwise Intersections:

$$|A \cap B| = \left[\frac{100}{6}\right] = 16$$

$$|A \cap C| = \left[\frac{100}{10}\right] = 10$$

$$|B \cap C| = \left[\frac{100}{15}\right] = 6$$

Triple Intersection:

$$|A \cap B \cap C| = \left[\frac{100}{30}\right] = 3$$

Now apply the PIE formula:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

=50+33+20-16-10-6+3=65

2. c) 80

Explanation:

Using the PIE formula:

|M∪P∪C|=|M|+|P|+|C|−|M∩P|−|M∩C|−|P∩C|+|M∩P∩C|

=40+35+30−15−10−8+5=80

3 c) 275

Explanation: Let:

A={Multiples of 4}=125

B={Multiples of 5}=100

NOTE

C={Multiples of 6}=83

Intersections:

$$|A \cap B| = \left[\frac{500}{20}\right] = 25$$

$$|A \cap C| = \left[\frac{500}{12}\right] = 41$$

$$|B \cap C| = \left[\frac{500}{30}\right] = 16$$

$$|A \cap B \cap C| = \left[\frac{500}{60}\right] = 8$$

Using PIE:

|A∪B∪C|=125+100+83-25-41-16+8=275

4. a) 145

Explanation:

Using the PIE formula:

|C∪F∪B|=90+70+50−30−20−15+10=145

5. a) 477

Explanation: Let:

A={Multiples of 3}=333

B={Multiples of 5}=200

C={Multiples of 7}=142

Intersections:

$$|A \cap B| = \left[\frac{1000}{15}\right] = 66$$

$$|A \cap C| = \left[\frac{1000}{21}\right] = 47$$

$$|B \cap C| = \left[\frac{1000}{35}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{105}\right] = 9$$

Now apply the PIE formula:

|A∪B∪C|=333+200+142-66-47-28+9=477

B. Short answer type question

1. State the Principle of Inclusion-Exclusion for two sets.

NOTE

2. How is the Principle of Inclusion-Exclusion useful in counting problems?
3. What is the formula for the union of three sets using PIE?
4. Explain why the Principle of Inclusion-Exclusion alternates between addition and subtraction.
5. In PIE, what does the term |A∩B∩C| represent?

Part 2

C. Long answer type question (200 words)

1. How many integers between 1 and 1000 are divisible by 4, 5, or 6? Explain using the Principle of Inclusion-Exclusion.
2. In a school of 200 students, 90 study Mathematics, 70 study Physics, and 60 study Chemistry. If 30 students' study both Mathematics and Physics, 25 study both Mathematics and Chemistry, 20 study both Physics and Chemistry, and 15 study all three subjects, how many students study at least one subject?
3. Out of 150 people surveyed, 60 like Tea, 70 like Coffee, and 50 like Juice. If 30 like both Tea and Coffee, 20 like both Tea and Juice, 15 like both Coffee and Juice, and 10 like all three, how many like at least one beverage?
4. Find the number of positive integers up to 500 that are divisible by 2, 3, or 5. Explain using PIE.
5. A college has 300 students. Among them, 120 play Cricket, 100 play Football, and 80 play Basketball. If 40 play both Cricket and Football, 30 play both Cricket and Basketball, 25 play both Football and Basketball, and 15 play all three sports, how many students participate in at least one sport?

Part 3

D. Long answer type question (300 words)

1. **In a city survey of 500 households, it was found that:**

   210 households have a car,

   180 households have a bike,

   150 households have a scooter.

   Additionally,

   80 households have both a car and a bike,

   60 households have both a car and a scooter,

   50 households have both a bike and a scooter,

   30 households have all three vehicles.

   Using the Principle of Inclusion-Exclusion, calculate:

NOTE

a) The number of households that have at least one vehicle.

b) The number of households that have none of these vehicles.

2. A class of 300 students was surveyed, and the following data was collected:

120 students play Chess,

150 students play Badminton,

130 students play Table Tennis.

Among these,

40 students play both Chess and Badminton,

50 students play both Chess and Table Tennis,

45 students play both Badminton and Table Tennis,

20 students play all three games.

Using PIE, determine:

a) The number of students who play at least one game.

b) The number of students who do not play any of these three games.

3. A company surveyed 400 employees about their skills and found that:

200 employees know Python,

180 employees know Java,

150 employees know C++.

Among them,

80 employees know both Python and Java,

60 employees know both Python and C++,

70 employees know both Java and C++,

40 employees know all three languages.

Using PIE, calculate:

a) The number of employees who know at least one programming language

b) The number of employees who do not know any of these three languages.

4. In a group of 600 students:

250 students like Mathematics,

200 students like Physics,

180 students like Chemistry.

Among these,

90 students like both Mathematics and Physics,

NOTE

80 students like both Mathematics and Chemistry,
70 students like both Physics and Chemistry,
40 students like all three subjects.

Using PIE, determine:

a) The number of students who like at least one subject.
b) The number of students who like exactly two subjects.

5. **A music festival was attended by 450 people. The following data was collected:**

200 people attended the Rock concert,
180 attended the Jazz concert,
150 attended the Pop concert.

Among them,

80 attended both Rock and Jazz,
70 attended both Rock and Pop,
60 attended both Jazz and Pop,
40 attended all three concerts.

Using PIE, calculate:

a) The total number of attendees who attended at least one concert.
b) The number of attendees who attended exactly two concerts.

UNIT 4
DERANGEMENT

NOTE

UNIT STRUCTURE

4.1 Introduction

4.2 Objective

4.3 Derangement

- **4.3.1 Definition of Permutation and Combination**
- **4.3.2 Problems on Permutation and Combination**
- **4.3.3 Definition of Derangement**
- **4.3.4 Problems on Derangement**

4.4 Summary

4.5 Glossary

4.6 Answer to check your progress

4.7 Suggested readings - OER

4.8 Reference

4.9 Exercise

4.1 INTRODUCTION

In the previous unit, you explored the basics of set operations and counting principles, including permutations and combinations.

These concepts help in arranging and selecting objects in different ways. Now, we introduce the concept of derangement, which deals with arrangements where no object appears in its original position. This principle is essential in combinatorics and real-world applications.

Building on this knowledge, derangement is a special type of permutation where no element remains in its original place. It is commonly used in problems like assigning tasks randomly without repetition. By solving problems on permutation, combination, and derangement, we develop a deeper understanding of advanced counting techniques.

This unit bridges basic counting principles with more complex arrangements using derangement. It covers the definition, fundamental properties, and problem-solving techniques related to derangement. Understanding this concept is crucial for solving real-world problems in probability, cryptography, and scheduling.

NOTE

4.2 OBJECTIVE

After studying this unit, you will be able to:

- To develop a solid foundation in the concepts of permutations and combinations, which serve as the basis for exploring derangements.
- To grasp the definition and properties of derangements, understanding how they differ from standard permutations.
- To apply the concept of derangements to practical scenarios, such as mismatched assignments, cryptographic problems, and logistical challenges.
- To analyze and compute arrangements under specific constraints, enhancing problem-solving skills in restricted combinatorial settings.
- To strengthen analytical and logical thinking by solving problems involving derangements and their applications in various mathematical and real-life contexts.

4.3 DERANGEMENT

4.3.1 Definition of Permutation and Combination?

A permutation is a distinct arrangement of numerous items, taken one at a time. The combination is a method of picking elements from a collection, the sequence of selection is irrelevant. That is, a permutation is used for a list of data and an arrangement is used for a group of data.

Permutation Formula: Selecting r elements without replacement from a set of n things—where order matters—is known as a permutation.

nPr = (n!) / (n − r)!

Combination Formula: The selection of r items from a set of n items, when order is irrelevant and replacement is not required, is called a combination.

$$ {}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} $$

Example 1: Find the number of permutations and combinations if $n = 12$ and

$r = 2$.

Solution: Given, $n = 12, r = 2$, Using the formula given above:

Permutation: $\boldsymbol{nPr} = (\boldsymbol{n}!) / (\boldsymbol{n} - \boldsymbol{r})!$

$= (12!) / (12 - 2)!$

$= 12! / 10!$

NOTE

$= (12\ x\ 11\ x\ 10!)/\ 10!\ =\ 132$

Combination: $\mathbf{nC_r} = \frac{\mathbf{n!}}{\mathbf{r!(n-r)!}}$

$$= \frac{12!}{2!\,(12-2)!}$$

$$= \frac{12!}{2!\,(10)!}$$

$$= \frac{12 \times 11 \times 10!}{2!\,(10)!} = 66$$

4.3.2 Problems on Permutation and Combination

A. Permutation Problems:

1. Arranging 5 books on ashelf:
   - The number of ways to arrange 5 books in a row is given by **5**!.
   - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
2. Forming a 3-digit number using the digits 1, 2, 3, 4, 5 (without repetition):
   - We select 3 digits and arrange them.
   - The number of ways is $\boldsymbol{P(5,3)} = \frac{\mathbf{5!}}{\mathbf{(5-3)!}}$
   - $P(5,3) = \frac{5!}{2!} = \frac{5\times4\times3\times2!}{2!} = 5 \times 4 \times 3 = 60$
3. Selecting and arranging 3 students in a row from a group of 10:
   - The number of ways to arrange 3 students from 10 is

$$\boldsymbol{P(10,3)} = \frac{\mathbf{10!}}{\mathbf{(10-3)!}}$$

   - $P(10,3) = \frac{10!}{7!} = \frac{10\times9\times8\times7!}{7!} = 10 \times 9 \times 8 = 720$
4. Creating different words using all the letters of "MATH":
   - The number of ways to arrange all 4 letters is **4!**
   - $4! = 4 \times 3 \times 2 \times 1 = 24$

B. Combination Problems:

1. Selecting 3 books from a set of 10:
   - The number of ways to choose 3 books is

$$\boldsymbol{C(10,3)} = \frac{\mathbf{10!}}{\mathbf{3!\,(10-3)!}}$$

NOTE

- $C(10,3) = \frac{10!}{3!7!} = \frac{10\times9\times8}{3\times2\times1} = 120$

2. Choosing 2 players for a doubles match from a team of 6:
   - The number of ways to choose 2 players is
   $$\boldsymbol{C(6,2) = \frac{6!}{2!\,(6-2)!}}$$
   - $C(6,2) = \frac{6!}{2!4!} = \frac{6\times5}{2\times1} = 15$
3. Forming a committee of 4 members from a group of 12 people:
   - The number of ways to choose 4 members is $\boldsymbol{C(12,4)} = \frac{\boldsymbol{12!}}{\boldsymbol{4!(12-4)!}}$
   - $C(12,4) = \frac{12\times11\times10\times9}{4\times3\times2\times1} = 495$
4. Picking any 5 cards from a deck of 52:
   - The number of ways to select 5 cards is
   $$\boldsymbol{C(52,5) = \frac{52!}{5!\,(52-5)!}}$$
   - $C(52,5) = \frac{52\times51\times50\times49\times48}{5\times4\times3\times2\times1} = 2{,}598{,}960$

4.3.3 Definition of Derangement

Derangement is a permutational arrangement that has no fixed points. In other terms, derangement can be defined as the permutation of the elements of a specific set so that every component of the entire set is found in its original location. Six persons in six seats can be arranged in six different ways. Now, how many ways can you rearrange them so that none of them are in their original positions? The question asks us for the deranged value of six people.

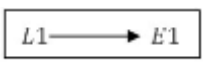
Let us take another example involving the insertion of letters into their respective envelopes.

Case 1: Consider one letter (L1) and one envelope (E1).

How many ways can the letter be put in the wrong envelope?

The answer will be zero because there is only one letter and one envelope.

There is no possibility to insert a single letter into the incorrect envelope.



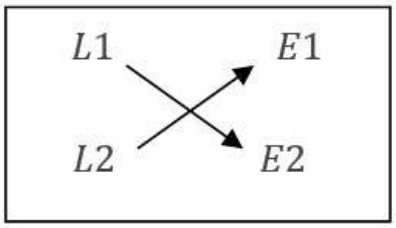
Case 2: Consider two letters, L1 and L2, and their respective envelopes, E1 and E2.

NOTE

In how many ways can each letter be put into the wrong envelope?

Will there only be one case?

When L1 is moved to E2, L2 will immediately go to E1.



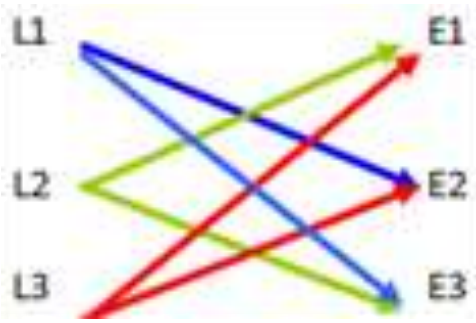
Thus, two letters can be placed in two envelopes so that none of them go into their proper envelopes in only one direction.

Case 3: Consider a case with three letters (L1, L2, L3) and three envelopes

(E1, E2, E3).

The number of ways a letter can be inserted into the wrong envelope is two, as demonstrated below:

(**Example:** L1 can be placed in E2 or E3)



In two distinct methods, each letter can be placed in three envelopes so that none of them end up in the correct envelope.

There are 9 methods to derange four letters, 44 ways to derange five, and 265 ways to derange six.

The calculation for the derangement value of "N" is as follows:

$$D_N = N! \, [1 - 1/1! + 1/2! - 1/3! \ldots\ldots 1/N!],$$

where "N" is the number of items to be de-arranged.

Example 1: In how many ways can you put 7 letters into their respective envelopes such that exactly 3 go into the right envelope?

Solution:

No of ways in which the 3 correct envelopes can be selected = $7C_3 = 35$

Derangement of the remaining 4 envelopes & letters = 9

NOTE

(derangement value for 4 is 9)

Total No of ways of arrangement = $9\,x\,35 = 315$.

Example 2: In how many ways can you form a dancing couple from 3 boys and

3 girls so that no boy dances with his respective girlfriend?

Solution: This is clearly a case of derangement of 3 boys and 3 girls.

The value can be calculated as D=3! (1-1/1! +1/2! -1/3!) = 6(2/6) =2 ways.

4.3.4 Problems on Derangement

Problem 1: Find the number of ways to derange (completely misplace) 4 distinct objects.

Solution:

The formula for the number of derangements ($!n$) of n objects is:

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

For n=4

$$!4 = 4! \sum_{k=0}^{4} \frac{(-1)^k}{k!}$$

Calculating step by step:

$$!4 = 24 \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 24 \times (1 - 1 + 0.5 - 0.1667 + 0.0417)$$

$$= 24 \times 0.375 = 9$$

Thus, the number of ways to derange 4 objects is **9**.

Problem 2: A mailman delivers 5 letters to 5 different people, but he places them randomly in their envelopes. In how many ways can none of the letters be placed in the correct envelope?

Solution:

This is a direct application of the derangement formula for $n = 5$

$$!5 = 5! \sum_{k=0}^{5} \frac{(-1)^k}{k!}$$

$$= 120 \times \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$= 120 \times (1 - 1 + 0.5 - 0.1667 + 0.0417 - 0.0083)$$

$$= 120 \times 0.3667 = 44$$

So, there are 44 ways to completely misplace all the letters.

NOTE

Problem 3: A teacher randomly distributes 6 assignments to 6 students. What is the probability that no student receives their correct assignment?

Solution:

The probability of a derangement is given by:

$P(\text{derangement}) = \frac{!n}{n!}$

For $n = 6$ we first compute $!6$

$$!6 = 6! \sum_{k=0}^{6} \frac{(-1)^k}{k!}$$

$$= 720 \times \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 720 \times (0.3681) = 265$$

Now, computing probability:

$P(\text{derangement}) = \frac{265}{720} \approx 0.3681$

Problem 4: A hotel mistakenly gives 7 guests their room keys at random. How many ways can they receive keys such that no one gets their own key?

Solution: Using the derangement formula for $n = 7$

$$!7 = 7! \sum_{k=0}^{7} \frac{(-1)^k}{k!}$$

$$= 5040 \times \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right)$$

$$= 5040 \times (0.3679) = 1854$$

Thus, the number of ways to completely misassign all 7 keys is 1854.

Problem 5: There are 4 guests at a dinner party. Their coats are randomly shuffled. In how many ways can exactly 2 guests receive their correct coat while the other 2 do not?

Solution:

Step 1: Choose 2 guests who will receive their correct coat.

There are $C(4,2) = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = 6\ ways.$

Step 2: The remaining 2 guests must receive incorrect coats.

The number of ways to derange 2 objects is $!2 = 1$.

Step 3 Multiply the choices

$$6 \times 1 = 6$$

So, there are 6 ways in which exactly 2 guests receive the correct coat while the other 2 do not.

NOTE

Activity 4.1

Exploring Permutations, Combinations, and Derangement

Objective:

To understand the concepts of permutations, combinations, and derangement through problem-solving and real-world applications.

Instructions:

1. **Part 1: Understanding Permutation & Combination**
   - List 3 real-life scenarios where permutation is used.
   - List 3 real-life scenarios where combination is used.
   - Solve the following problems:
     a) In how many ways can 5 different books be arranged on a shelf?
     b) A committee of 3 members is to be selected from a group of 7 people. In how many ways can this be done?
2. **Part 2: Exploring Derangement**
   - Consider 4 letters (A, B, C, D) placed in 4 envelopes labeled $(1, 2, 3, 4)$. How many ways can they be placed so that no letter is in its correct envelope?
   - Solve: In how many ways can 5 people sit in 5 chairs such that no one sits in their assigned seat?
3. **Reflection Questions:**
   - What are the key differences between permutations and combinations?
   - How does derangement differ from standard permutations?
   - Discuss how derangement can be applied in a real-world scenario, such as secret santa gift exchanges.

Expected Outcome:

Students will gain practical problem-solving skills in permutations, combinations, and derangement and understand their real-life applications.

☑ Check Your Progress 4.1

Notes: a) Space is given below for your answer.
b) Check your answer with the one given at the end of this unit.

1. Define derangement and provide an example.

……
……
……

2. Every permutation of a set is a derangement. (True or False).

.....

3. What is the value of !3 (the number of derangements for 3 objects)?

a) 2 b) 3 c) 4 d) 6

4. The number of derangements of n objects is denoted by _____

4.4 SUMMARY

This unit introduced the concept of n!n!n! permutations and combinations, which are fundamental techniques used in counting and arranging objects in mathematics. Students learned the definition of permutation as the arrangement of objects in a specific order and combination as the selection of objects where order does not matter. Various numerical problems helped develop the ability to calculate arrangements and selections using standard formulas and logical reasoning. The unit also explained how these concepts are applied in solving practical counting problems related to seating arrangements, team selection, scheduling, and probability.

The unit further focused on the concept of derangement, which refers to arrangements in which no object remains in its original position. Students explored the definition, properties, and methods of solving derangement problems through examples and exercises. By practicing these problems, learners improved their analytical thinking and developed a better understanding of advanced counting principles. Important formulas and techniques were used to determine the number of incorrect arrangements in different situations.

A major learning outcome of this unit is the ability to distinguish between permutations, combinations, and derangements and apply the correct method according to the problem requirement. These concepts are highly relevant in fields such as computer science, cryptography, probability theory, operations research, and data analysis, where systematic counting and arrangement methods are essential. Understanding derangements also supports deeper study in discrete mathematics and combinatorics.

Students are encouraged to further explore advanced topics such as circular permutations, probability applications of derangements, and real-life optimization problems. Regular practice of numerical problems and logical reasoning exercises will strengthen problem-solving skills and improve mathematical accuracy and confidence.

NOTE

4.5 GLOSSARY

- **Permutation**: The arrangement of objects in a specific order. For example, arranging the letters "ABC" as "CAB" or "BCA" are permutations.
- **Combination**: The selection of objects without considering their order. For instance, choosing two letters from "ABC" results in combinations like "AB" or "AC."
- **Derangement**: A special type of permutation where no element appears in its original position.
- **Subfactorial** ($!n$): The notation used to represent the number of derangements for n objects.
- **Factorial** ($n!$): The product of all positive integers up to n, used extensively in permutations and combinations.
- **Recursive Formula**: A mathematical expression that calculates the current term based on previous terms, such as the recursive formula for derangements.
- **Inclusion-Exclusion Principle**: A method used in combinatorics to calculate derangements by systematically adding and subtracting overlapping cases.
- **Constraint**: A restriction or condition applied to a problem, like the condition in derangements that no object remains in its original position.
- **Probability of Derangement**: The likelihood that a random permutation of n objects is a derangement, which asymptotically approaches $\frac{1}{e}$.
- **Real-World Application**: Practical scenarios where derangements are applied, such as assigning tasks without overlapping responsibilities or distributing gifts without self-selection.

4.6 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 4.1

1. **Answer:** A derangement is a permutation of a set in which no element appears in its original position.

   Example: For the set $\{1, 2, 3\}$, the derangements are:

   $(2, 3, 1)$ and $(3, 1, 2)$ — because in both cases, no number is in its original position.

2. **Answer:** False.

Not every permutation is a derangement. A derangement requires that no element stays in its original position, which is not true for all permutations.

3. **Answer:** Correct option: (a) 2

   Explanation: For 3 objects, there are 2 derangements:

   - (2, 3, 1)
   - (3, 1, 2)

4. **Answer:** The number of derangements of n objects is denoted by $!n$ (read as "subfactorial n").

4.7 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1244	

4.8 REFERENCE

- https://www.geeksforgeeks.org/permutations-and-combinations/
- https://www.khanacademy.org/math/statistics-probability/counting-permutations-and-combinations
- https://byjus.com/free-cat-prep/derangement/
- https://www.geeksforgeeks.org/permutations-and combinations/

4.9 EXERCISE

Part 1

A. Multiple choice question

1. What is a derangement?

a) A combination where all elements are in their original position

b) A permutation where no element remains in its original position

c) A selection of elements in random order

d) A permutation with at least one element in its original position

NOTE

2. How many derangements are there for 3 objects (!3)?

a) 2

b) 3

c) 4

d) 6

3. Which of the following is the probability that a random permutation of n objects is a derangement as n becomes very large?

a) 0

b) 1

c) $\frac{1}{e}$

d) $\frac{1}{n!}$

4. What is the recursive formula for calculating derangements?

a) $!n = n! - !(n-1)$

b) $!n = (n-1)(!(n-1) + !(n-2))$

c) $!n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$

d) $!n = n! + n$

5. If there are 4 people, in how many ways can they exchange seats such that no one sits in their original seat?

a) 4

b) 8

c) 9

d) 24

Answers with Explanations:

1. **b) A permutation where no element remains in its original position**

Explanation: A derangement is defined as a special permutation where no element appears in its initial position, satisfying a specific constraint.

2. **a) 2**

Explanation: Using the formula $!n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$, for n=3:

$$!3 = 3!\,(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}) = 6(1 - 1 + 0.5 - 0.1667) = 2$$

NOTE

3. c) $\frac{1}{e}$

Explanation: The probability of a random permutation being a derangement approaches $\frac{1}{e}$ (~0.3679) as n increases due to the inclusion-exclusion principle in combinatorics.

4. b) $!n = (n-1)(!(n-1) + !(n-2))$

Explanation: This is the recursive formula for derangements, derived from combinatorial principles.

5. c) 9

Explanation: Using the derangement formula for n=4n = 4:

$$!4 = 4!\,(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}) = 24(1 - 1 + 0.5 - 0.1667 + 0.0417) = 9$$

B. Short answer type question

1. What is a derangement in permutations?
2. How many derangements exist for a set of 4 distinct objects?
3. What is the formula for the number of derangements of nn objects?
4. If there are 5 letters and 5 envelopes, in how many ways can the letters be placed in the wrong envelopes?
5. What is the derangement of a set of 2 elements?

Part 2

C. Long answer type question (200 words)

1. Define derangement and derive the formula for the number of derangements of n distinct objects. Use this formula to find the number of derangements for $n = 6$.
2. There are 5 letters and 5 corresponding envelopes. In how many ways can the letters be placed in the envelopes such that no letter goes into its correct envelope?
3. A company has 7 employees, each assigned a specific locker. Due to a mistake, the locker keys are randomly distributed. In how many ways can this distribution be done such that no employee gets the key to their own locker?
4. A teacher randomly distributes 4 different assignments to 4 students. What is the probability that no student receives their correct assignment? Solve using the concept of derangements and probability.
5. In a party, 6 guests check their coats at the entrance. When leaving, the coats are randomly returned. What is the probability

NOTE

that exactly 4 guests receive the wrong coat while 2 receive the correct coat?

Part 3

D. Long answer type question (300 words)

1. Derive the formula for the number of derangements $!n$ of n distinct objects using the principle of inclusion-exclusion. Explain each step in detail and apply the formula to find the number of derangements for $n = 7$

2. A post office mistakenly shuffles 6 letters into 6 envelopes, each labeled for a specific recipient.
   (a) Find the total number of ways the letters can be placed in the envelopes.
   (b) Find the number of ways in which none of the letters are placed in their correct envelopes (i.e., the number of derangements for $n = 6$).
   (c) What is the probability that all letters are placed incorrectly?

3. A professor randomly distributes 8 different assignments to 8 students.
   (a) Compute the number of ways the professor can distribute the assignments so that no student receives their own assignment.
   (b) Determine the probability that at least one student gets their correct assignment.
   (c) If there were 10 students instead of 8, how would the probability change?

4. A hotel has 10 guests; each assigned a unique locker. By mistake, the locker keys are shuffled and given randomly to the guests.
   (a) Find the number of ways the keys can be distributed so that no guest receives their correct key.
   (b) Determine the probability that exactly 2 guests receive the correct keys while the rest do not.

UNIT 5
PROPOSITION

NOTE

UNIT STRUCTURE

5.1 Introduction
5.2 Objective
5.3 Proposition
5.3.1 Definition of Proposition
5.3.2 Kind of sentences or propositional
5.4 Logical Connective
5.4.1 Truth Table
5.4.2 Types of Propositions based on truth value
5.5 Logical Equivalence
5.6 Summary
5.7 Glossary
5.8 Answer to check your progress
5.9 Suggested readings - OER
5.10 Reference
5.11 Exercise
5.12 Feedback form

5.1 INTRODUCTION

In the previous unit, you explored the basics of set theory, including operations on sets and their representations. These fundamental concepts are essential for understanding logical structures in mathematics. Now, we shift our focus to propositions, which form the foundation of mathematical logic. Propositions help us analyze statements based on their truth values.

Building on this knowledge, this unit introduces the concept of propositions, logical connectives, and truth tables. We will classify different types of propositions and examine how their truth values determine logical validity. Understanding logical equivalence is crucial in reasoning, proof construction, and computer science applications.

This unit bridges set theory with mathematical logic, enabling a deeper understanding of formal reasoning. By studying logical connectives, truth tables, and logical equivalence, you will develop skills in evaluating logical arguments. These concepts are widely used in artificial intelligence, computer programming, and automated decision-making.

NOTE

5.2 OBJECTIVE

After studying this unit, you will be able to:

- Analyse the concept of propositions and their truth values.
- Differentiate between simple and compound propositions.
- Apply logical connectives such as AND, OR, NOT, IMPLICATION, and BICONDITIONAL.
- Determine logical equivalence and identify tautologies and contradictions.
- Use propositional logic in mathematical proofs and computer science applications.
- Identify and differentiate between tautologies, contradictions, and contingencies based on their truth values.

5.3 PROPOSITION

5.3.1 Definition of Proposition

A proposition or sentence is defined as a declarative statement that may be True or False, but not both. If a proposition is a true statement, its True Values are True (represented by T), and if not, they are False (represented by F). Additionally, the numbers 1 and 0 are used to symbolize it. For proposition variables, lowercase letters (P, Q, R, etc.) are utilized. There are only two possible values for propositional variables: "true" (T or 1) and "false" (F or 0).

Example 1.

[a] Following sentences are propositions:

1. The sun rises in the East and sets in the West. (TRUE)
2. 'B' is a vowel. (FALSE)

[b] Following sentences are not propositions:

1. What time is it? (Interrogative)
2. Please go from here. (Request)
3. Close the door. (Command)
4. What a hot day! (Exclamation)

Example 2.

[a] Following sentences are propositions.:

1. Madhya Pradesh's capital is Bhopal. (TRUE)
2. 5 + 5 = 2. (FALSE)

[b] Following sentences are not propositions:

1. What is your name? **(**Interrogative)

NOTE

2. Go out and play. (Order)
3. May God help you? (Wish)
4. Come here. (Imperative)

5.3.2 Kind of Sentences or Propositional

Usually, sentences are of two kinds.

(a) Simple statement: A basic sentence has feeling connected.

(b) Compound statement: A compound statement is a single statement that contains many components or sub-statements.

5.4 LOGICAL CONNECTIVE

Logical connectives are symbols or words used to combine or modify statements in logical reasoning. They include conjunction (AND), disjunction (OR), negation (NOT), implication (IF-THEN), and biconditional (IF AND ONLY IF). Each connective defines a specific way in which the truth values of individual propositions are related, forming compound propositions. Logical connectives are the building blocks of formal logic and are crucial in constructing arguments, reasoning, and performing logical operations.

Logical connectives are symbols or words used in logic to connect individual propositions or statements to form more complex expressions. They define the relationships between the truth values of the propositions and are essential in constructing logical arguments and reasoning.

The symbols used to construct compound propositions are known as Logical Connectives or Logical Operators. To illustrate connectedness, utilize the fundamental symbols listed below.

TABLE OF LOGICAL CONNECTIVES WITH THEIR SYMBOLS

S.no.	Connective Word	Name of the Connective	Symbols	Order
1.	Not	Negation	$\sim$ or $\neg$	1
2.	and	Conjunction	$\wedge$	2
3.	or	Disjunction	$\vee$	3
4.	if……then	Conditional	$\Rightarrow$ or $\rightarrow$	4
5.	if and only if	Bi-conditional	$\Leftrightarrow$ or $\rightarrow$	5

Example 1: Express the notation of statements given below:

NOTE

(i) What soever I say, he refuses from that.

(ii) I shall reach the station in time, otherwise I shall miss the train.

(iii) I shall go to Delhi, but I shall not see the zoo.

(iv) In the position when he is ill, I shall wire.

(v) Whenever I take leave, I fall ill.

Solution:

(i) Let p ≡ I say something.

q ≡ He refuses.

Statement is: $p \rightarrow q$

("If I say something, he refuses")

(ii) Let p ≡ I reach the station in time.

q ≡ I miss the train.

Statement is: $p \vee q$

("I reach the station in time or I miss the train")

(iii) Let p ≡ I go to Delhi.

q ≡ I see the zoo.

Statement is: $p \wedge \sim q$

("I go to Delhi, but I don't see the zoo")

(iv) Let p ≡ He is ill.

q ≡ I wire.

Statement is: $p \Rightarrow q$

("If he is ill, I will wire")

(v) Let p ≡ I take leave.

q ≡ I fall ill.

Statement is: $p \Rightarrow q$

("If I take leave, I fall ill")

Examples 2:

1. Conjunction (AND)

- Symbol: ∧
- Meaning: The statement is true only if both propositions are true.
- Example:

NOTE

- Let **P** = "It is raining."
- Let **Q** = "I have an umbrella."
- **P ∧ Q**: "It is raining AND I have an umbrella."
- Truth Value: True only when both **P** and **Q** are true.

2. Disjunction (OR)

- Symbol: ∨
- Meaning: The statement is true if at least one of the propositions is true.
- Example:

- **P ∨ Q**: "It is raining OR I have an umbrella."
- Truth Value: True when either **P** or **Q** (or both) are true.

3. Negation (NOT)

- Symbol: ¬
- Meaning: The negation of a statement reverses its truth value.
- Example:
  - Let **P** = "It is raining."
  - **¬P**: "It is NOT raining."
  - Truth Value: True if **P** is false, and false if **P** is true.

4. Implication (IF-THEN)

- Symbol: →
- Meaning: If the first proposition is true, then the second must also be true for the statement to be true.
- Example:
  - **P → Q**: "If it is raining, then I have an umbrella."
  - Truth Value: False only if **P** is true and **Q** is false.

6. Biconditional (IF AND ONLY IF)

- Symbol: ↔
- Meaning: The statement is true when both propositions have the same truth value.
- Example:
  - **P ↔ Q**: "It is raining IF AND ONLY IF I have an umbrella."
  - Truth Value: True when both **P** and **Q** are either true or false.

NOTE

Logical connectives are widely used in mathematics, computer science, and reasoning to structure complex logical statements.

5.4.1 Truth Table

A truth table is a tabular representation used to determine the truth value of logical expressions based on all possible combinations of truth values of their constituent propositions. It systematically lists every possible scenario and evaluates the resulting truth values for compound statements. Truth tables are instrumental in verifying the validity of logical expressions, analyzing arguments, and understanding logical relationships between propositions. The number of truth functions is always 2^n, where n is the number of components in a statement.

Table. No. 1.1 Truth Table of Negation

Truth Table for ¬ p

p	¬ *p*
T	F
F	T

Example,

p: "Bhopal is the capital of M.P."

¬ **p**: "Bhopal is not the capital of M.P."

[b] Conjunction (∧):

Table. No. 1.2 Truth Table of Conjunction

Truth Table for $(p \wedge q)$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:

p: "Ram is healthy."

q: "He has blue eyes."

NOTE

p∧q: “Ram is healthy and he has blue eyes.”

Thus. when p and q both is ‘true’ then resultant value is ‘true.’

[c] Disjunction(∧)**:**

Table. No. 1.3 Truth Table of Disjunction

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

p: “It is cold.”

q: “It is raining.”

p ∨ q: “It is cold or it is raining.”

Therefore, the resultant value is "False" when both *p* and *q* are "False."

[d] Implication or Conditional(⇒ **or** ⟶)**:**

Table. No. 1.4 Truth Table of Conditional

Truth Table for $p \Rightarrow q$

p	**q**	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example,

p: “Today It is raining.”

q: “I will take tea.”

p ⇒ **q**: “If today it is raining then I will take tea.”

Therefore, the resultant value is "false" when both *p* and *q* are "false."

NOTE

[e] Biconditional or bi-implication (⇔ or ↔)**:**

Table. No. 1.5 Truth Table of Biconditional

Truth Table for $(p \Leftrightarrow q)$

p	q	$(p \Leftrightarrow q)$
T	T	T
T	F	F
F	T	F
F	F	T

Example,

p: "Today I will take tea."

q: "It is raining."

p ⇔ q: "Today I will take tea if and only if it is raining."

Therefore, the outcome is "true" if both *p* and *q* are "true." Additionally, if both *p* and *q* are "false," the outcome is "true."

[f] NAND(↑) **:** It means 'not of and'.

Table. No. 1.6 Truth Table of NAND

Truth Table for $(p \uparrow q)$

p	q	$(p \uparrow q)$
T	T	F
T	F	T
F	T	T
F	F	T

[g] NOR or Joint Denial (↓)**:** It means 'not of or'.

Table. No. 1.7 Truth Table of NOR

Truth Table for $(p \downarrow q)$

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

[h] XOR (Exclusive OR)(⊕)**:** A compound statement of *p* and *q* written as (p ⊕ q).

Table.No.1.6 Truth Table of XOR

Truth Table for $(p \oplus q)$

p	q	$(p \oplus q)$
T	T	F
T	F	T
F	T	T
F	F	F

5.4.2 Types of Propositions Based on Truth Value

Propositions are classified based on their truth values. A tautology is always true, regardless of the truth values of its components. A contradiction is always false, regardless of its components. A contingency may be true or false depending on the truth values of its components. This classification is significant in evaluating the nature of logical statements and identifying logical equivalences or inconsistencies within arguments.

There are three types of propositions when classified according to their truth values:

- **Tautology**:
  - A compound statement that makes sense for every possible truth value of its variables is called a tautology.
  - Its truth table's final column contains merely T (Truth).
- **Contradiction:**
  - A compound proposition that is logically incorrect (false) for every possible truth value of its variables is said to as contradictory. One term for contradiction is "logically false."
  - It has only F (False) in last column of its truth table.
- **Contingency**:
  - A compound proposition that is neither a tautology nor a contradiction is referred to as contingent.
  - It has both T (True) and F (False) in last column of its truth table.

Example 1. Prove that the tautology $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$ exists.

Solution: The $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$ truth table is

NOTE

Truth Table as below

p	q	~p	$(p \Rightarrow q)$	$(\sim p \vee q)$	$(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Example 2: Prove by truth table that given equation is Contradiction:

$(p \vee q) \wedge (\sim p \rightarrow (\sim q))$

Solution: Let $(p \vee q), (\sim p \rightarrow (\sim q))$ be respectively denoted by R, S.

The truth table is as below

p	q	R p ∨ q	~p	~q	S ~p→ (~ q)	(R ∧S)
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

Hence the given equation is Contradiction.

Example 5: Prove by truth table that given equation is Contingency:

$(p \Rightarrow q) \Rightarrow (p \wedge q)$

Solution (a):

Truth Table as below

P	q	$(p \Rightarrow q)$	$(p \wedge q)$	$(p \Rightarrow q) \Rightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Since all entries of last column contains both T (True) and F (False).

Hence the given equation is Contingency.

NOTE

5.5 LOGICAL EQUIVALENCE

Logical equivalence occurs when two statements or expressions have identical truth values under all possible conditions. This means that the two statements, though possibly expressed differently, represent the same logical content. Logical equivalence is determined using truth tables or logical laws and plays a crucial role in simplifying logical expressions and validating logical arguments.

Logical connectives are widely used in mathematics, computer science, and reasoning to structure complex logical statements.

In mathematics, logical equivalence is commonly represented by a double arrow

($\Leftrightarrow$ *or* $\leftrightarrow$) or triple lines ($\equiv$). If the truth values of two statements (or propositions) are consistently the same, they are said to be logically similar. The expression $'P \equiv Q'$ indicates that two assertions P and Q are logically equal.

Example 1. Demonstrate the logical equivalence of $(p \vee p) \equiv p$.

Solution: Truth Table for $(p \vee p) \equiv p$

p	p	$(p \vee p)$	$(p \vee p) \equiv p$
T	T	T	T
F	F	F	T

Example 2. Clearly demonstrate the logical equivalence of $\sim(p \vee q) \equiv \{(\sim p) \wedge (\sim q)\}$.

Solution: $\sim(p \vee q), \{(\sim p) \wedge (\sim q)\}$ truth table, take A and B, respectively

p	q	$(\sim p)$	$(\sim q)$	$(p \vee q)$	$\sim(p \vee q)$	$\{(\sim p) \wedge (\sim q)\}$	$A \equiv B$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

NOTE



Activity 5.1

Constructing Compound Propositions

Objective: To learn how to form compound propositions using logical connectives.

Instructions:

1. **Consider the following simple propositions:**
   - P: "It is raining."
   - Q: "I will take an umbrella."
   - R: "The ground is wet."
2. **Use the logical connectives AND (∧), OR (∨), NOT (¬), IMPLICATION (→), and BICONDITIONAL (↔) to form compound propositions.**

Tasks:

- Form a compound proposition using AND (∧).
- Form a compound proposition using OR (∨).
- Negate a given proposition using NOT (¬).
- Create a conditional statement using IMPLICATION (→).
- Form a biconditional statement using BICONDITIONAL (↔).

Example Answers:

- AND: "It is raining and I will take an umbrella" → $P \wedge Q$
- OR: "It is raining or the ground is wet" → $P \vee R$
- NOT: "It is not raining" → $\neg P$
- IMPLICATION: "If it is raining, then I will take an umbrella" → $P \rightarrow Q$
- BICONDITIONAL: "I will take an umbrella if and only if it is raining" → $Q \leftrightarrow P$

Discussion:

- Explain the meaning of each compound proposition.
- Discuss how these logical structures are used in reasoning and problem-solving.

Expected outcome: This activity helps in understanding how logical connectives combine statements to form meaningful propositions.

NOTE

☑ Check Your Progress 5.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a proposition with an example.

.....

.....

.....

2. The sentence "Close the door" is a proposition. (True or False).

.....

3. Which of the following is a valid proposition?

a) How are you? b) 5 + 2 = 7

c) Please sit down d) Open the book

4. A proposition that is always true is called a _____.

5.6 SUMMARY

This unit introduced the fundamental concept of a proposition in logic and explained how logical reasoning is used to analyze statements and arguments. Students learned the definition of a proposition as a declarative sentence that can be classified as either true or false. The unit also discussed different kinds of sentences and identified which statements qualify as propositions and which do not, such as questions, commands, or exclamatory sentences. In addition, the concept of logical connectives was explored, including conjunction, disjunction, negation, conditional, and biconditional statements, which are used to combine or modify propositions in logical expressions. The unit further explained truth tables and demonstrated how they help determine the truth values of compound propositions under different conditions. Students also studied the types of propositions based on truth values, including tautology, contradiction, and contingency, and understood their importance in logical analysis. Another major topic covered was logical equivalence, where students learned how different logical statements can express the same meaning and produce identical truth values. By the end of the unit, students developed the ability to analyze logical statements systematically, construct truth tables, and identify equivalent propositions. These concepts are highly relevant in mathematics, computer science, programming, artificial intelligence, digital circuit design, and decision-making processes, where logical thinking and accurate reasoning are essential. Understanding propositions and logical equivalence also strengthens problem-solving and critical-thinking skills that are valuable in academic and professional environments. Students are encouraged to continue exploring advanced topics in symbolic logic, Boolean algebra, and formal reasoning to deepen their understanding of logical systems and their practical applications in technology and everyday life.

NOTE

5.7 GLOSSARY

- **Proposition** – A declarative statement that is either true (T) or false (F) but not both.
- **Truth Value** – The value assigned to a proposition, indicating whether it is **true (T) or false (F)**.
- **Simple Proposition** – A proposition that does not contain any logical operators (e.g., "The Earth is round.").
- **Compound Proposition** – A proposition formed by combining two or more simple propositions using logical connectives (e.g., "It is raining and cold.").
- **Negation (¬P)** – The logical opposite of a proposition. If P is true, then ¬P (not P) is false, and vice versa.
- **Logical Connectives** – Symbols used to combine propositions:
- **Tautology** – A proposition that is always true, regardless of truth values (e.g., "P ∨ ¬P").
- **Contradiction** – A proposition that is always false, regardless of truth values (e.g., "P ∧ ¬P").
- **Logical Equivalence** – Two propositions are logically equivalent if they always have the same truth values in every situation.
- **Predicate** – A statement containing variables that become a proposition when specific values are substituted (e.g., "x > 5" becomes a proposition when x is assigned a number).

5.8 ANSWER TO CHECK YOUR PROGRESS-POSSIBLE ONE

Check your progress 5.1

1. **Answer:** A proposition is a declarative sentence that is either true or false, but not both.

   Example: "5 + 2 = 7" is a proposition because it is a true statement.

2. **Answer:** False.

   "Close the door" is a command, not a declarative sentence, so it is not a proposition.

3. **Answer:** Correct option: (b) 5 + 2 = 7

   Explanation: It is a declarative sentence with a definite truth value (true).

4. **Answer:** A proposition that is always true is called a tautology.

5.9 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1245	

5.10 REFERENCE

- https://ggc-discrete-math.github.io/logic.html
- https://sites.millersville.edu/bikenaga/math-proof/truth-tables/truth-tables.html
- https://www.codeguage.com/courses/logic/propositional-logic-truth-tables
- https://www.scribd.com/document/400239714/1
- https://byjus.com/maths/truth-table/

5.11 EXERCISE

Part 1

A. Multiple choice question

1. **Which of the following is not a proposition?**

   a) 2 + 2 = 4

   b) What is your name?

   c) The Earth orbits the Sun

   d) 7 is a prime number

2. **The truth value of $p \wedge q$ is true only when:**

   a) Both p and q are false

   b) Either p or q is true

   c) Both p and q are true

   d) p is true and q is false

3. **Which of the following represents the negation of proposition p?**

   a) p ∨ q

NOTE

b) ¬p

c) p ∧ q

d) p → q

4. If p is true and q is false, what is the truth value of $p \rightarrow q$?

a) True

b) False

c) Cannot be determined

d) Sometimes true

5. Which of the following is a tautology?

a) p∧¬p

b) p∨¬p

c) p→¬p

d) ¬(p∨q)

Answers with Explanations:

1. b) What is your name?

Explanation: It's a question, not a declarative sentence with a truth value.

2. c) Both p and q are true

Explanation: "AND" (∧) is true only when both components are true.

3. b) p∨¬p

Explanation: ¬p denotes the logical negation of p.

3. b) False

Explanation: Implication p→qp \rightarrow qp→q is false when p is true and q is false.

4. b) p∨¬pp \lor \neg pp∨¬p

Explanation: This expression is always true regardless of the truth value of p.

B. Short answer type question

1. What is a compound proposition?
2. Define a tautology.
3. What does the logical connective “¬” represent?

NOTE

Part 2

C. Long answer type question (200 words)

1. Define and explain different types of logical connectives with examples.
2. Construct a truth table for p→qp \rightarrow qp→q and explain its truth values.
3. Describe tautology, contradiction, and contingency with suitable examples.

Part 3

D. Long answer type question (300 words)

1. Explain the meaning of proposition in detail. Discuss types of propositions based on truth value with truth tables.
2. What are logical connectives? Explain the use of AND, OR, NOT, IMPLICATION, and BICONDITIONAL with symbolic notation, truth tables, and real-life examples.
3. Discuss the difference between tautology, contradiction, and contingency. Give at least two examples each and verify using truth tables.

DR. C.V. RAMAN UNIVERSITY

Chhattisgarh, Bilaspur AN A/SECT GROUP UNIVERSITY

Approved by : PCI | AICTE | NCTE | BCI | Member of : AIU | Joint Committee : (UGC | DEB | AICTE) | Recognized by : MSC | A NAAC Accredited University

DR. C. V. RAMAN UNIVERSITY (C.G.)

SLM FEEDBACK FORM

PROGRAMME : _____

COURSE : _____

BLOCK : _____

UNIT : _____

Dear Student,

While studying the units of this block, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill up and send us the following questionnaire, which pertains to this block. If you find the space provided insufficient, kindly use a separate sheet.

Please mail to aisectcontent@aisect.org/ for online submission please scan QR code:

Questionnaire

1. **Name** : _____

2. **Enrolment No.** : | | | | | | | | | | | |

3. **How many hours did you need for studying the units?** : | |

4. **Please give your reactions to the following items based on your reading of the block:**

Items	Excellent	Very Good	Good	Poor	Give specific Examples (If poor)
Presentation Quality					
Language and Style					
Illustrations					
Used (diagrams, tables, etc.)					
Conceptual Clarity					
Check Your Progress Questions					
Suggested Readings - OER					

5. **Any other comments:**

_____

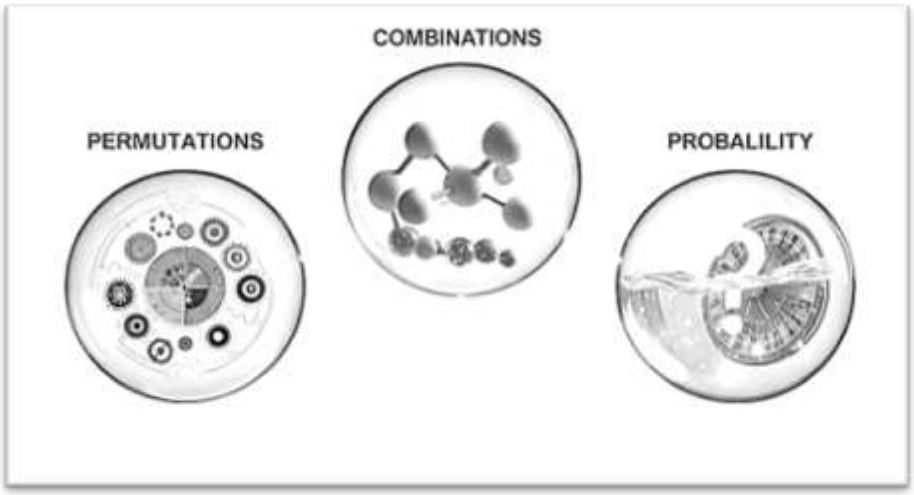
_____

_____

_____

***

BLOCK II	**PERMUTATIONS, COMBINATIONS & PROBABILITY**	**89-148**
Unit 6	**Permutation and Combination**	**91**
Unit 7	**Probability**	**111**
Unit 8	**Probability Distribution**	**131**



This block presents essential mathematical tools and concepts for understanding counting techniques, uncertainty, and the analysis of random phenomena. It begins with the study of **permutations and combinations**, which provide systematic methods for counting and determining the number of possible arrangements or selections of objects under different conditions. These techniques form the foundation of combinatorial analysis and are widely used in fields such as computer science, operations research, and statistics. Building upon these counting principles, the block introduces the concept of **probability**, which offers a mathematical framework for measuring and quantifying the likelihood of events occurring in both deterministic and random experiments. Learners will explore fundamental probability rules, methods of calculating probabilities, and the interpretation of probabilistic outcomes in practical situations. The block further examines **probability distributions**, which describe how probabilities are allocated across all possible outcomes of a random variable. Through the study of discrete and continuous distributions, students gain insight into patterns of variation, risk assessment, and predictive modeling. By integrating counting methods, probability theory, and probability distributions, this block develops analytical and problem-solving skills that are essential for applications in statistics, data science, artificial intelligence, business analytics, engineering, and informed decision-making in real-world contexts.

UNIT 6
PERMUTATION, COMBINATION

NOTE

UNIT STRUCTURE

6.1 Introduction

6.2 Objective

6.3 The Rules of Sum and Product

6.3.1 Definition of Rules of Sum and Product

6.4 Permutations, Combinations

6.4.1 Factorial Notation

6.4.1.1 Permutation

6.4.1.2 Restricted Permutation

6.4.1.3 Permutations of things not all different

6.4.1.4 Permutation with repetition allowed

6.4.1.5 Circular Permutation

6.4.2 Combinations

6.5 Binomial and Multinomial Theorems

6.5.1 Binomial Theorem statement

6.5.2 Multinomial Theorem statement

6.6 Combinations with Repetitions

6.7 Summary

6.8 Glossary

6.9 Answer to check your progress

6.10 Suggested readings - OER

6.11 Reference

6.12 Exercise

6.1 INTRODUCTION

In the previous unit, you explored the basics of counting techniques and their significance in problem-solving. Now, we extend these ideas by introducing the fundamental rules of sum and product. These principles form the foundation for understanding permutations and combinations. Mastering them will enhance your ability to analyze various counting scenarios.

Building on this knowledge, this unit introduces advanced counting methods, including factorial notation, permutations, and combinations. You will also learn about the binomial and multinomial theorems, which are essential in algebra and probability. These concepts are widely used in real-world applications such as cryptography and data organization.

NOTE

This unit bridges fundamental counting principles with more complex topics like permutations, combinations, and binomial expansions. You will explore how these methods apply to different types of arrangements and selections. By the end, you will be able to solve problems involving counting efficiently and accurately.

6.2 OBJECTIVE

After studying this unit, you will be able to….

- Apply the Rules of Sum and Product to solve counting problems.
- Define and compute permutations and combinations in various scenarios, including restricted and repeated cases.
- Use factorial notation to simplify and calculate arrangements and selections efficiently.
- Explain and apply the Binomial and Multinomial Theorems in algebraic expansions.
- Solve problems involving combinations with repetitions and their real-world applications.

6.3 THE RULES OF SUM AND PRODUCT

6.3.1. Definition of Rules of Sum and Product

[A] Sum Rule

One of two events can happen in m + n ways if one can happen in m ways and another in n ways, but they cannot happen simultaneously. In general, if

$E_i(i = 1, 2, 3 \ldots k)$ are k events that cannot occur simultaneously and can occur in n_i ways, then one of the k events can occur in $n_1 + n_2 + n_3 + \cdots + n_k$ ways.

Example 1. If a class has 14 boys and 12 girls, calculate the number of methods to choose one student as class representative.

Solution. Using the sum rule, there are $14 + 12 = 26$ options to choose one student (boy or girl) as class representative.

Example 2. If a student is admitted to four different engineering institutions and five medical colleges, calculate the number of ways he can choose one of the colleges listed above.

Solution. Using the sum rule, there are $4 + 5 = 9$ options for selecting one of the colleges.

[B] Product Rule

If there are m possible outcomes for one event and n possible outcomes for another, and the number of ways the second event occurs is

NOTE

unrelated to after the first event occurs, the two events can happen at the same time in mn ways. In general, if $E_i(i = 1, 2, \ldots . k)$ are k events and if E_1 can occur in n_1 ways, E_2 can occur in n_2 ways (no matter how E_1 occurs), E_3 can occur in n_3 ways (no matter how E_1 and E_2 occur),…., E_k occurs in n_k ways (no matter how $k-1$ events occur), then the k events can occur simultaneously in $n_1 \times n_2 \times n_3 \times \ldots \ldots \times n_k$ ways.

Example 1. Three people enter the automobile, which has five seats. To show how many ways can they take their seats?

Solution. The first individual has a choice of five seats and may sit in any of them. So, there are five ways to occupy the first seat. The second individual has a choice of four seats. Similarly, the third individual gets the option of three seats. Hence, the required number of ways for all three people to seat is $5 \times 4 \times 3 = 60$.

Example 2. There are four roads from city X to Y and five roads from city Y to Z, find:

(i) How many ways can you get from city X to city Z via city Y?

(ii) What are the various round-trip routes from city X to Y, Z, Y, and back to X?

Solution. (i) In going from city X to Y, any of the 4 roads may be taken. In the same way in going from city Y to Z, any of the 5 roads may be taken. thus, by the product rule, there are $5 \times 4 = 20$ ways to travel from city X to Z via city.

(ii) There are two ways to complete a round-trip trip:

1. From city X to city Y

2. From city Y to Z

3. From Z to Y city

4. From Y city to X

Thus, we can be performed in 4 ways to perform 1, 5 ways to perform 2, 5 ways to performed 3 and 4 ways to perform 4.

By product rule, there are $4 \times 5 \times 5 \times 4 = 400$ round-trip routes.

6.4 PERMUTATION, COMBINATION

6.4.1. Factorial Notation ($|n$ *or* n$!$),

When $n!$ is multiplied by the first n natural integers, $n! = 1.2.3 \ldots (n-1)\, n$

NOTE

Example,

$5! = 1.2.3.4.5$
$= 5.4.3.2.1 = 120$

$4! = 1.2.3.4.5 = 4.3.2.1 = 24$

Clearly, $\because 0! = 1$,

$1! = 1$

$n! = 1.2.3.\ldots..(n-2)(n-1)n$

$= \{1.2.3.\ldots..(n-2)(n-1)\}n$

$= (n-1)!\, n$

$= n.\,(n-1)!$

$= n.\,(n-1)(n-2)!.$

$\frac{n!}{r!} = n(n-1)(n-2)\ldots\ldots.(r+1)\ and$

$\frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots.\ldots.(n-r+1)$

Example: $\frac{5!}{2!} = 5.4.3 = 60$,

$\frac{3!}{6!} = \frac{1}{6.\,5.\,4} = \frac{1}{120}$

6.4.2 Permutation

Permutations are the numerous arrangements that can be made from a given collection of items by selecting some or all of them at once. The number of permutations of n distinct items taken at the same time is given by

$P(n,r)$ or ${}^{n}P_{r.}$ were, $r \le n$

The distinct arrangements of the three letters a, b, c taken two at a time are ab, bc, ca, ba, cb, ac; thus, the total number of arrangements (or permutations) of all three things taken two at a time is $P(3,2) = 6$.

The value of $P(n,r)$ or $P\_(r.)$ were, $r \le n$, the number of permutations of n different items taken r at a time is given by.

$${}^{n}P_{r.} = P(n,r) = \frac{n!}{(n-r)!}$$

The formula "P (n, r) = $\frac{n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-r+1)(n-r)!}{(n-r)!}$"

Hence, " $P(n,r) = n(n-1)(n-2)\ldots\ldots(n-r+1)$"

Note 1. The total number of permutations of n different items at once is.

"${}^{n}P_{n} = P(n,n) = n(n-1)(n-2)\ldots\ldots.3.2.1 = n!$"

NOTE

Note 2. Value of 0 ! = 1

By putting $P(n,n) = \frac{n!}{(n-n)!}$

$$n! = \frac{n!}{0!} \quad i.e., 0! = \frac{n!}{n!} = 1$$

Example 1. Determine the value for the following Permutation.

$(a)\ {}^4P_2$ $(b)\ {}^9P_3$ $(c)\ {}^{20}P_2$ $(d)\ {}^{52}P_4$

Solution:

$$(a)\ {}^4P_2 = \frac{4!}{(4-2)!} = \frac{4\times3\times2!}{2!} = 12$$

$$(b)\ {}^9P_3 = \frac{9!}{(9-3)!} = \frac{9\times8\times7\times6!}{6!} = 504$$

$$(c)\ {}^{20}P_2 = \frac{20!}{(20-2)!} = \frac{20\times19\times18!}{18!} = 380$$

$$(d)\ {}^{52}P_4 = \frac{52!}{(52-4)!} = \frac{52\times51\times50\times49\times48!}{48!} = 6497400$$

Example 2. How many variable names of 8 letters can be constructed from a,b,c,d,e,f,g,h,i condition that no letter should be repeated?

Solution: There are nine letters to choose from, and eight must be selected.
The total number of variable names containing eight letters is

$${}^9P_8 = \frac{9!}{(9-8)!} = \frac{9!}{1!} = 9!$$

6.4.2 1. Restricted Permutation

1. The number of permutations of n distinct items taken r at the same time in which k objects do not occur is $P(n-k,r)$.The number of permutations of n distinct items taken r at the same time in which k particular objects do not occur is $P(n-k,r)$.
2. The number of permutations of n different items taken r at a time, with k distinct objects always present, is represented as $P(n-k,r-k)\times P(r,k)$.
3. When two or more things do not happen at the same time.

Example 1. How many permutations of 10 things chosen four at a time will result in:

(a) Two things always occurring

(b) Never occurring?

Solution.

(a) Keeping aside the **two things that will always occur**, the number of permutations of $10-2=8$ items taken two at a time is ${}^8P_2$ Two

NOTE

items can occupy any of the four spaces, allowing for a total of ${}^4P_2$ arrangements. Therefore, total numbers of permutation are ${}^8P_2 \times {}^4P_2 = 8 \times 7 \times 4 \times 3 = 672$

(b) Leaving away the two things that will **never occur**, the number of permutations of 8 things taken 4 at a time is ${}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$

Example 2. How many ways can 7 boys and 5 girls be positioned in a row such that no two girls can sit together?

Solution. Because there are no restrictions on boys, let first determine the places of seven boys.

Their positions are indicated as follows.

$$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7 \times$$

7 boys can be placed in 7! ways, but if 5 girls sit at positions indicated by x, no two of them will sit together.

There are 8 possible seating arrangements for 5 girls
i.e., ${}^8P_5$ ways

The required number of ways to seat 7 boys and 5 girls in a row without two girls sitting together is $= {}^8P_5 \times 7!$.

Example 3. How many ways can four boys and four girls sit in a row, alternately?

Solution

Case I. If a boy sits in first position: Such arrangements are going to be of the type that $B\ G\ B\ G\ B\ G\ B\ G$

There were four seats for four boys, so they could be arranged in four different ways i.e. 4!,. Again, there are four seats for four girls, so they can arrange is four different ways i.e., 4!. Therefore, total number of ways seated is $4! \times 4!$.

Case II. When a girl seats first, the different seating arrangements are arranged as follows:$G\ B\ G\ B\ G\ B\ G\ B$

The total number of possibilities in their scenario is $4! \times 4!$

The required number of ways is $4! \times 4! + 4! \times 4! = 1152$.

6.4.2.2. Permutation of Thing Not All Different

To determine the number of ways in which n things can be organized among themselves, taking all of them at once, while p of them is of one kind, q of them is of another kind, r of them are of a third kind, and the remainder are all different.

Let x be the needed number of permutations. Now, replace p similar items with p distinct items that are also different from others. These p distinct items can be allowed among themselves in $p!$ ways without altering the placements of the others.

NOTE

If p similar items of an identical kind have been substituted by p unique items, the total number of permutations is $x \times p!$.

If we substitute q similar items with q distinct items for any one of the $x \times p!$ permutations, the number of permutations without disrupting the reset arrangement is going to be $q!$.

If both replacements are completed, the number of permutations is equal to x multiplied by p and q.

Similarly, replacing r similar items with r distinct items result in $x \times p! \times q! \times r!$ permutations.

Each $x \times p! \times q! \times r!$ permutation is a combination of n distinct items.

$\therefore x \times p! \times q! \times r! = n!$

Hence, $x = \frac{n!}{p!\ q!\ r!}$

Example 1. How many ways can the letters of the word $BANANA$ be placed.?

Solution. The word $BANANA$ contains six letters, three of that are identical ($3A's$), two of them were that same ($2N's$), and the last one is distinct.

Hence, the needed number of permutations = $(6!)/(3!\ 2!) = 60$.

Example 2: A coin is tossed six times. In how many ways can we get four heads and two tails?

Solution. Tossing a coin six times or six coins at once yields the same number of configurations. The number of configurations with four heads and two trails out of six is $(6!)/(4!\ 2!) = 15$.

6.4.2.3. Permutation with Repetition Allowed

At a given time m, there are n^r permutations of n distinct items taken r, where items can be repeated any number of times.

Example 1. How many ways can three rewards be split among four when a boy can receive any number of rewards?

Solution. Each of the three rewards can be awarded in four different ways, as a boys can get any number of rewards.

Therefore, there are numerous ways to distribute all awards.

$=4 \times 4 \times 4 = 4^3 = 64$.

Example 2. How many ways may five software projects be assigned to six final-year students if none of them are assigned to the same student?

Solution. We have five projects and six students.

Each project can be distributed in six different ways.

NOTE

Thus, the number of possibilities to assign 5 projects is = $6 \times 6 \times 6 \times 6 \times 6 = 6^5$.

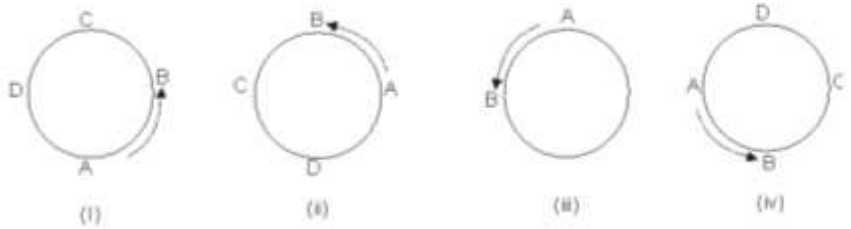
The number of ways that all projects are assigned to the same student equals 6.

Allocating 5 projects to six students gives an overall maximum of 7770 possible combinations i.e., $6^5 - 6 = 7770$,

6.4.2.4. Circular Permutation

The permutation described above are more correctly called linear permutation because the objects are placed in a line (or row). When we arrange them in a circle, their number and permutation change. Collection of an items organized in a circle is a circular permutation.

Consider four people: $A, B, C,$ and $D,$ who will be positioned in a circle. The following configurations are identical since none of the four positions $A, B, C, or\ D$ have changed relative positions.



However, in linear configurations, all four configurations are distinct.



Thus, circular permutations differ only when the relative order of the object's changes; otherwise, they are identical.

It is possible to determine the number of permutations of n objects in a circle by holding one element constant and arranging the remaining $(n-1)$ components in

$(n-1)!$ ways. This is because the number of circular permutations is dependent on the relative positions of the items.

Total number of circular arrangements for n different elements is equal to $(n-1)!$

Example 1. Five men and four women can be seated at a circular table in how many ways?

(a) there are no restrictions, (b) all four girls sit together, (c) all four of the girls do not sit together. (d) two girls never sit together.?

Solution.

(a) Nine people in all $(5 + 4 = 9)$. There are $9 - 1! = 8!$ options for these people to sit at the round table.

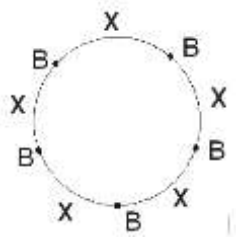
NOTE

(b) We get 5 + 1 = 6 people if we consider 4 girls to be one person. These six people can sit around a table in $(6-1)! = 5!$ But there are four ways to arrange four girls among themselves. Therefore, $5! \times 4$ is the necessary number of ways!

(c) $8! - 5! \times 4!$ = the number of arrangements when all four girls are not seated together minus the total number of arrangements without restrictions.

(d) As there are no restrictions on boys, we first set up five seats at a round table, which can be done in five ways $((5-1)! = 4!)$. There are currently five spots available for four girls, as shown by ×. So, four girls can sit in ${}^5P_4 = 5!$ Ways

Therefore, the necessary number of ways is $4! \times 5!$.



6.4.3 Combination

Combinations are the various groupings or choices that can be constructed from a given collection of items by taking some or all of them at once, regardless of the order.

The number of combinations of n different things taken r $(\leq n)$ at a time is denoted by $C(n,r)$ $or\ {}^n C_r$. The selection of two letters from three letters a, b, c are $ab\ bc\ ca$ and thus, the number of combinations of 3 letters taken 2 at a time is $C(3,2) = 3$.

The number of combinations of 4 letters a, b, c, d taken two at a time is $C(4,2) = 6$ and they are ab, ac, ad, bc, bd, cd.

The value of $C(n,r)$

$$\therefore x \times r! = P(n,r)$$

$$= \frac{n!}{(n-r)!}$$

$$\therefore x = \frac{P(n,r)}{r!}$$

$$= \frac{n!}{(n-r)!}$$

or $C(n,r) = \frac{n!}{r!\,(n-r)!}$ ..... (1)

Cor I. $C(n,n) = C(n,0) = 1$

Putting $r = n$ in (1)

NOTE

$$C(n, n) = \frac{n!}{n!\,(n-n)!} = \frac{1}{0!} = 1$$

Putting $r = 0$ in (1)

$$C(n, 0) = \frac{n!}{0!\,(n-0)!} = \frac{n!}{0!\;\; n!} = 1$$

Properties of ${}^{n}C_r$ *or* $C(n, r)$

Example 1. Evaluate the following

(i) ${}^{6}C_3$ (ii) ${}^{10}C_8$

Solution.

(i) ${}^{6}C_3 = \frac{6 \times 5 \times 4 \times 3!}{3!(6-3)!} = \frac{120}{3!} = 20$

(ii) ${}^{10}C_8 = {}^{10}C_2 = \frac{10!}{(10-2)!2!} = \frac{10 \times 9 \times 8!}{8! \times 2!} = 45.$ [$\because$ ${}^{n}C_r = {}^{n}C_{n-r}$]

Example 2. There are two women and five males. How many ways can a three-person committee be constituted to include (i) exactly one woman and (ii) at least one woman?

Solution. (i) We must pick two males out of five and one woman out of two.

The number of methods for choosing a single woman = ${}^{2}C_1 = 2$

The number of methods for choosing two males = ${}^{5}C_2 = \frac{5.4}{2} = 10$.

$\therefore$ The committee can be formed in $2 \times 10 = 20\ ways$.

(ii) The committee can be formed in the following ways

(a) 2 men and 1 woman

(b) 1 man and 2 women

For (a) 2 men out of 5 men and 1 woman out of 2 women can be selected in ${}^{5}C_2 \times {}^{5}C_1\ ways$.

For (b) 1 man out of 5 men and 2 women out of 2 women can be selected in ${}^{5}C_1 \times {}^{2}C_2\ ways$.

$\therefore$ The total number of committee formation methods

$= {}^{5}C_2 \times {}^{2}C_1 + {}^{5}C_1 \times {}^{2}C_2 = 20 + 5 = 25.$

6.5 BINOMIAL AND MULTINOMIAL THEOREM

NOTE

6.5.1. Binomial Theorem Statement

The binomial theorem is the method of expanding an expression that has been raised to any finite power. A binomial theorem is a powerful tool of expansion which has applications in Algebra, probability, etc.

- **Binomial Expression:** A binomial expression is an algebraic expression that contains two dissimilar terms.

  Ex.., $a + b$, $a^3 + b^3$, etc.

- **Binomial Expansion**

  Important points to remember

  - Total number of terms in the expansion of-

  $(x + y)^n$ is $(n + 1)$.

  - Sum of exponents of x and y is always n.
  - nC_0, nC_1, $nC_2, \ldots\ldots$, nC_n are called binomial coefficients and represented by $C_0, C_1, C_2 \ldots\ldots, C_n$
  - The binomial coefficients, which are equidistant from the beginning and from the ending, are equal,

    i.e., $nC_0 = nC_n$, $nC_1 = nC_{n-1}$, $nC_2 = nC_{n-2} \ldots\ldots\ldots\ldots\ldots\ldots$ etc.

- **Binomial Expansion Formula:**

  Suppose $n \in N, x, y, \in R$ then $(x + y)^n = n\sum_{r=0} nC_r\ x^{n-r} \cdot y^r$ were,

  $$nC_r = \frac{n!}{(n-r)!r!}$$

Example1: Expand $(x/3 + 2/y)^4$

Sol:

$$\left(\frac{x}{3}+\frac{2}{y}\right)^4 = 4C_0\left(\frac{x}{3}\right)^4 + 4C_1\left(\frac{x}{3}\right)^3\left(\frac{2}{y}\right) + 4C_2\left(\frac{x}{3}\right)^2\left(\frac{2}{y}\right)^2 + 4C_3\left(\frac{x}{3}\right)\left(\frac{2}{y}\right)^3 + 4C_4\left(\frac{2}{y}\right)^4$$

$$= \frac{x^4}{81} + \frac{8x^3}{27y} + \frac{8x^2}{3y^2} + \frac{32x}{3y^3} + \frac{16}{y^4}$$

Example2: $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5$

Sol: We have

$$(x + y)^5 + (x - y)^5 = 2[5C_0\ x^5 + 5C_2\ x^3 y^2 + 5C_4\ xy^4]$$

$$= 2(x^5 + 10\,x^3 y^2 + 5xy^4)$$

Now $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5 = 2[(\sqrt{2})^5 + 10(\sqrt{2})^3(1)^2 + 5(\sqrt{2})(1)^4]$

$$= 58\sqrt{2}$$

NOTE

- **Properties of Binomial Coefficient:**

  Binomial coefficients refer to the integers, which are coefficients in the binomial theorem. Some of the most important properties of binomial coefficients are:

  - $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$
  - $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$
  - $C_0 - C_1 + C_2 - C_3 \ldots + (-1)^n . nC_n = 0$
  - $nC_1 + 2.nC_2 + 3.nC_3 + \ldots + n.nC_n = n.2^{n-1}$
  - $C_1 - 2C_2 + 3C_3 - 4C_4 + \ldots + (-1)^{n-1} C_n = 0 \qquad for\, n > 1$
  - ${C_0}^2 + {C_1}^2 + {C_2}^2 + \ldots {C_n}^2 = [(2n)! / (n!)^2]$

Example1: If $(1 + x)^{15} = a_0 + a_1 x + \ldots\ldots + a_{15} x^{15}$ then, find the value of

$$\sum_{r=1}^{15} r.\frac{a_r}{a_{r-1}}$$

Sol:

$$\sum_{r=1}^{15} r.\frac{a_r}{a_{r-1}} = 1.\frac{a_1}{a_0} + 2.\frac{a_2}{a_1} + 3.\frac{a_3}{a_2} + \cdots \ldots\ldots\ldots + 15.\frac{a_{15}}{a_{14}}$$

$$= \frac{C_1}{C_0} + 2.\frac{C_2}{C_1} + 3.\frac{C_3}{C_2} + \cdots \ldots\ldots\ldots + 15.\frac{C_{15}}{C_{14}}$$

$$= 15 + 14 + 13 + \cdots \ldots\ldots\ldots\ldots\ldots + 1$$

$$= \frac{[15(15+1)]}{2} = 120$$

6.5.2. Multinomial Theorem

The binomial theorem is extended to polynomials with more than two terms by the multinomial theorem. It explains how a multinomial (sum of several terms) expands its powers.

For any positive integer n the multinomial theorem states:

Using the binomial theorem, we have

$$(x + a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

The number of possibilities to select k elements from a set of n elements is represented by the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Where, k is an index running from 0 to n.

x^{n-k} and a^k are the respective powers of x and a.

NOTE

Example: Expanding $(x + a)^2$

Solution: If $n = 2$ the expansion is:

$$(x + a)^2 = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

$$= \binom{2}{0} x^2 a^0 + \binom{2}{1} x^1 a^1 + \binom{2}{2} x^0 a^2 = x^2 + 2xa + a^2$$

This result can be generalized in the following form:

$$(x_1 + x_2 + \ldots + x_m)^n = \sum_{k_1 + k_2 + \ldots + k_m = n} \frac{n!}{k_1!.k_2! \ldots k_m!} {x_1}^{k_1} {x_2}^{k_2} \ldots \ldots {x_m}^{k_m}$$

where:

n is the total power of the multinomial.

m is the number of terms in the multinomial.

$k_1, k_2, \ldots, k_m$ are non-negative integers such that $k_1 + k_2 + \ldots + k_m = n$

The coefficient $\frac{n!}{k_1!.k_2!\ldots k_m!}$ is the multinomial coefficient.

Example: Expanding $(x + y + z)^3$

To expand $(x + y + z)^3$apply the multinomial theorem:

Solution:

1. $n = 3, m = 3$ (terms: x, y, z).
2. Compute the terms for all combinations of $k_1, k_2, , k_3$ such that $k_1 + k_2 + \ldots + k_m = 3$

Expansion:

$$(x + y + z)^3 = \sum_{k_1 + k_2 + k_3 = 3} \frac{3!}{k_1!.k_2!k_3!} x^{k_1} y^{k_2} z^{k_3}$$

Terms:

For $k_1 = 3, k_2 = 0, k_3 = 0. \frac{3!}{3!.0!0!} x^3 y^0 z^0 = x^3$

For $k_1 = 2, k_2 = 1, k_3 = 0. \frac{3!}{2!.1!0!} x^2 y^1 z^0 = 3x^2 y$

For $k_1 = 1, k_2 = 2, k_3 = 0. \frac{3!}{1!.2!0!} x^1 y^2 z^0 = 3xy^2$

For $k_1 = 0, k_2 = 0, k_3 = 3. \frac{3!}{0!0!3!.} x^0 y^0 z^3 = z^3$

Final Expansion:

$$(x + y + z)^3 = x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3$$

NOTE

6.6 COMBINATION WITH REPETITION

Combination with Repetition ("Multisets"):

Combinations with repetition involve selecting items from a set where:

1. Order does **not** matter.

2. Repetition of items is **allowed**.

We call this selecting from 'multisets.' The following formula provides the number of methods to repeatedly choose r things from a list of n different items:

$$C(n + r - 1, r) = \frac{(n + r - 1)!}{[r!\ (n - 1)!]}$$

This is the number of ways that r identical things can be divided into n different groups.

Examples:

Example1: Distributing Candies

You have 4 types of candies: Chocolate (C), Mint (M), Strawberry (S), and Lemon (L). You want to distribute 6 candies to a child, allowing repetition of candy types. How many ways can this be done?

Solution:

1. Think of this as finding the number of ways to distribute 6 candies into 4 categories.

2. Use the formula for combinations with repetition:

$C(n + r - 1, r) = C(4 + 6 - 1, 6) = C(9, 6)$

3. Compute $C(9, 6)$:

$C(9, 6) = \frac{9!}{[6!\ (9-6)!]} = \frac{(9 \times 8 \times 7)}{(3 \times 2 \times 1)} =$ 84 ways to distribute the candies

Example 2. How many ways can four people share twenty identical apples?

Solution. The required number

$$= {}^{20+4-1}C_{4-1} \qquad (r = 20, n = 4)$$

$$= {}^{23}C_3 = \frac{23 \times 22 \times 21}{3!} = 1771$$

Example 3. Find the number of ways to distribute 20 identical balls in 4 different boxes so that so box remains empty.

Solution. The required number

$$= {}^{20-1}C_{4-1}\ (r = 20, n = 4)$$

$$= {}^{19}C_3 = \frac{19 \times 18 \times 17}{3!} = 969.$$

NOTE

Activity 6.1

Expansions and Distributions

Objective:

To apply the Binomial and Multinomial Theorems, Combinations with Repetitions to algebraic and distribution problems.

Instructions:

Solve the following problems by using the appropriate theorem or formula.

1. **Binomial Theorem Statement:**

   Expand $(a + b)^5$using the Binomial Theorem.

2. **Binomial Coefficient Calculation:**

   Find the coefficient of x^3y^2 in the expansion of $(x + y)^5$.

3. **Multinomial Theorem Statement:**

   Expand $(x + y + z)^3$using the Multinomial Theorem.

4. **Multinomial Coefficient Calculation:**

   Find the coefficient of $x^2y^1z^1$ in the expansion of $(x + y + z)^4$

   Combinations with Repetitions:

   A shop sells 4 different flavors of ice cream. In how many ways can a customer buy 6 ice creams if they can choose any flavor in any quantity?

5. **Combinations with Repetitions in Real Life:**

   A bakery sells 3 types of cookies: chocolate, vanilla, and strawberry. If a customer wants to buy 8 cookies, in how many ways can they select the cookies?

Expected Outcome:

Students will gain a deeper understanding of binomial and multinomial expansions, as well as how combinations with repetitions work in real-world scenarios.

☑ Check Your Progress 6.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define the Rule of Sum in counting principles

.....

.....

.....

2. In a circular permutation of n objects, the number of possible arrangements is $n!$ True or False:

.....

NOTE

3. The number of ways to arrange 5 books on a shelf is:

a) 120 b) 24

c) 60 d) 5

4. The number of ways to choose r objects from nnn objects without considering order is given by the formula _____.

6.7 SUMMARY

The unit on **Permutation and Combination** introduced important counting techniques used in mathematics, probability, statistics, and decision-making. It began with the **Rules of Sum and Product**, which explain how to count outcomes when events occur separately or together. Students learned the definition and practical use of these rules in solving counting problems efficiently. The unit then explored **Permutations and Combinations**, including **factorial notation**, which forms the foundation of arranging and selecting objects. Different types of permutations were studied, such as simple permutations, restricted permutations, permutations of objects that are not all different, permutations with repetition allowed, and circular permutations. Students also learned the concept of **Combinations**, where the order of selection does not matter, along with methods for solving related numerical problems. The unit further covered the **Binomial and Multinomial Theorems**, which help expand algebraic expressions and calculate coefficients in polynomial expansions. In addition, **Combinations with Repetitions** explained how objects can be selected when repetition is allowed. By the end of the unit, students developed the ability to apply mathematical reasoning to counting, arranging, and selecting objects systematically. These concepts are highly relevant in real-life situations such as scheduling, coding, cryptography, business planning, probability analysis, genetics, and computer science algorithms. Understanding permutations and combinations also supports advanced learning in statistics, data science, and operations research. Students are encouraged to practice a variety of problems to strengthen their logical thinking and problem-solving abilities. Further exploration through textbooks, online tutorials, quizzes, and real-world applications can help deepen understanding and improve analytical skills in combinatorial mathematics.

6.8 GLOSSARY

- **Counting Principles:** Basic rules (sum and product) used to determine the total number of possible outcomes in a scenario.
- **Factorial** ($n!$): The product of all positive integers up to n, used in various counting techniques.

NOTE

- **Permutation:** The arrangement of a set of objects in a specific order.
- **Combination:** The selection of objects from a set where order does not matter.
- **Circular Permutation:** Arranging objects in a circular manner where rotations of the same order count as one.
- **Binomial Coefficients:** The coefficients in the expansion of $(x + y)^n$, represented as $\binom{n}{k}$
- **Multinomial Coefficients:** The coefficients in the expansion of expressions with more than two terms, such as $(x_1 + x_2 + \cdots \ldots \ldots + x_m)^n$
- **Restricted Arrangements:** Arrangements where certain conditions, such as grouping of elements, must be followed.
- **Repetitive Selection:** A counting method where elements can be selected multiple times, common in permutations and combinations with repetition.
- **Applications of Counting Principles:** These concepts are widely used in probability, computer science, cryptography, and data organization.

6.9 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 6.1

1. **Answer:** The Rule of Sum states that if there are m ways to do one task and n ways to do another different task, and only one of them can be done at a time, then the total number of ways to do either task is: $m + n$

   Example: If you can travel by 3 trains or 2 buses, then you have $3 + 2 = 5$ ways to travel.

2. **Answer:** False.

   In a circular permutation, the number of arrangements of n objects is: $(n - 1)!$

   Because one object is fixed as a reference.

3. **Answer:** Correct option: (a) 120

   Explanation: The number of linear arrangements of 5 books is:

   $$\mathbf{5! = 5 \times 4 \times 3 \times 2 \times 1 = 120}$$

NOTE

4. **Answer:** $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

6.10 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1246	

6.11 REFERENCE

- https://brilliant.org/wiki/rule-of-sum-and-rule-of-product-problem-solving/
- https://discrete.openmathbooks.org/dmoi3/sec_counting-combperm.html
- https://math.libretexts.org/Courses/Monroe_Community_College/Supplements_for_Discrete_Judy_Dean/01%3A_Groups_I/1.04%3A_Permutations
- https://byjus.com/maths/permutations-and-combinations-questions/
- https://math.mit.edu/~fgotti/docs/Courses/Combinatorial%20Analysis/4.%20Binomial%20Theorem/Binomial% 20Theorem.pdf
- https://www.scribd.com/presentation/433013181/Session-22-ppt

6.12 EXERCISE

Part 1

A. Multiple choice question:

1. **How many ways can 5 different letters be arranged in a row?**

   a) 120

   b) 60

   c) 30

   d) 25

2. **In how many ways can 6 different people be seated around a circular table?**

   a) 720

NOTE

b) 120

c) 60

d) 5

3. How many ways can a committee of 3 people be selected from a group of 7?

a) 35

b) 42

c) 21

d)14

4. What is the number of ways to arrange the letters in the word "APPLE"?

a) 120

b) 60

c) 30

d) 5

5. Which of the following is a correct expansion of $(x+y)^3$ $(x+y)^3$ $(x+y)^3$ using the Binomial Theorem?

a) $x^3 + y^3$

b) $x^3 + 3x^2y + 3xy^2 + y^3$

c) $x^3 + 2xy + y^3$

d) $x^3 + 3x2y + y^3$

Answers with Explanations:

1. (a) 120

Explanation: The number of ways to arrange n different objects in a row is $n!$. Here, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

2. (d) 5

Explanation: In a circular arrangement, the number of ways to arrange n objects is $(n-1)!$. For 6 people, it is $(6-1)! = 5! = 120$

3. (a) 35

Explanation: The formula for combinations is

$$C\ (n,r)\ = \frac{n!}{r!\ \times (n-r)!}$$

NOTE

Here,

$$C\ (7,3) = \frac{7!}{3!\ \times (7-3)!} = \frac{7!}{3!\ \times 4!} === \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

4. (b) 60

Explanation: The formula for permutations of objects with repetitions is $\frac{n!}{p1!p2!\ldots pk!}$. For "APPLE" (5 letters, with P appearing twice), the calculation is $\frac{5!}{2!} = \frac{120}{2} = 60$

5. (b) $x^3 + 3x^2y + 3xy^2 + y^3$

Explanation: The Binomial Theorem states that

$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k . For\ (x + y)^3$, it expands to: $x^3 + 3x^2y + 3xy^2 + y^3$

B. Short answer type question

1. What is factorial notation?
2. How many ways can 4 different objects be arranged in a row?
3. Define a permutation with repetition.

Part 2

C. Long answer type question (200 words)

1. Explain the Rule of Product with an example.
2. Derive the formula for the number of ways to arrange nnn distinct objects in a row.
3. Explain the concept of restricted permutations with an example.

Part 3

D. Long answer type question (300 words)

1. Explain in detail the difference between permutations and combinations with suitable examples.
2. Derive the formula for permutations when some objects are identical.
3. Discuss the applications of the Binomial and Multinomial Theorems in real-life scenarios.

UNIT 7
PROBABILITY

NOTE

UNIT STRUCTURE

7.1 Introduction
7.2 Objective
7.3 Discrete Probability
7.3.1. Definition of Probability
7.3.2 Random Experiment
7.3.3 Sample Point:
7.3.4 Sample Space:
7.3.5. Events:
7.3.6. Independent Events:
7.3.7. Dependent Events:
7.3.8. Impossible and sure events
7.3.9 Simple and Compound events
7.3.10 Complementary Events:
7.4 Random Variables
7.4.1 Definition of random variable:
7.4.2. More examples of random variables in probability
7.5 Repeated Trials
7.5.1. Definition of repeated trials
7.5.2 More examples of repeated trials in probability
7.6 Summary
7.7 Glossary
7.8 Answer to check your progress
7.9 Suggested readings - OER
7.10 Reference
7.11 Exercise

7.1 INTRODUCTION

In the previous unit, you explored the basics of counting principles, permutations, and combinations, which are essential for understanding probability. Probability theory helps quantify uncertainty in various real-world situations. It plays a crucial role in statistics, machine learning, and risk analysis. In this unit, you will learn about probability concepts such as random experiments, sample space, and events.

Building on this knowledge, this unit introduces discrete probability, which deals with finite or countable outcomes. You will learn about sample spaces, independent and dependent events, and types of probability events. Additionally, the concept of random variables and

NOTE

repeated trials will be explored with examples. These ideas form the foundation for advanced statistical methods.

This unit bridges the gap between counting techniques and probability by explaining how outcomes of random experiments are analyzed mathematically. You will explore key concepts such as simple and compound events, complementary events, and the difference between independent and dependent events. The study of random variables and repeated trials further enhances the understanding of probability applications.

7.2 OBJECTIVE

After studying this unit, you will be able to….

- Define probability and explain its significance in analyzing uncertain events.
- Describe Random Experiments and Sample Spaces.
- Explain independent, dependent, impossible, sure, simple, compound, and complementary events with examples.
- Define the concept of random variables.
- Explain Repeated Trials in Probability.

7.3 DISCRETE PROBABILITY

7.3.1 Definition of Probability

A metric that quantifies the likelihood of a given occurring by dividing the number possible events by the number of favorable outcomes. The chance of the event occurring as an expression of its the possibility of happening.

A number between 0 and 1 is used to represent it, with 0 denoting that the event is impossible and 1 denoting that it will happen.

The event's probability P The formula used to compute an occurrence is:

$$P(A) = \frac{(Total\ number\ of\ favorable\ outcomes)}{(Total\ number\ of\ possible\ outcomes)}$$

Examples:

Example 1: When flipping a fair coin, what is the likelihood of receiving a head?

Solution: Total outcomes (head or tail).= 2

Favorable outcomes (Head) = 1

P(Head) = 1 / 2 = 0.5

Thus, there is a 0.5 chance of receiving a head.

NOTE

Example 2: What is the probability of rolling a 3 on a standard 6-sided die?

Solution: Total outcomes 6-sided die = $(1, 2, 3, 4, 5, 6)$

Favorable outcome (rolling a 3) = 1

$P(3) = 1 / 6 \approx 0.167$

Thus, the likelihood of rolling a three is roughly 0.167.

Example 3: A bag contains 5 red balls, 3 green balls, and 2 blue balls. If a ball is randomly drawn, what is the probability of drawing a green ball?

Solution: Number of green balls = 3

Total number of balls = $5\,(red) + 3\,(green) + 2\,(blue) = 1$

The formula for the chance of drawing a green ball is

$$P = \frac{Total\ number\ of\ green\ balls}{Total\ number\ of\ balls}$$

$$= 3 / 10 = 0.3\ or\ 30\%$$

Therefore, the probability of drawing a green ball is 0.3 or 30%.

7.3.2. Random Experiment

A procedure or activity that has one or more possible results but whose conclusion is unpredictable and cannot be predicted with confidence is called a random experiment.

Examples:

1. **Flipping a Coin:** There are only two possible outcomes while flipping a fair a coin: heads (H) or tails (T). A coin toss is regarded as a random experiment since it is impossible to anticipate the outcome beforehand. For this experiment, the set of possible outcomes is $\{H, T\}$.

2. **Rolling a Dice:** Someone can roll a six-sided die and get one, two, three, four, five, or six. You cannot forecast the precise number that will occur in any roll, but each outcome has an equal probability of happening. With $\{1, 2, 3, 4, 5, 6\}$ as the range of potential results, which renders the experiment random.

7.3.3. Sample Point

One potential result of an experiment that is random is called a sample point. It stands for one particular outcome from the sample space, which is the collection of all potential possibilities.

NOTE

Example:

1. **Flipping a Coin:** "Heads" (H) and "tails" (T) are the sample points in the flipping experiment. One conceivable result is represented by each of these.
2. **Rolling a Die:** In this experiment, a six-sided die is rolled using sample points

   $1, 2, 3, 4, 5, and\ 6$. Every integer in the sample space represents a distinct outcome.

7.3.4. Sample Space:

The collection of all potential results from a random experiment is known as the sample space. It includes every potential result that can occur, making it a complete representation of the experiment's possible outcomes.

Examples:

1. **Flipping a Coin:** When flipping a coin, the sample space is $S = \{H, T\}$

   where T stands for tails and H for heads. Every conceivable result of the coin toss is included in this sample space.
2. **Rolling a Die:** The set of all sample points is the sample space, as if we were tossing a coin. Thus, when a six-sided die is rolled.

   Possible outcomes of the die are sample points: $1, 2, 3, 4, 5, 6$.

   'The sample space is the set of all sample points': $S = \{1, 2, 3, 4, 5, 6\}$.

7.3.5. Event

In probability theory, an event is a specific outcome or set of outcomes from a random experiment. Typically, events are shown as subsets of a sample space, which is the collection of all potential outcomes. The probability of an occurrence is an evaluation of the possibility that it could happen during the experiment.

Thus, an event in probability is any collection of outcomes from a random experiment, and it can range from a single outcome to multiple possible outcomes. For example, in a coin toss, possible events include landing on heads or tails.

Examples:

1. **Getting an Even Number When Rolling a Die**

   If a fair die is rolled. How likely is it that you will receive an even number?

   Next: **Space of Samples** (S): $\{1, 2, 3, 4, 5, 6\}$

   Event (E): Since an even number is rolled, $E = \{2, 4, 6\}$

NOTE

Out of six possible outcomes, three are favorable $(2, 4, and\ 6)$.

Probability of Event

$$E = \frac{(Number\ of\ favorable\ outcomes)}{(Total\ outcomes)} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Thus, probability of rolling an even number is 0.5.

2. Drawing an Ace from a Deck of Cards

Let us say we draw a single card from a 52-card deck.

How likely is it that an Ace will be drawn?

Then: **Sample Space** ($\boldsymbol{S}$): All 52 cards in the deck.

Event (E): Drawing an Ace. A pack of cards has 4 Aces, a single from each suit.

Out of 52 possible outcomes, there are only four favorable possibilities (also known as the 4 Aces).

$$E = \frac{(Total\ number\ of\ favorable\ results)}{(Total\ outcomes)} = \frac{4}{52} = \frac{1}{13} \approx 0.077$$

is the probability of the event.

Thus, probability of drawing an Ace is approximately 0.077 *or* 7.7%.

7.3.6. Independent Event

According to probability theory, two events are considered independent if the likelihood of one happening does not change when the other happens. Events A and B are mathematically independent if: $P(A \cap B) = P(A) \cdot P(B)$

This indicates that the product of the probabilities of the two events alone determines the likelihood that they will occur together.

Examples

1. Tossing Two Coins

What is the likelihood of receiving a head on the first coin and a tail on the second if we were tossing two fair coins? Are these things happening on their own?

Solution: Let A be the "head" on the first coin and B be the "tail" on the second.

$P(A)$ = Probability of getting a head on the first coin = $1/2$

$P(B)$ = Probability of getting a tail on the second coin = $1/2$

$P(A \cap B)$ = Probability of receiving a tail on the second coin and a head on the first

$= (1/2) \times (1/2) = 1/4$

NOTE

Since $P(A \cap B) = P(A) \cdot P(B)$, the events A and B are independent.

Thus, the probability of getting a head on the first coin and a tail on the second coin is $1/4 = 0.25$.

2. Rolling a Die and Tossing a Coin

If we roll a fair six-sided die and toss a fair coin, what is the probability of rolling a 4 and getting heads? Are these events independent?

Solution: Let A = 'Rolling a 4' and B = 'Getting heads'.

$P(A)$ = Probability of rolling a $4 = 1/6$

$P(B)$ = Probability of getting heads $= 1/2$

$P(A \cap B)$ = Probability of rolling a 4 and getting heads = $(1/6) \times (1/2) = 1/12$

Since $P(A \cap B) = P(A) \cdot P(B)$, the events A and B are independent.

Thus, the probability of rolling a 4 and getting heads is $1/12 \approx 0.0833$.

7.3.7. Dependent Event

If the occurrence of one event influences the likelihood of the other event occurring, then the two events are said to be dependent. In these situations, the likelihood that both events will occur simultaneously is determined by:

$$P(A \cap B) = P(A).P(B/A)$$

Here, $P(B/A)$ is the conditional probability of event B given that event A has occurred.

Examples

1. Drawing Cards Without Replacement

What is the probability of drawing two access consecutively from a standard 52-card deck without replacement?

Solution: Thus,

let A ="First card is an Ace" and B = "Second card is an Ace".

$P(A)$ = Probability of the first card being an Ace = $\frac{4}{52} = \frac{1}{13}$

After one Ace is drawn, only 3 Aces are left, and the total number of cards becomes 51.

$P(B/A)$ = Probability of the second card being an Ace given the first card is an Ace = $\frac{3}{51}$

NOTE

$$P(A \cap B) = P(A).P(B/A) = \frac{1}{13}.\frac{3}{51} = \frac{3}{663} = \frac{1}{221}$$

Hence, the probability of drawing two Aces consecutively is $\frac{1}{221}$ approx 0.0045 *or* 0.45%.

2. Selecting Marbles from a Bag Without Replacement

Let us say there are three red and five blue marbles in a bag.

If two marbles are drawn one after the other without replacement, what is the probability that the first marble is red and the second marble is blue?

Solution: Thus,

let A = "First marble is red" and B = "Second marble is blue".

$P(A)$ = Probability of the first marble being red = 3/8

After a red marble is drawn, there are 2 red marbles and 5 blue marbles left, making a total of 7 marbles.

$P(B/A)$ = Probability of the second marble being blue given the first marble is red $= \frac{5}{7}$

$$P(A \cap B) = P(A).P(B/A) = \frac{3}{8}.\frac{5}{7} = \frac{15}{56}$$

Hence, the probability of the first marble being red and the second marble being blue is $\frac{15}{56}$ approx. 0.2679 *or* 26.79%.

7.3.8. Impossible and Sure Event

A. Impossible Event: An impossible event is an event that can never occur in any situation or under any circumstances. Its probability is always 0.

Examples:

1. Rolling a 7 on a standard six-sided die (since the die only has numbers 1–6).
2. Drawing a red card from a standard deck of playing cards after removing all red cards.

B. Sure Event: A sure event is an event that is guaranteed to occur under any situation. Its probability is always 1.

1. Rolling a number less than 7 on a standard six-sided die (all outcomes, 1–6, satisfy this condition).
2. Drawing a card from a standard deck of 52 cards (you are guaranteed to get a card).

NOTE

7.3.9. Simple and Compound Event

A. Simple Event: A simple event consists of only one outcome of an experiment. It cannot be broken down further into simpler components.

Examples:

1. Rolling a 3 on a standard six-sided die (only one possible outcome: 3).
2. Selecting the "King of Hearts" from a 52-card deck (just one card matches this requirement).

B. Compound Event: A compound event consists of two or more simple events combined. It can occur in multiple ways and involves more than one possible outcome.

Examples:

1. Rolling an 'odd number' on a six-sided die (possible outcomes: 1, 3, or 5).

   Choosing a "face card" from a regular deck of cards (the Jack, Queen, or King of any suit are possible results).

7.3.10 Complementary Event

Complementary events are two events that are mutually exclusive (cannot happen simultaneously) and together cover all possible outcomes of an experiment. The probability of the complementary events adds up to 1.

Formula: $P(A) + P(A') = 1$

Examples:

1. **Tossing a coin:**

   Event A: Getting a head.

   Complement A′: Getting a tail.

   Together, these events cover all possible outcomes.

2. **Rolling a die:**

   Event A: Rolling a number greater than 4 (outcomes: 5, 6).

   Complement A′ : Rolling a number less than or equal to 4 (outcomes: $1, 2, 3, 4$).

   Together, these events cover all possible outcomes.

Activity 7.1

Understanding Probability through a Dice Experiment

Objective:

To explore the concepts of probability, sample space, and different types of events using a six-sided die.

Materials Required:

- A six-sided die
- Pen and paper

Instructions:

1. Roll the die 20 times and record each outcome.
2. List the sample space for rolling a single die.
3. Classify different events from your recorded data:
   - A simple event (e.g., rolling a 3).
   - A compound event (e.g., rolling an even number).
   - An impossible event (e.g., rolling a 7).
   - A sure event (e.g., rolling a number between 1 and 6).
4. Calculate the probability of each type of event based on your recorded outcomes and compare them with theoretical probabilities.

Discussion Questions:

- What was the observed probability of rolling a 6 compared to the theoretical probability?
- How does increasing the number of trials affect the probability values?
- Can you identify any independent or dependent events based on this experiment?

7.4 RANDOM VARIABLE

7.4.1. Definition of Random Variable

A random variable is a numerical value determined by the outcome of a random experiment. It is a function that maps each outcome of the sample space to a real number.

Types of Random Variable:

1. **Discrete Random Variable:** Takes countable values (e.g., integers).
2. **Continuous Random Variable:** Takes any value within a range or interval

   (e.g., real numbers).

NOTE

Examples:

1. Discrete Random Variable:

- *Experiment:* Flipping a coin three times.
- *Random Variable X:* The quantity of heads acquired.
- *Possible values:* $X = 0, 1, 2, 3$.

2. Continuous Random Variable:

- **Experiment:** Measuring the height of students in a class.
- **Random Variable Y:** The height of a randomly chosen student (in cm).
- **Possible values:** Y can take any real number within a reasonable range

  (e.g., 140 cm to 200 cm).

7.4.2. More Examples of Random Variables in Probability

Example 1: Rolling Two Dice

Define a random variable X as the sum of the numbers on two six-sided dice that you throw.

Solution:

Possible values of X: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Probability distribution of X:

$P(X = 2) = 1/36$ (only one way: $1 + 1$).

$P(X = 7) = 6/36 = 1/6$ (ways: $1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, 6 + 1$).

Probabilities for other values are calculated similarly based on combinations.

Conclusion: The total of the numbers on two six-sided dice that you toss is what you designate as a random variable X.

Example 2: Drawing Balls from a Bag

A bag contains 3 red balls and 2 blue balls. You draw one ball, record its color, return it, and repeat the process 10 times. Define a random variable Y as the number of red balls drawn.

Solution:

Y can take values $\{0, 1, 2, \ldots, 10\}$.

Y follows a binomial distribution:

Probability of success (drawing a red ball) $p = 3/5$.

Probability of failure $1 - p = 2/5$.

NOTE

$P(Y = k) = (10\ choose\ k).p^k.(1-p)^{(10-k)}$, where k is the number of red balls drawn.

Conclusion: Y is a discrete random variable with a binomial probability distribution.

7.5 REPEATED TRIALS

7.5.1. Definition of Repeated Trials

Repeated Trials in probability which refer to the outcomes of events that repeat one or more times. The probability of success (p) and failure (q) are used to determine the total probability of n successes in a trial.

It also refers to the repetition of an experiment or observation to ensure the reliability and accuracy of results. By conducting multiple trials, we reduce the effects of random errors or anomalies and obtain a more consistent and valid outcome.

we can find the probability of various outcomes.

If a trial is repeated n time, the probability of $'r'$ successes and failures $'n - r'$ will be: $p^r q^{n-r}$

Were, p = probability of success

q = probability of failure

An experiment that repeats identically is known as **repeated trials** experiment in probability. A repeated trial in probability **reduces the errors**, increases the **reliability**, and **accuracy** of an experiment.

Examples:

3. Flipping a Coin:

- Suppose you are testing whether a coin is fair (i.e., has equal chances of landing heads or tails).
- You flip the coin 10 times and record the outcomes.
- To ensure your results are reliable, you repeat the process (e.g., flipping the coin another 10 times) and compare the frequencies of heads and tails across the trials.

4. Measuring Reaction Time:

- A person is asked to press a button as quickly as possible after seeing a light. Their reaction time is measured in milliseconds.
- To minimize variations caused by fatigue, distraction, or other factors, the test is repeated 5 times, and the average reaction time is calculated for a more reliable result.

NOTE

For example:

The probability of tossing a coin thrice

We are aware that a coin can have either a head (H) or a tail (T). $P(H) = 1/2$ is the likelihood that Head will occur. The probability of the occurrence of Tail is

$P(T) = 1/2.$

The probability of Head (H) and Tail (T) outcomes is equal because both have equal chances to occur.

The outcomes of tossing a coin thrice are

$(H,H,H),(H,H,T),(H,T,H),(T,H,H),(T,T,H),(T,H,T),(H,T,T)$, and (T,T,T).

The outcomes of getting two heads and one tail = $(H,H,T),(H,T,H),(T,H,H)$

Probability of each outcome will be $(1/2 \times 1/2 \times 1/2) = (1/2)^3$

Total probability of getting two heads and one tail = $3 \times (1/2)^3$

The outcomes of tossing a coin twice are $(H,H),(H,T),(T,H),(T,T)$.

7.5.2. More Examples of Repeated Trials in Probability

Example 1: Rolling a Die

You roll a six-sided die multiple times to estimate the probability of rolling a '4.'

Steps:

1. Perform 50 trials and record the outcomes.

2. Suppose '4' appears 8 times in these 50 trials.

Solution: Estimate the probability of rolling a '4':

$P(4)$ = Number of times 4 appears / Total number of trials

$P(4) = 8 / 50 = 0.16$

Conclusion: The probability of rolling a '4' based on repeated trials is approximately 16%.

Example 2: Drawing Cards from a Deck

You draw one card from a standard 52-card deck, record whether it is a heart, and replace the card. This is repeated 40 times.

Steps:1. Out of 40 trials, you observe that a heart is drawn 12 times.

Solution: Estimate the probability of drawing a heart:

$P(Heart)$ = Number of hearts drawn / Total number of trials

$$P(Heart) = 12 / 40 = 0.3$$

NOTE

Conclusion: The probability of drawing a heart based on repeated trials is approximately 30%. This is close to the theoretical probability of $13/52 = 0.25$ (25%).

Activity 7.2

Exploring Random Variables and Repeated Trials

Objective:

To understand random variables and the effect of repeated trials on probability outcomes.

Materials Required:

- A coin
- A pen and paper

Instructions:

1. Define a random variable for a coin toss experiment. Let:
   - $X = 1$ if the coin lands on heads.
   - $X = 0$ if the coin lands on tails.
2. Toss the coin 30 times and record the outcomes in terms of X.
3. Compute the relative frequency of heads and tails after every 5 tosses.
4. Plot a graph of the relative frequency of heads versus the number of trials.
5. Compare your results with the theoretical probability

$$(P(Heads) = 0.5, P(Tails) = 0.5).$$

Discussion Questions:

- How does the frequency of heads and tails change as the number of tosses increases?
- What happens to the relative frequency of heads when the number of trials is very large?

Why do we use random variables in probability? Give examples from real-life applications.

☑ Check Your Progress 7.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a random experiment ?

.....

NOTE

……

……

2. The probability of an impossible event is always 1. True or False:

3. What is the probability of getting a head when flipping a fair coin?

a) 0 b) 0.25

c) 0.5 d) 1

4. The total probability of all possible outcomes in a sample space is always _____.

7.6 SUMMARY

This unit introduced the fundamental concepts of probability and helped learners understand how uncertainty and chance can be measured mathematically. It began with the definition of probability and explained important terms such as random experiments, sample points, sample spaces, and events. Students learned how probability is used to describe the likelihood of outcomes occurring in different situations. The unit also covered independent and dependent events, showing how the occurrence of one event may or may not affect another event. Additional concepts such as impossible events, sure events, simple events, compound events, and complementary events were discussed to build a complete understanding of probability theory.

$$P(E) = \frac{Number\ of\ Favorable\ Outcomes}{Total\ Number\ of\ Outcomes}$$

The unit further introduced random variables and explained how numerical values can be assigned to outcomes of random experiments. Learners explored several examples of random variables used in everyday probability problems. The concept of repeated trials was also studied, emphasizing experiments that are performed multiple times under similar conditions. Students gained knowledge about how repeated trials help in predicting outcomes and analyzing patterns in probability experiments.

By the end of this unit, students should be able to identify different types of events, construct sample spaces, calculate basic probabilities, and distinguish between independent and dependent events. They should also understand the role of random variables and repeated trials in statistical analysis and decision-making. These concepts are highly relevant in real-life applications such as weather forecasting, business risk analysis, insurance, games, scientific experiments, and data analysis.

NOTE

Students are encouraged to further explore advanced topics in probability and statistics to strengthen their analytical and problem-solving skills. Practicing numerical problems, conducting simple probability experiments, and studying real-world applications can help deepen understanding and improve confidence in the subject.

7.7 GLOSSARY

- **Probability:** A measure of the likelihood of an event occurring, expressed as a value between 0 and 1.
- **Random Experiment:** A process or action that leads to uncertain outcomes.
- **Sample Space:** The set of all possible outcomes of a random experiment.
- **Sample Point:** A single possible outcome of a random experiment.
- **Event:** A subset of the sample space, representing one or more possible outcomes.
- **Independent Event:** An event whose occurrence is not affected by another event.
- **Dependent Event:** An event whose probability is influenced by the occurrence of another event.
- **Complementary Event:** The event that consists of all outcomes not included in the given event.
- **Random Variable:** A function that assigns numerical values to the outcomes of a random experiment.
- **Repeated Trials:** Multiple occurrences of the same experiment to observe probability patterns and distributions.

7.8 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 7.1

1. **Define a random experiment.**

   Answer: A random experiment is an experiment or process that can be repeated under identical conditions, but its outcome cannot be predicted with certainty in advance.

   Example: Tossing a coin — we do not know whether it will land on heads or tails.

2. **The probability of an impossible event is always 1. (True or False)**

   Answer: False.

   The probability of an impossible event is 0, not 1.

   Example: Rolling a 7 on a standard die is impossible.

NOTE

3. **What is the probability of getting a head when flipping a fair coin?**

   Answer: Correct option: (c) 0.5

   Explanation: A fair coin has two equally likely outcomes — Head or Tail.

   So, P(Head) = 1/2 = 0.5

4. **The total probability of all possible outcomes in a sample space is always ___.**

   Answer: The total probability of all possible outcomes is always 1.

   Explanation: The total probability of all possible outcomes is always 1 because one outcome must occur. In a coin toss, P(Head) + P(Tail) = 1.

7.9 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1247	

7.10 REFERENCE

- https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_probability.htm
- https://www.geeksforgeeks.org/probability-in-maths/
- https://www.probabilitycourse.com/chapter3/3_1_6_solved3_1.php
- https://byjus.com/maths/random-variable/
- https://www.javatpoint.com/repeated-trials-in-probability
- https://byjus.com/maths/random-experiments/
- https://byjus.com/maths/sample-space/
- https://byjus.com/maths/types-of-events-in-probability/
- https://www.vedantu.com/jee-main/maths-difference-between-independent-and-dependent-events
- https://unacademy.com/content/jee/study-material/mathematics/impossible-events/
- https://www.slideshare.net/slideshow/repeated-trials-probability/66386618

NOTE

7.11 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following is a simple event?

a) Getting an even number on a die

b) Drawing a red card from a deck

c) Tossing a coin and getting heads

d) Getting a sum of 7 when rolling two dice

2. If two events A and B are independent, then which of the following is true?

a) $P(A \cap B) = P(A) + P(B)$

b) $P(A \cap B) = P(A) \times P(B)$

c) $P(A \cap B) = P(A) - P(B)$

d) $P(A \cap B) = P(A) \div P(B)$

3. In a deck of 52 cards, what is the probability of drawing a King?

a) 1/26

b) 1/13

c) 1/52

d) 1/4

4. A bag contains 3 red balls and 7 blue balls. A ball is drawn at random. What is the probability that it is red?

a) 3/10

b) 7/10

c) ½

d) 1/3

5. Which of the following is an example of a dependent event?

a) Tossing a coin twice

b) Rolling a die and getting a 6

c) Drawing two cards from a deck without replacement

d) Drawing a single card from a deck

Answers with Explanations:

1. c) Tossing a coin and getting heads

Explanation: A simple event consists of only one outcome. Getting heads on a coin toss is a single, indivisible outcome, unlike other options that involve multiple outcomes.

NOTE

2. b) $P(A \cap B) = P(A) \times P(B)$

Explanation: For independent events, the probability of their intersection (both happening together) is the product of their individual probabilities.

3. b) 1/13

Explanation: There are 4 Kings in a standard deck of 52 cards. The probability is calculated as 4/52 = 1/13.

4. a) 3/10

Explanation: The total number of balls is 3 + 7 = 10. Since 3 of them are red, the probability of drawing a red ball is 3/10.

5. c) Drawing two cards from a deck without replacement

Explanation: In this case, the probability of drawing the second card depends on what was drawn first, making the events dependent.

B. Short answer type question

1. Define probability in simple terms.
2. What is an impossible event?
3. Give an example of a dependent event.
4. What is the probability of a sure event?
5. Define a sample space with an example.

Part 2

C. Long answer type question (200 words)

1. Explain the concept of independent and dependent events with suitable examples.
2. Define and differentiate between simple and compound events in probability.
3. What is a complementary event? How is it useful in probability calculations?
4. Describe the importance of sample space and its role in probability theory.
5. Explain the significance of repeated trials in probability with real-world examples.

Part 3

D. Long answer type question (300 words)

1. Discuss the fundamental principles of probability and explain the role of discrete probability distributions.

NOTE

2. Explain in detail the difference between theoretical and experimental probability with real-world applications.
3. What is a random variable? Explain its types and significance in probability theory.
4. Discuss the real-life applications of probability in various fields such as medicine, finance, and engineering.
5. What is the probability of drawing at least one Ace from a deck of 52 cards when drawing 5 cards? Explain with a proper probability approach.

UNIT 8
PROBABILITY DISTRIBUTION

UNIT STRUCTURE

8.1 Introduction
8.2 Objective
8.3 Probability Distributions
8.3.1 Definition of Probability Distribution
8.3.1.1. Binomial Distribution
8.3.1.2. Poisson Distribution
8.3.1.3. Exponential Distribution
8.3.1.4. Bernoulli's Distribution
8.3.1.5. Normal Distribution
8.4 Summary
8.5 Glossary
8.6 Answer to check your progress
8.7 Suggested readings - OER
8.8 Reference
8.9 Exercise
8.10 Feedback form

8.1 INTRODUCTION

In the previous unit, you explored the basics of probability, including random experiments, events, and probability calculations. You learned how different types of events influence outcomes and the role of probability in real-world scenarios. These concepts form the foundation for understanding probability distributions. In this unit, we extend this knowledge to explore various probability distributions and their applications.

Building on this knowledge, we introduce probability distributions, which describe how probabilities are assigned to different outcomes of a random variable. You will learn about key distributions such as Binomial, Poisson, and Normal distributions. Each distribution has unique characteristics and is used in different statistical analyses. Understanding these distributions helps in modeling real-world data effectively.

This unit bridges the gap between theoretical probability and practical applications by introducing probability distributions. A probability distribution defines the likelihood of different outcomes of a random process. From simple distributions like Bernoulli's to more complex

NOTE

ones like Normal and Exponential distributions, this unit will equip you with essential tools for statistical analysis and decision-making.

8.2 OBJECTIVE

After studying this unit, you will be able to….

- Describe the concept of probability distributions
- Define and differentiate between various probability distributions, including Binomial, Poisson, Exponential, Bernoulli, and Normal distributions.
- Apply probability distributions to real-world problems and analyze their significance in different scenarios.
- Derive and interpret the probability mass function (PMF) and probability density function (PDF) for various distributions.
- Compute probabilities, mean, variance, and standard deviation for different probability distributions.

8.3 PROBABILITY DISTRIBUTION

8.3.1. Definition of Probability Distribution:

A **probability distribution** is a mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment. It describes how the total probability of 1 is distributed among all possible values of a random variable.

The important probability distributions are:

1. Binomial distribution
2. Poisson distribution
3. Exponential distribution
4. Bernoulli's distribution
5. Normal distribution

8.3.1.1 Binomial Distribution:

The binomial distribution is a discrete probability distribution that describes the number of successes in a fixed number of independent trials, where each trial has only two possible outcomes: success or failure. The probability of success (p) remains constant for all trials.

Properties of Binomial Distribution:

1. Fixed number of trials (n).
2. Each trial is independent.

NOTE

3. Each trial has only two outcomes: success or failure.

4. The probability of success (p) and failure ($q = 1 - p$) is constant.

Probability Mass Function (PMF) or Binomial distribution formula is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where:

- $P(X = k)$ is the probability of k successes in n trials,
- n is the total number of trials,
- k is the number of successes,
- p is the probability of success in a single trial,
- $(1 - p)$is the probability of failure.
- $C(n,k) = \binom{n}{k} = \frac{n!}{(k!(n-k)!)}$ is the binomial coefficient.

Example 1. Tossing a Coin

A coin is tossed 5 times. Find the probability of getting exactly 3 heads, assuming the coin is fair ($p = 0.5$).

Solution: $n = 5, k = 3, p = 0.5$

$P(X = 3) = C(5,3) * (0.5)^3 * (0.5)^2$

$$P(X = 3) = \frac{5!}{3!\,(5-3)!} * (0.5)^3 * (0.5)^2$$

$P(X = 3) = 10 * 0.125 * 0.25 = 0.3125$

The probability of getting exactly 3 heads is 0.3125.

Example2: A fair coin is flipped 10 times. What is the probability of getting exactly 6 heads?

Solution: To find the probability of getting exactly 6 heads when a fair coin is flipped 10 times, we can use the binomial probability formula:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- $n = 10$ is the number of trials (coin flips),
- $k = 6$ is the number of desired heads,
- $p = 0.5$ is the probability of getting heads on any single flip,
- $\binom{n}{k}$ is the binomial coefficient, representing the number of ways to choose k successes (heads) from n trials.

NOTE

Step-by-step calculation:

1. Binomial Coefficient Calculation:

$$\binom{10}{6} = \frac{10!}{6!\,(10-6)!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

2. Applying the Binomial Formula:

$$P(X = 6) = \binom{n}{6}(0.5)^6(0.5)^4 = 210 \times (0.5)^{10}$$

3. Simplifying the Expression:

$$P(X = 6) = 210 \times \frac{1}{1024} = \frac{210}{1024} \approx 0.2051$$

Thus, the probability of getting exactly 6 heads when flipping a fair coin 10 times is approximately 0.2051, or 20.51%.

Example 3: Quality Control

A factory produces light bulbs, and 90% of them are defect-free ($p = 0.9$). Determine the likelihood that exactly eight of the ten bulbs in a randomly chosen sample are defect-free.

Solution: $n = 10, k = 8, p = 0.9$

$$\mathrm{P(X = 8)} = \mathrm{C}(10,8) * (0.9)^8 * (0.1)^2$$

$$\mathrm{P(X = 8)} = \frac{10!}{8!\,(10-8)!} * (0.9)^8 * (0.1)^2$$

$$\mathrm{P(X = 8)} = 45 * 0.43046721 * 0.01 = 0.1937$$

The probability that exactly 8 bulbs are defect-free is 0.1937.

8.3.1.2 Poisson Distribution:

Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space, given that the events occur independently and at a constant average rate. The events must be rare or infrequent compared to the size of the interval.

Probability mass function (PMF) or Poisson Distribution Formula is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Were,

e is the base of the natural logarithm (about 2.71828),

k is the number of events, and

λ is the average number of events (rate).

NOTE

Examples

Example 1: Calls to a Call Center

A call center receives an average of 3 calls per minute ($\lambda = 3$). What is the probability that the call center will receive exactly 5 calls in one minute?

Solution: We are given:

- $\lambda = 3$,
- $k = 5$

Using the Poisson distribution formula:

$$P(X = 5) = \frac{3^5 e^{-3}}{5!}$$

First, calculate the components:

$3^5 = 243$

$e^{-3} \approx 0.0498$

$5! = 120$

Now substitute these values into the formula:

$$P(X = 5) = \frac{243 \times 0.0498}{120} = \frac{12.1}{120} \approx 0.1008$$

So, the probability that exactly 5 calls will be received in one minute is approximately 0.1008 *or* 10.08%.

Example 2: Bus Arrivals at a Station

A bus station observes an average of 2 buses arriving every 10 minutes ($\lambda = 2$).What is the probability that exactly 1 bus will arrive in the next 10 minutes?

Solution: We are given:

- $\lambda = 2$
- $k = 1$

Using the Poisson distribution formula:

$$P(X = 1) = \frac{2^1 e^{-2}}{1!}$$

First, calculate the components:

- $2^1 = 2$,
- $e^{-2} \approx 0.1353$,
- $1! = 1$

NOTE

Now substitute these values into the formula:

$$P(X = 1) = \frac{1 \times 0.1353}{2} = 0.2706$$

So, the probability that exactly 1 bus will arrive in the next 10 minutes is approximately 0.2706 *or* 27.06%.

8.3.1.3 Exponential Distribution

The exponential distribution is a continuous probability distribution often used to model the time until an event occurs. It is defined by its probability density function (PDF):

$$f(x;\ \lambda) = \begin{cases} \lambda . e^{-\lambda x} & for\ x\ \geq\ 0 \\ 0 & for\ x\ <\ 0 \end{cases}$$

Where:

- $\lambda\ >\ 0$ is the rate parameter (inverse of the mean, $\lambda\ =\ 1/\mu$).
- x is the random variable representing time or distance until the event occurs.

Examples

1. Time Between Arrivals

A call center receives calls at an average rate of 3 calls per minute ($\lambda\ =\ 3$). The time between successive calls follows an exponential distribution.

The probability that the next call arrives within 10 seconds ($x\ =\ 10/60\ =\ 1/6$ minutes) is:

$$P\left(X\ \leq \frac{1}{6}\right)\ =\ 1\ -\ e^{-\lambda x}\ =\ 1\ -\ e^{-\left(\lambda.\frac{1}{6}\right)}\ =\ 1\ -\ e^{-(-0.5)}$$

2. Lifespan of a Machine Component

The lifespan of a machine component is exponentially distributed with an average lifespan of 5 years ($\mu\ =\ 5\ \rightarrow\ \lambda\ =\ 1/5\ =\ 0.2$).

The probability that the component lasts more than 8 years ($x\ =\ 8$) is:

$$P(X\ >\ 8)\ =\ e^{-\lambda x}\ =\ e^{-(-0.2\ .8)}\ =\ e^{(-1.6)}.$$

8.3.1.4 Bernoulli's Distribution

Bernoulli's distribution is a discrete probability distribution that models a single trial with two possible outcomes: success (with probability p) or failure (with probability $1\ -\ p$).

It is defined by its probability mass function (PMF):

$$P(X\ =\ x) = \begin{cases} p & if\ x = 1 \\ (1\ -\ p) & if\ x = 0 \end{cases}$$

NOTE

Where:

X is the random variable representing the outcome (1 for success, 0 for failure).

p is the probability of success, with $0 \leq p \leq 1$.

The expected value and variance are:

Mean: $\mathrm{E}[X] = p$

Variance: $Var(X) = p(1-p)$

Examples

1. **Tossing a Coin**

   A fair coin is tossed, where success (heads) has a probability of $p = 0.5$. The probability of getting heads $(X = 1)$ is:

   $$P(X = 1) = p = 0.5.$$

   The probability of getting tails $(X = 0)$ is:

   $$P(X = 0) = 1 - p = 0.5.$$

2. **Manufacturing Defect**

   In a manufacturing process, the probability of a product being defective is $p = 0.02$.

   The distribution can model whether a randomly selected product is defective

   $(X = 1)$ or **non-defective** $(X = 0)$.

   The probability that the product is defective is:

   $$P(X = 1) = p = 0.02.$$

   The probability that the product is not defective is:

   $$P(X = 0) = 1 - p = 0.98.$$

8.3.1.5. Normal Distribution:

Normal Distribution also known as the Gaussian distribution, is a continuous probability distribution that is symmetric about its mean, showing that data near the mean are more frequent in occurrence than data far from the mean. In a normal distribution, the mean, median, and mode is all equal.

The formula for the normal distribution's probability density function (PDF) is given by:

$$f(x;\mu,\sigma) = (\frac{1}{\sqrt{(2\pi\sigma^2}})e^{-(x-\mu)^2/(2\sigma^2)}$$

Where:

e is Euler's number (approximately 2.71828)

NOTE

μ is the mean (average) of the distribution,

σ (sigma) is the standard deviation.

σ^2 (sigma squared) is the variance.

x is the random variable for which the PDF is being evaluated.

The normal distribution is often referred to as the "bell curve" due to its bell-shaped appearance.

Examples

1. **Human Heights**

   Human heights within a population often follow a normal distribution.

   - For example, if the average height (μ) of adult males in a region is 170 cm with a standard deviation (σ) of 10 cm:
   - Most men have heights between 160 cm and 180 cm (within 1 standard deviation).
   - Heights greater than 200 cm or less than 140 cm are very rare.

2. **Exam Scores**

   Scores in standardized tests (like SAT or IQ tests) are often normally distributed.

   - Suppose the average score (μ)is 100 with a standard deviation (σ) of 15:
   - 68% of the students score between 85 and 115.
   - Only 2.5% of students score below 70 or above 130.

Activity 8.1

Understanding Probability Distributions through a Coin Toss Experiment

Objective:

To explore the concept of probability distributions and understand how different distributions model real-world situations.

Materials Required:

- A fair coin
- Pen and paper

Instructions:

1. Define the random variable (X):
   - Let X represent the number of heads obtained when flipping a coin 5 times.

NOTE

2. Perform the experiment:
   - Toss the coin 5 times and record the number of heads.
   - Repeat the experiment 20 times and note the results.
3. Construct the probability distribution:
   - Count how many times you obtained 0, 1, 2, 3, 4, and 5 heads.
   - Find the relative frequency (number of occurrences/total trials).
4. Compare with the Binomial Distribution:
   - Calculate the theoretical probability of getting k heads in 5 tosses using the Binomial formula:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $n = 5$ and $p = 0.5$.

   - Compare your experimental results with the expected probabilities.

Discussion Questions:

- How closely does your experimental probability match the theoretical Binomial probability?
- What happens when you increase the number of trials?
- Can you think of real-life scenarios where Binomial distribution is applicable?

☑ Check Your Progress 8.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is a probability distribution?

.....

.....

.....

2. The sum of all probabilities in a probability distribution is always 1. True or False:

3. Which of the following distributions is used to model the number of occurrences of an event in a fixed interval of time or space?

NOTE

A) Binomial Distribution

B) Poisson Distribution

C) Normal Distribution

D) Exponential Distribution

4. The _____ distribution is often called the "bell curve" because of its symmetric shape.

8.4 SUMMARY

This unit introduced the concept of Probability Theory and focused on different types of probability distributions used to describe uncertain events and random variables. Students learned the meaning and importance of a probability distribution, which explains how probabilities are assigned to possible outcomes of a random experiment. The unit covered major distributions such as the Binomial Distribution, Poisson Distribution, Exponential Distribution, Bernoulli's Distribution, and Normal Distribution, along with their characteristics, formulas, and practical uses. These distributions help in understanding patterns of data, predicting outcomes, and making statistical decisions in various fields.

A major learning outcome of this unit is the ability to identify different probability distributions and understand when each distribution should be applied. Students learned that the Bernoulli Distribution deals with experiments having only two outcomes, while the Binomial Distribution is useful for repeated independent trials. The Poisson Distribution helps in analyzing rare events occurring within a fixed interval, and the Exponential Distribution explains waiting time between events. The Normal Distribution, often called the bell-shaped curve, plays a significant role in statistics because many real-life data sets follow this pattern. Students also developed an understanding of concepts such as probability functions, mean, variance, and distribution behavior.

The knowledge gained from this unit has wide practical applications in business, economics, engineering, healthcare, research, and data analysis. Probability distributions are used in quality control, risk assessment, forecasting, traffic analysis, insurance studies, and scientific research. Understanding these concepts also provides a strong foundation for advanced topics in statistics, machine learning, and data science. Students are encouraged to further explore statistical software, practical problem-solving techniques, and real-world case studies to strengthen their understanding and analytical skills in probability and statistics.

NOTE

8.5 GLOSSARY

- **Probability Distribution:** A mathematical function that describes the likelihood of different outcomes for a random variable.
- **Random Variable:** A numerical outcome of a random experiment, classified as discrete or continuous.
- **Binomial Distribution:** A discrete probability distribution used for experiments with two outcomes (success/failure).
- **Poisson Distribution:** A distribution that models rare events occurring within a fixed time or space interval.
- **Exponential Distribution:** A continuous probability distribution that describes the waiting time between Poisson-distributed events.
- **Bernoulli Distribution:** A probability distribution of a single experiment with only two possible outcomes.
- **Normal Distribution:** A symmetric, bell-shaped continuous distribution used in statistics and probability theory.
- **Mean (Expected Value):** The average outcome of a probability distribution.
- **Variance:** A measure of the spread or dispersion of a probability distribution.
- **Standard Deviation:** The square root of variance, indicating how much individual values deviate from the mean.

8.6 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 8.1

1. **What is a probability distribution?**

   Answer: A probability distribution is a function or a table that gives the probabilities of all possible outcomes of a random variable in a sample space.

   It shows how the total probability (which equals 1) is distributed among various possible outcomes.

2. **The sum of all probabilities in a probability distribution is always 1. (True or False)**

   Answer: True.

   In a valid probability distribution, the sum of the probabilities of all possible outcomes is always 1.

NOTE

3. Which of the following distributions is used to model the number of occurrences of an event in a fixed interval of time or space?

Answer: Correct option: (B) Poisson Distribution

Explanation: The Poisson distribution is used to model counting events (like number of arrivals, calls, etc.) over a fixed interval of time, distance, or space.

4. The _____ distribution is often called the "bell curve" because of its symmetric shape.

Answer: The Normal distribution is often called the "bell curve" because of its symmetrical, bell-shaped appearance.

Explanation: The Normal distribution is called the “bell curve” because of its symmetrical bell-shaped appearance.

8.7 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1248	

8.8 REFERENCE

- https://byjus.com/maths/probability-distribution/
- https://byjus.com/maths/binomial-distribution/
- https://byjus.com/maths/poisson-distribution/
- https://byjus.com/maths/exponential-distribution/
- https://byjus.com/maths/bernoulli-trials-binomial-distribution/
- https://byjus.com/maths/normal-distribution/
- https://www.shiksha.com/online-courses/articles/normal-distribution-definition-and-examples/
- https://statisticsbyjim.com/probability/bernoulli-distribution/

NOTE

- https://www.probabilitycourse.com/chapter4/4_2_2_exponential . php
- https://www.cuemath.com/data/poisson-distribution/

8.9 EXERCISE

Part 1

A. Multiple choice question

1. **Which of the following distributions is used to model binary outcomes (success/failure)?**

   a) Normal Distribution

   b) Bernoulli Distribution

   c) Poisson Distribution

   d) Exponential Distribution

2. **If a normal distribution has a mean of 50 and a standard deviation of 10, what is the probability that a randomly selected value falls within one standard deviation from the mean?**

   a) 50%

   b) 68%

   c) 95%

   d) 99.7%

3. **Which of the following probability distributions is used to model rare events, such as the number of calls received at a call center in an hour?**

   a) Normal Distribution

   b) Binomial Distribution

   c) Poisson Distribution

   d) Bernoulli Distribution

4. **The sum of the probabilities in a probability distribution must always be:**

   a) 0

   b) 1

   c) Less than 1

   d) Greater than 1

5. **What shape does the normal distribution curve have?**

   a) Skewed Left

   b) Skewed Right

NOTE

c) Bell-shaped

d) U-shaped

Answers with Explanations:

1. **B) Bernoulli Distribution**

   Explanation: The Bernoulli distribution models an experiment with only two possible outcomes, such as flipping a coin (heads/tails) or passing/failing an exam.

2. **B) 68%**

   Explanation: According to the empirical rule, about 68% of the values in a normal distribution lie within one standard deviation of the mean.

3. **C) Poisson Distribution**

   Explanation: The Poisson distribution is used to model the probability of a certain number of events occurring in a fixed interval, given a constant average rate.

4. **B) 1**

   Explanation: A probability distribution represents the likelihood of all possible outcomes, and their total probability must sum to 1.

5. **C) Bell-shaped**

   Explanation: The normal distribution is symmetric and forms a bell-shaped curve, with the highest probability density around the mean.

B. Short answer type question

1. Define the Bernoulli distribution.
2. What is the mean of a normal distribution?
3. Write the probability mass function of a Binomial distribution.
4. What type of distribution is used for modeling the waiting time between two consecutive events?
5. Name one real-life example where the Poisson distribution is used.

Part 2

C. Long answer type question (200 words)

1. Explain the difference between discrete and continuous probability distributions with examples.
2. Derive the probability mass function of the Poisson distribution.

NOTE

3. How does the normal distribution approximate the binomial distribution? Explain with an example.
4. Describe the properties and applications of the Exponential distribution.
5. Compare and contrast Binomial and Poisson distributions in terms of their assumptions and real-world applications.

Part 3

D. Long answer type question (300 words)

1. Derive the mean and variance of the Binomial distribution.
2. Discuss the characteristics of the Normal distribution and explain its importance in statistics.
3. Derive the formula for the Exponential distribution and explain its real-life applications.
4. Explain the Central Limit Theorem and how it relates to probability distributions.
5. Solve a real-world problem using the Poisson distribution and explain its significance.

DR. C.V. RAMAN UNIVERSITY

Chhattisgarh, Bilaspur AN A/SECT GROUP UNIVERSITY

Approved by : PCI | AICTE | NCTE | BCI | Member of : AIU | Joint Committee : (UGC | DEB | AICTE) | Recognized by : UGC | A NAAC Accredited University

DR. C. V. RAMAN UNIVERSITY (C.G.)

SLM FEEDBACK FORM

PROGRAMME : _____

COURSE : _____

BLOCK : _____

UNIT : _____

Dear Student,

While studying the units of this block, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill up and send us the following questionnaire, which pertains to this block. If you find the space provided insufficient, kindly use a separate sheet.

Please mail to aisectcontent@aisect.org/ for online submission please scan QR code:

Questionnaire

1. **Name** : _____

2. **Enrolment No.** : | | | | | | | | | | | |

3. **How many hours did you need for studying the units?** : | |

4. **Please give your reactions to the following items based on your reading of the block:**

Items	Excellent	Very Good	Good	Poor	Give specific Examples (If poor)
Presentation Quality					
Language and Style					
Illustrations					
Used (diagrams, tables, etc.)					
Conceptual Clarity					
Check Your Progress Questions					
Suggested Readings - OER					

5. **Any other comments:**

_____

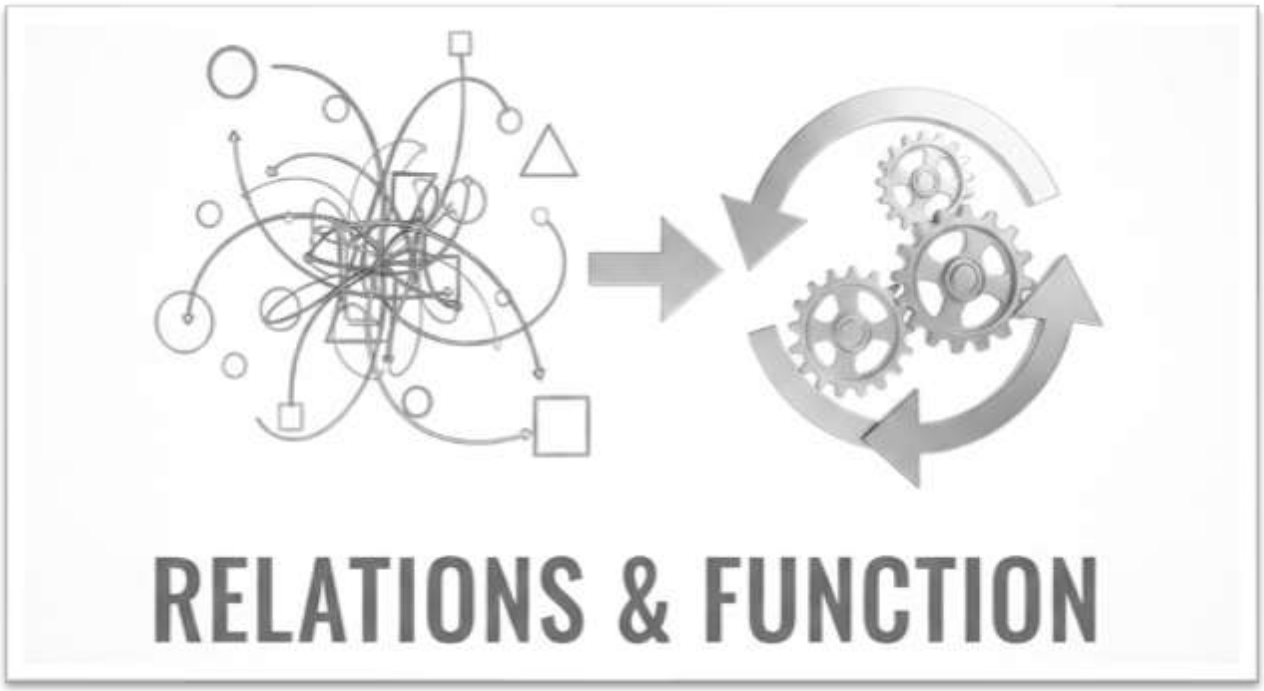
_____

_____

_____

***

BLOCK III	RELATIONS & FUNCTION	149-240
Unit 9	Cartesian Product and Relations	151
Unit 10	Operations and Properties of Relations	169
Unit 11	Important types of Relations	181
Unit 12	Lattices	201
Unit 13	Function	217



This block provides an in-depth study of relationships within sets and mappings between sets. It starts with Cartesian product and relations, establishing how ordered pairs form relationships. Next, the block examines operations and properties of relations, helping learners understand reflexive, symmetric, transitive, and similar characteristics. This is followed by important types of relations, enhancing classification and reasoning skills. The block then explores lattices, which are special partially ordered sets with wide applications in algebra and computer science. Finally, the concept of functions is introduced, detailing mappings between sets and emphasizing their mathematical and computational significance.

UNIT 9
CARTESIAN PRODUCT AND RELATIONS

UNIT STRUCTURE

9.1 Introduction
9.2 Objective
9.3 Cartesian (Cross) Product of Set
- **9.3.1. Definition of Cartesian Product**
- **9.3.2. Concept of Ordered Pairs**
- **9.3.3. Mathematical Definition**
- **9.3.4. Notation and Representation**

9.4 Relation
- **9.4.1. Definition of Relation**
- **9.4.2. Concept of Relation in Mathematics**
- **9.4.3. Representation of Relations**
- **9.4.4. Domain, Codomain, and Range**
- **9.4.5. Types of Relations (General Overview)**

9.5 Summary
9.6 Glossary
9.7 Answer to check your progress
9.8 Suggested readings – OER
9.9 Reference
9.10 Exercise

9.1 INTRODUCTION

In the previous unit, you explored the basics of sets and their operations, laying the groundwork for understanding mathematical structures. You learned about subsets, power sets, and fundamental set operations. Now, we extend these ideas to Cartesian products and relations. These concepts help establish meaningful connections between elements of different sets.

Building on this knowledge, this unit introduces the Cartesian (cross) product of sets, which forms the basis for defining relations. A relation describes how elements of one set correspond to elements of another set. By understanding domains, codomains, and ranges, you will gain insights into different types of relations. These concepts are essential in various fields, including computer science and discrete mathematics.

This unit bridges set theory and advanced mathematical concepts by exploring the representation and classification of relations. You will

NOTE

learn about ordered pairs, graphs, and matrix representations of relations. Understanding different types of relations, such as reflexive, symmetric, and transitive, will enhance your analytical skills. These ideas play a crucial role in logic, database management, and graph theory.

9.2 OBJECTIVE

After studying this unit, you will be able to:

- Construct the Cartesian Product of Sets:
- Define and Recognize Relations in Mathematics:
- Represent Relations Using Various Methods:
- Identify Domain, Codomain, and Range of Relations:
- Classify and Differentiate Types of Relations:

9.3 CARTESIAN (CROSS) PRODUCT OF SET

3.3.1 Definition of Cartesian Product:

The **Cartesian product** of two sets A and B, denoted as $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

Mathematically:

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

If either A or B is empty, then $A \times B = \emptyset$.

[A] EXAMPLES OF CARTESIAN PRODUCT

Example 1: Let $A = \{1,2\}$ and $B = \{x, y\}$. Find $A \times B$:

Solution: $A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$.

This represents all combinations of elements from A and B.

Example 2:

Let $A = \{a, b\}$ and $B = \{1,2,3\}$.Find $A \times B$**:**

Solution: $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

This lists all ordered pairs where the first element comes from A and the second from B.

Example 3: Let $A = \{3,4,5\}$ and $B = \{p, q\}$. Find $A \times B$:

Solution: $A \times B = \{(3, p), (3, q), (4, p), (4, q), (5, p), (5, q)\}$.

This set includes all possible pairs where the first element comes from A and the second comes from B.

Example 4: Let $A = \{x, y\}$ and $B = \{a, b, c\}$ Find $B \times A$:

Solution: $B \times A = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}$.

NOTE

Here, we form pairs with the first element from B and the second element from A, showing order in Cartesian products.

9.3.2. Concept of Ordered Pairs

An ordered pair is a pair of elements written in a specific order, typically as (a, b), where:

- The first element comes from the first set.
- The second element comes from the second set.
- The order of elements matters, i.e., $(a, b) \neq (b, a)$ in general

Example: If $A = \{1, 2\}$ and $B = \{x, y\}$,

then ordered pairs formed are: $(1, x), (1, y), (2, x), (2, y)$

This ensures that each element from A is paired with each element from B in a fixed order.

[A] Basic Ordered Pairs Examples

1. **Mathematical Representation**
   - $(3,5)$
   - $(-2,4)$
   - $(0,0)$
2. **Coordinates in a Cartesian Plane**
   - $(2,3)$ represents a point where $x = 2$ and $y = 3$.
   - $(-4,1)$ represents a point where $x = -4$ and $y = 1$.
3. **Real-Life Examples**
   - **Chessboard Position**: The position of a chess piece can be written as (row, column).
   - **Example**: $(2, D)$ means 2nd row, D column.
   - **GPS Location**: Latitude and Longitude are ordered pairs.
   - **Example:**(40.7128, -74.0060) represents New York City.
4. **Set Theory and Relations**
   - A relation between students and their grades:

     $\{(A, 85), (B, 90), (C, 78)\}$, where A, B, C are students and the second value is their grade.
5. **Function Mapping Example**
   - Consider a function $f(x) = 2x + 1$, some ordered pairs will be:
   - $\{(1,3), (2,5), (3,7)\}$, where the first value is input x and the second is the output $f(x)$.

9.3.3. Mathematical Definition of Cartesian Product

The **Cartesian product** of two sets A and B is the set of all **ordered pairs** (a, b), where aa belongs to A and b belongs to B. It is denoted by $A \times B$

NOTE

Mathematical Notation:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example:

Let $A = \{1,2\}$ and $B = \{a, b, c\}$, then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

[A] Properties of Cartesian Product:

1. Non-Commutative: $A \times B \neq B \times A$ in general.
2. Cardinality: If |A|=m and |B|=n, then $| A \times B | = m \times n$
3. Cartesian Product with an Empty Set: If either A or B is empty, then A×B=∅

[B] Functions and Relations as Cartesian Products

(a) Student Marks (Mapping Students to Scores)

- Let $S = \{Alice, Bob, Charlie\}$ be a set of students.
- Let $M = \{85,90,78\}$ be their respective marks.
- The Cartesian product $S \times M$ could include: $\{(Alice, 85), (Bob, 90), (Charlie, 78)\}$
- This represents a relation between students and their marks.

(b) Travel Routes (Mapping Cities to Flight Destinations)

- Let $C_1 = \{Delhi, Mumbai\}$ be a set of starting cities.
- Let $C_2 = \{London, New York\}$ be a set of destinations.
- The Cartesian product $C_1 \times C_2$ gives all possible flights:

$$\{(Delhi, London), (Delhi, New York), (Mumbai, London),$$
$$(Mumbai, New York)\}$$

- Ea
- ch ordered pair represents a direct flight.

[C] Cartesian Product in Computer Science

(a) Database Tables (Relational Databases)

- Consider a **Student Table**:

Student_ID	Name
101	John
102	Mary

NOTE

- **Course Table**:

Course_ID	Course_Name
C1	Math
C2	Science

- The Cartesian product (Student Table × Course Table) results in:

$\{(101, John, C1, Math), (101, John, C2, Science), (102, Mary, C1, Math),$

$(102, Mary, C2, Science)\}$

- This forms a join operation in databases.

[D] Real-World Applications of Cartesian Product

(a) Seating Arrangements in a Classroom

- Suppose a classroom has rows $A = \{1,2,3\}$ and columns $B = \{X, Y\}$.
- The Cartesian product A×B gives all possible seat locations:

$$A \times B = \{(1, X), (1, Y), (2, X), (2, Y), (3, X), (3, Y)\}$$

Here, each ordered pair represents a seat position (row, column).

(b) Playing Cards (Deck of 52 Cards)

- A standard deck has 4 suits $S = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$
- Each suit has 13 ranks $R = \{A, 2,3, \dots, 10, J, Q, K\}$
- The set of all cards is given by the Cartesian product:

$$S \times R = \{(\heartsuit, A), (\heartsuit, 2), \dots, (\spadesuit, K)\}$$

- Each ordered pair represents a unique card in the deck.

(c) GPS Coordinates (Latitude, Longitude Pairs)

- The location of any place on Earth is represented as an ordered pair (latitude, longitude).

Example:

- New York City → $(40.7128°N, 74.0060°W)$
- London → $(51.5074°N, 0.1278°W)$
- The set of all locations can be seen as a Cartesian product of latitude values and longitude values.

9.3.4. Notation and Representation of Cartesian Product

The Cartesian Product of two sets A and B is the set of all ordered pairs (a, b), where aa is from set A and b is from set B. It is denoted by:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

NOTE

1. Notation of Cartesian product

Example: Let $A = \{1,2\}$ and $B = \{a, b, c\}$, then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

2. Representation in Tabular Form

For $A = \{x, y\}$ and $B = \{1,2,3\}$:

A	B	Ordered Pair
x	1	(x,1)
x	2	(x,2)
x	3	(x,3)
y	1	(y,1)
y	2	(y,2)
y	3	(y,3)

Thus, $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

3. Representation in Cartesian Plane (Graphical Representation)

For $A = \{1,2\}$ and $B = \{3,4\}$, we plot the points on a 2D plane:

$$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$$

These are plotted as points:

- (1,3)
- (1,4)
- (2,3)
- (2,4)

Each point represents an **ordered pair**.

4. Representation in Set Form

If $A = \{p, q\}$ and $B = \{m, n\}$, then:

$$A \times B = \{(p, m), (p, n), (q, m), (q, n)\}$$

5. Matrix Representation

If $A = \{1,2\}$ and $B = \{a, b, c\}$, the Cartesian product can be arranged in matrix form:

	a	b
1	$(1, a)$	$(1, b)$
2	$(2, a)$	$(2, b)$

NOTE

[A] Cartesian Product of Two Finite Sets

If A and B are finite sets, then their Cartesian product contains a finite number of ordered pairs.

Mathematical Definition: For two finite sets A and B, their Cartesian product is: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

The total number of elements in A×B is given by: $| A \times B | = | A | \times | B |$

where $| A |$ and $| B |$ denote the number of elements in sets A and B.

Example:

Let $A = \{1,2,3\}$ and $B = \{x, y\}$, then: $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

Since $| A | = 3$ and $| B | = 2$, we get: $| A \times B | = 3 \times 2 = 6$

So, there are 6 ordered pairs in A×B.

[B] Cartesian Product of Infinite Sets

If either A or B is an infinite set, then A×B is also infinite.

Mathematical Definition: If A or B is infinite, then: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

will have infinitely many ordered pairs.

Examples:

(a) Cartesian Product of a Finite and an Infinite Set

Let A={1,2,3} (finite) (set of natural numbers, infinite). Then:

$$A \times B = \{(1,1), (1,2\text{and B} = \text{N}), (1,3), \dots, (2,1), (2,2), (2,3), \dots, (3,1), (3,2), (3,3), \dots\}$$

This set is infinite.

(b) Cartesian Product of Two Infinite Sets

If both sets are infinite, say A=N and B=N , then:

$$N \times N = \{(1,1), (1,2), (1,3), \dots, (2,1), (2,2), (2,3), \dots\}$$

This is the set of all pairs of natural numbers, which is countably infinite.

[C] Visual Representation of Cartesian Product

(a) Representation Using a Table

For $A = \{1,2\}$ and $B = \{a, b, c\}$, we can represent $A \times B$ in a table:

A	B	Ordered Pair
1	a	(1, a)
1	b	(1, b)

NOTE

1	c	(1, c)
2	a	(2, a)
2	b	(2, b)
2	c	(2, c)

This gives:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

(b) Representation Using a Graph

For $A = \{1,2\}$ and $B = \{3,4\}$, the Cartesian product is:

$$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$$

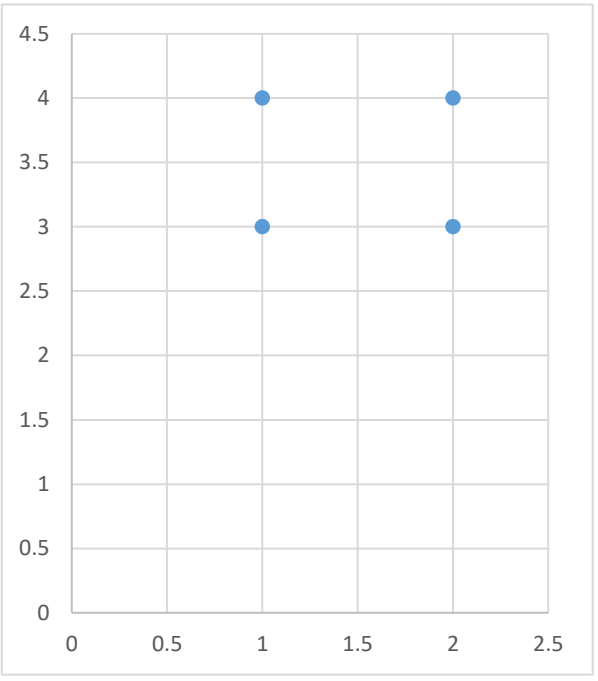
This can be plotted on a 2D Cartesian plane, where:

- The x-axis represents elements of A.
- The y-axis represents elements of B.

Graphical Representation:

Each ordered pair is represented as a point on the Cartesian plane.

X-(A)	Y-(B)
1	3
1	4
2	3
2	4



Here:

- (1,3), (1,4), (2,3), and (2,4) are plotted as points.

NOTE

9.4. RELATION

9.4.1 Definition of Relation

A **relation** in abstract algebra is a subset of the Cartesian product of two sets. If A and B are two sets, then a relation R from A to B is defined as $R \subseteq A \times B$,

where $A \times B = \{(a, b) \mid a \in A, b \in B\}$. If $(a, b) \in R$, we say aa is related to b under the relation R, often written as aRb.

[A] EXAMPLES

Example 1: Divisibility Relation

Let $A = \{1,2,3,4\}$, and define the relation R as aRb if $a \mid b$ (i.e., *a divides b*).

Solution: $A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), \ldots, (4,4)\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Example 2: Equality Relation

Let $A = \{x, y, z\}$, and define R as aRb if $a = b$.

Solution: $A \times A =$ $\{(x,x), (x,y), (x,z), (y,x), (y,y), (y,z), (z,x), (z,y), (z,z)\}$

$R = \{(x,x), (y,y), (z,z)\}$.

Example 3: Congruence Modulo Relation

Let $A = \{0,1,2,3,4\}$, and define R as aRb if $a \equiv b(mod\ 2)$

(i.e., a and b have the same parity).

Solution: $A \times A = \{(0,0), (0,1), (0,2), \ldots, (4,4)\}$.

$R = \{(0,0), (0,2), (0,4), (2,0), (2,2), (2,4), (4,0), (4,2), (4,4), (1,1), (1,3),$

$(3,1), (3,3)\}$.

Example 4: Less Than Relation

Let $A = \{1,2,3\}$, and define R as aRb if $a < b$.

Solution: $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

$R = \{(1,2), (1,3), (2,3)\}$.

9.4.2. Concept of Relation in Mathematics

1. Definition of a Relation

A relation in mathematics is a connection between elements of two sets. It is a subset of the Cartesian product of two sets.

NOTE

2. Mathematical Definition

Let A and B be two sets. A **relation** R from A to B is a subset of the Cartesian product A×B

R⊆A×B

If (a,b)∈R, we say that "a is related to b" and write it as: $a\ R\ b$

3. Example of a Relation

Example 1: Relation Between Students and Their Marks

Let:$A = \{Alice,\ Bob,\ Charlie\}$ (students)

- $B = \{85,90,78\}$ (marks)
- Relation R represents "Student scored this mark":

$$R = \{(Alice, 85), (Bob, 90), (Charlie, 78)\}$$

Here, RR is a **subset of A×B**.

Example 2: Divisibility Relation

Let $A = \{1,2,3,4,5,6\}$ and define a relation R where:

$$R = \{(a, b) \mid a\ divides\ b\}$$

If $B = \{2,4,6\}$, then the relation includes:

$$R = \{(1,2), (1,4), (1,6), (2,4), (2,6), (3,6)\}$$

9.4.3. Representation of Relations

(a) Tabular Form

A (Input)	B (Output)	Relation (A → B)
1	2	(1,2)
1	4	(1,4)
1	6	(1,6)
2	4	(2,4)
2	6	(2,6)
3	6	(3,6)

(b) Graphical Representation (Arrow Diagram)

For $A = \{1,2,3\}$ and $B = \{4,5,6\}$, where:

$$R = \{(1,4), (2,5), (3,6)\}$$

We can represent it using an arrow diagram:

A → B

1 → 4

NOTE

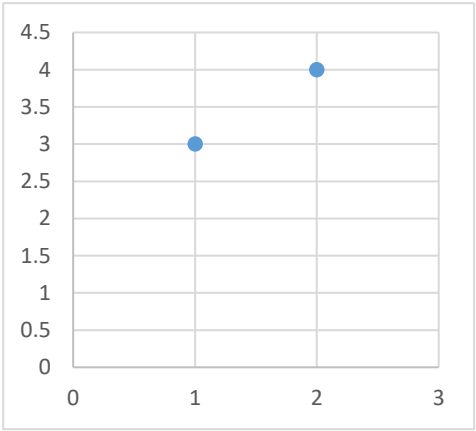
$2 \rightarrow 5$

$3 \rightarrow 6$

(c) Graph Representation

If $A = \{1,2\}$ and $B = \{3,4\}$, and the relation is $R = \{(1,3),(2,4)\}$, we can plot the points on a graph.

A	B
1	3
2	4



Each * represents an ordered pair in R.

9.4.4. Domain, Codomain, and Range in a Relation

In a relation RR from a set AA to a set BB, three important terms are used:

1. **Domain**: The set of all first elements (inputs) in the relation.
2. **Codomain**: The set in which the second elements (outputs) are defined.
3. **Range**: The set of actual second elements (outputs) that appear in the relation.

1. Domain

The **domain** of a relation RR is the set of all first elements (inputs) from the ordered pairs.

Example:

Let R={(1,A),(2,B),(3,C)}R = \{(1, A), (2, B), (3, C)\}.

- The **domain** is {1,2,3}\{1, 2, 3\}.

2. Codomain

The **codomain** is the set that contains all possible second elements (outputs), even if some elements are not actually used.

NOTE

Example:

Let $A = \{1,2,3\}$ and $B = \{A, B, C, D\}$ and the relation is:

$$R = \{(1, A), (2, B), (3, C)\}$$

- The **codomain** is $B = \{A, B, C, D\}$

Even though D is in B, it does not appear in the relation.

3. **Range**

The **range** is the set of all second elements (outputs) that actually appear in the relation.

Example:

For $R = \{(1, A), (2, B), (3, C)\}$:

- The range is $\{A, B, C\}$ since these values appear in the relation.
- The codomain B includes D, but D is not in the range.

Example with Numbers

Let $R = \{(1,4), (2,5), (3,6)\}$, where:

- $A = \{1,2,3\}$ (input set)
- $B = \{4,5,6,7,8\}$ (output set)

Then:

- **Domain** = {1,2,3} (first elements)
- **Codomain** = {4,5,6,7,8} (all possible outputs)
- **Range** = {4,5,6} (actual outputs)

Key Differences

Term	Definition	Example
Domain	Set of all first elements (inputs)	{1,2,3}
Codomain	The set in which outputs are defined	{4,5,6,7,8}
Range	Set of actual second elements (outputs)	{4,5,6}

9.4.5. Types of Relations (General Overview)

1. **Empty Relation**: No elements are related.
$$R = \emptyset$$
2. **Universal Relation**: Every element of A is related to every element of B.
$$R = A \times B$$
3. **Identity Relation**: Each element of A is related to **itself**.
$$R = \{(a, a) \mid a \in A\}$$

NOTE

4. **Inverse Relation**: If $(a, b) \in R$, then the inverse relation $R - 1$ contains (b, a)
5. **Reflexive, Symmetric, and Transitive Relations**: Important for defining equivalence relations.



Activity 9.1

Exploring Cartesian Products and Relations

Objective: To understand the Cartesian product of sets and how relations are formed and represented.

Materials Needed:

- Pen and paper
- A calculator (optional)

Instructions:

1. **Cartesian Product Exercise:**
   - Consider two sets:

   $A = \{1,2,3\}$ and $B = \{a, b\}$
   - Write down the Cartesian product $A \times B$
   - How many ordered pairs are there?
2. **Defining a Relation:**
   - Define a relation R from set A to set B, where each number in A is related to a letter in B if the number is odd.
   - Write down the set of ordered pairs representing this relation.
3. **Representation of Relations:**
   - Draw a diagram (arrow representation) to show the relation you defined.
   - Represent the relation using a matrix and a directed graph.
4. **Domain, Codomain, and Range Identification:**
   - Identify the domain, codomain, and range of your relation RR.
5. **Types of Relations (Discussion Question):**
   - Can the relation you defined be classified as reflexive, symmetric, or transitive? Why or why not?
   - Discuss with a partner or write a short explanation.

Conclusion:

This activity helps in visualizing how Cartesian products form the basis of relations and how they can be represented in different ways.

NOTE

By working through this, you will gain a deeper understanding of mathematical relations and their properties.

☑ Check Your Progress 9.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define the Cartesian product of two sets with an example.

.....

.....

.....

2 The range of a relation is always equal to its codomain. True or False:

3 If $A = \{1, 2\}$ and $B = \{a, b, c\}$, how many elements are there in $A \times B$?

a) 2 b) 3

c) 5 d)6

4. The set of all first elements in a relation is called the _____ of the relation.

9.5 SUMMARY

The unit on Cartesian Product and Relations introduced the fundamental concepts used to study relationships between sets in mathematics. Students learned about the Cartesian (Cross) Product of sets, which combines elements from two sets to form ordered pairs. The concept of ordered pairs was explained as an arrangement where the order of elements matters, helping students understand how elements are connected within sets. The mathematical definition, notation, and representation of Cartesian products were also discussed, enabling learners to write and interpret expressions such as A×BA \times BA×B. The unit further explored the concept of relations, defining a relation as a subset of a Cartesian product that describes connections between elements of two sets. Different methods of representing relations, including ordered pairs, tables, arrow diagrams, and graphs, were introduced for better understanding and visualization. Students also learned the important concepts of domain, codomain, and range, which help identify the input values, possible outputs, and actual outputs of a relation. In addition, a general overview of different types of relations was provided to help students recognize how mathematical relationships behave in various situations. By the end of the unit,

NOTE

students developed the ability to form Cartesian products, identify and represent relations, and analyze their properties systematically. These concepts are highly useful in advanced mathematics, computer science, database management, logic, and real-life applications involving connections between objects, people, or data. Understanding relations also builds a strong foundation for studying functions and higher mathematical structures. Students are encouraged to practice more examples and explore real-world applications of relations to strengthen their conceptual understanding and problem-solving skills.

9.6 GLOSSARY

- **Set** – A well-defined collection of distinct objects or elements.
- **Cartesian Product**– The set of all ordered pairs formed by taking one element from each of two sets.
- **Relation**– A subset of a Cartesian product that defines a relationship between elements of two sets.
- **Domain** – The set of all first elements of the ordered pairs in a relation. It represents the input values of the relation.
- **Codomain**– The set in which the second elements of the ordered pairs must lie. It represents the possible outputs of the relation.
- **Range**– The set of actual second elements (outputs) that appear in the ordered pairs of a relation. It is a subset of the codomain.
- **Representation of Relations**– Ways to visualize a relation, such as arrow diagrams, matrices, and directed graphs.
- **Reflexive Relation**– A relation where every element is related to itself.
- **Symmetric Relation**– A relation where if (a, b) exists, then (b, a) also exists.
- **Transitive Relation**– A relation where if (a, b) and (b, c) exist, then (a,c)(a, c) must also exist.

9.7 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 9.1

1. **Define the Cartesian product of two sets with an example.**

   Answer: The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

   Example: If $A = \{1, 2\}$ and $B = \{x, y\}$, then

   $$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

NOTE

2. The range of a relation is always equal to its codomain. (True or False)

Answer: False.

The range is the set of actual outputs (second elements in the relation), while the codomain is the set of all possible outputs. They can be different.

3. If A = {1, 2} and B = {a, b, c}, how many elements are there in A × B?

Answer: Correct option: (d) 6

Explanation: |A×B|=|A|×|B|=2×3=6|A × B| = |A| × |B| = 2 × 3 = 6

4. The set of all first elements in a relation is called the _____ of the relation.

Answer: The set of all first elements in a relation is called the **domain** of the relation.

9.8 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1249	

9.9 REFERENCE

- https://www.geeksforgeeks.org/cartesian-product-of-sets/
- https://byjus.com/maths/cartesian-products-of-sets/
- https://www.cuemath.com/algebra/cartesian-product/
- https://byjus.com/maths/relations-and-its-types/
- https://study.com/academy/lesson/relation-in-math-definition-examples.html

9.10 EXERCISE

Part 1

A. Multiple choice question

1. If $A = \{x, y\}$ and $B = \{1, 2, 3\}$, how many elements will be in $A \times B$?

b) 2

NOTE

b) 3

c) 5

d) 6

2. What is the domain of the relation $R = \{(1,2),(3,4),(5,6)\}$?

a) {2, 4, 6}

b) {1, 3, 5}

c) {1, 2, 3, 4, 5, 6}

d) {(1,2), (3,4), (5,6)}

3. Which of the following relations is reflexive?

a) $R = \{(1,1),(2,2),(3,3),(1,2)\}$

b) $R=\{(1,2),(2,3),(3,1)\}$

c) $R=\{(1,1),(2,2),(3,3)\}$

d) $R=\{(1,2),(2,1),(3,3)\}$

4. If a relation is both symmetric and transitive, then it must be:

a) Reflexive

b) Equivalence

c) Antisymmetric

d) None of these

5. The range of the relation $R = \{(2,3),(4,5),(6,3),(8,5)\}$ is:

a) {2,4, 6, 8}

b) {3, 5}

c) {3, 5, 6, 8}

d) {2, 3, 4, 5, 6, 8}

Answers with Explanations:

1 (d) 6

Explanation: The Cartesian product $A \times B$ consists of all ordered pairs,

so $| A |\times| B | = 2 \times 3 = 6$.

2. (b) {1, 3, 5}

Explanation: The domain consists of the first elements of the ordered pairs.

3. (c) $R = \{(1,1),(2,2),(3,3)\}$

Explanation: A reflexive relation must contain all pairs (a,a) for all elements in the set.

4. (b) Equivalence

Explanation: An equivalence relation is one that is reflexive, symmetric, and transitive.

NOTE

5. (b) {3, 5}

Explanation: The range consists of the second elements (outputs) of the ordered pairs.

B. Short answer type question

1. What is the Cartesian product of two sets?
2. Define a relation in mathematics.
3. What is the domain of a relation?
4. Give an example of a symmetric relation.
5. What are the different ways to represent a relation?

Part 2

C. Long answer type question (200 words)

1. Explain the Cartesian product of two sets with an example. How is it used in defining relations?
2. Define a relation. How can a relation be represented using matrices and graphs?
3. Differentiate between the domain, codomain, and range of a relation with suitable examples.

Part 3

D. Long answer type question (300 words)

1. Discuss in detail the Cartesian product of sets and its significance in defining relations. Provide multiple examples to illustrate the concept.
2. Define relations and explain the various ways they can be represented, including arrow diagrams, matrices, and graphs. Give detailed examples for each method.
3. What are reflexive, symmetric, and transitive relations? Explain with real-life applications and examples for better understanding.

UNIT 10
OPERATIONS AND PROPERTIES OF RELATIONS

NOTE

UNIT STRUCTURE

10.1 Introduction
10.2 Objective
10.3 Operation on Relation
- **10.3.1 Union of Relations**
- **10.3.2 Intersection of Relations**
- **10.3.3 Composition of Relations**
- **10.3.4 Inversion (or Reversal) of a Relation**

10.4 Properties of Relation
- **10.4 1 Reflexive Relation**
- **10.4 2 Symmetric Relation**
- **10.4 3 Transitive Relation**
- **10.4 4 Antisymmetric Relation**

10.5 Binary Relation on a Set
- **10.5.1 Definition of Binary Relation on a Set**
- **10.5.2 Examples of Binary Relations**

10.5 Summary
10.6 Glossary
10.7 Answer to check your progress
10.8 Suggested readings – OER
10.9 Reference
10.10 Exercise

10.1 INTRODUCTION

In the previous unit, you explored the basics of relations, including their definition, representation, domain, codomain, and types. These concepts helped establish connections between elements of different sets. Understanding relations is fundamental in various mathematical applications. Now, we extend this knowledge to operations and properties of relations.

Building on this knowledge, this unit introduces operations on relations, such as union, intersection, and composition. It also explores properties like reflexivity, symmetry, transitivity, and equivalence. Additionally, the concept of binary relations on a set will be discussed with examples. These ideas are widely used in logic, graph theory, and computer science.

NOTE

This unit bridges the study of basic relations and their advanced mathematical applications by examining binary relations in depth. You will learn how to analyze and manipulate relations using set operations and understand their essential properties. Mastering these concepts is crucial for understanding functions, databases, and discrete structures.

10.2 OBJECTIVE

After studying this unit, you will be able to:

- Explain the concept of operations on relations, including union, intersection, composition, and inverse of a relation.
- Identify and analyze the properties of relations, such as reflexivity, symmetry, transitivity, and equivalence.
- Define and explain binary relations on a set with appropriate examples.
- Represent binary relations using arrow diagrams, matrices, and directed graphs.
- Apply the knowledge of relations to real-world problems, including databases, networking, and logic systems.

10.3 OPERATION ON RELATION

Operation on a relation refers to how a binary relation interacts with operations like union, intersection, composition, or inversion. A binary relation R on a set A is a subset of the Cartesian product A×A. The operation can apply to manipulate or combine relations in different ways.

10.3.1. Union of Relations

Combines the pairs from both relations

The union of two relations R and S on a set A is denoted $R \cup S$ and is defined as:

$$R \cup S = \{(a, b) \mid (a, b) \in R \ \ or \ \ (a, b) \in S\}$$

Example 1: Union of Relations

Let $A = \{1,2,3\}$, and consider the relations R and S on A:

$R = \{(1,2), (2,3)\}$

$S = \{(1,3), (3,2)\}$

The union $R \cup S$ is: $R \cup S = \{(1,2), (2,3), (1,3), (3,2)\}$

NOTE

10.3.2. Intersection of Relations

Keeps only the common pairs from both relations.

The intersection of two relations R and S on a set A is denoted $R \cap S$ and is defined as: $R \cap S = \{(a,b) \mid (a,b) \in R \text{ and } (a,b) \in S\}$

Example 1: Intersection of Relations

Let $A = \{1,2,3\}$, and consider the relations R and S on A:

$R = \{(1,2),(2,3)\}$

$S = \{(1,3),(2,3)\}$

The intersection $R \cap S$ is: $R \cap S = \{(2,3)\}$

10.3.3. Composition of Relations:

Forms new pairs by "connecting" matching elements from one relation to another.

The composition of two relations R and S on a set A is denoted $R \circ S$ and is defined as: $R \circ S = \{(a,c) \mid \exists\, b \in A$ such that $(a,b) \in R$ and $(b,c) \in S\}$.

Example 1: Composition of Relations

Let $A = \{1,2,3\}$, and consider the relations R and S on A:

- $R = \{(1,2),(2,3)\}$
- $S = \{(2,3),(3,1)\}$

The composition $R \circ S$ is: $R \circ S = \{(1,3),(2,1)\}$

10.3.4. Inversion (or Reversal) of A Relation:

Reverses the direction of each pair in the relation.

The inverse (or reversal) of a relation R, denoted R^{-1}, is defined as:

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$ `

Example 1: Inversion of a Relation

Let $A = \{1,2,3\}$, and consider the relation R on A:

- $R = \{(1,2),(2,3)\}$

The inverse $R^{-1}$is: $R^{-1} = \{(2,1),(3,2)\}$

10.4 PROPERTIES OF RELATION

Relations have various properties that define their characteristics on a set A. A **binary relation** R on a set A is a subset of $A \times A$. These properties can coexist (e.g., reflexive, symmetric, and transitive together define an equivalence relation), or they may be mutually exclusive depending on the relation.

A relation may satisfy one or more of the following properties:

NOTE

10.4.1. Reflexive Relation

A relation R on A is reflexive if every element of A is related to itself, i.e.,$(a, a) \in R \; \forall a \in A$.

Example 1: Reflexive Relation

Let $A = \{1,2,3\}$, and let $R = \{(1,1), (2,2), (3,3), (1,2)\}$

Check if R is reflexive:

$(1,1) \in R, (2,2) \in R, (3,3) \in R$.

Since all diagonal elements are in R, R is reflexive.

10.4.2. Symmetric Relation

A relation R on A is symmetric if: $(a, b) \in R \implies (b, a) \in R \forall a, b \in A$.

Example 1: Symmetric Relation

Let $A = \{1,2,3\}$, and let $R = \{(1,2), (2,1), (2,3), (3,2)\}$.

Check if R is symmetric:

For $(1,2) \in R, (2,1) \in R$.

For $(2,3) \in R, (3,2) \in R$.

Since all pairs are symmetric, R is symmetric.

10.4.3. Transitive Relation

A relation R on A is transitive if: $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R \quad \forall a, b, c \in A$.

Example 1: Transitive Relation

Let $A = \{1,2,3\}$, and let $R = \{(1,2), (2,3), (1,3)\}$.

Check if R is transitive: $(1,2) \in R$ and $(2,3) \in R \implies (1,3) \in R$

Since this condition holds for all possible triples, R is transitive.

10.4.4. Antisymmetric Relation

A relation R on A is antisymmetric if: $(a, b) \in R$ and $(b, a) \in R \Longrightarrow a = b \quad \forall a, b \in A$.

Example 1: Antisymmetric Relation

Let $A = \{1,2,3\}$, and let $R = \{(1,2), (2,3), (1,1), (2,2)\}$

Check if R is antisymmetric:

For $(1,2) \in R$ and $(2,1) \notin R$, the condition holds.

No pair (a, b) exists with both (a, b) and (b, a)(unless $a = b$.

Therefore, R is antisymmetric.

10.5 BINARY RELATION ON A SET

NOTE

10.5.1 Definition of Binary Relation on a Set

A binary relation R on a set A is a subset of the Cartesian product $A \times A$. In other words, a binary relation R on A is a collection of ordered pairs (a, b), where $a, b \in A$ and $(a, b) \in R$ indicates that a is related to b under R.

Mathematically: $R \subseteq A \times A$.

A binary relation can exhibit different properties, such as being reflexive, symmetric, antisymmetric, or transitive, depending on its behavior.

10.5.2 Examples of Binary Relation

Example 1: Reflexive Relation

Let $A = \{1,2,3\}$. A binary relation R is reflexive if $(a, a) \in R$ for all $a \in A$.

Solution: A reflexive relation on A must include $(1,1), (2,2)$, and $(3,3)$. One possible relation is:

$$R = \{(1,1), (2,2), (3,3), (1,2)\}.$$

Example 2: Symmetric Relation

Let $A = \{x, y\}$. A binary relation R is symmetric if $(a, b) \in R$ implies $(b, a) \in R$.

Solution: A symmetric relation on A could be: $R = \{(x, x), (y, y), (x, y), (y, x)\}$.

Example 3: Antisymmetric Relation

Let $A = \{1,2,3\}$. A binary relation R is antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ together imply $a = b$.

Solution: An antisymmetric relation on A could be:

$R = \{(1,1), (2,2), (3,3), (1,2)\}$.

Here, $(1,2) \in R$ but $(2,1) \notin R$, satisfying antisymmetric.

Example 4: Transitive Relation

Let $A = \{a, b, c\}$. A binary relation R is transitive if $(a, b) \in R$ and $(b, c) \in R$ together imply $(a, c) \in R$.

Solution: A transitive relation on A could be:

$R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$.

NOTE

Activity 10.1

Exploring Binary Relations and Their Properties

Objective:

To understand the concept of binary relations, their representation, and properties such as reflexivity, symmetry, and transitivity.

Materials Required:

- Pen and paper
- A set of elements (Example: $A = \{1,2,3\}$
- Grid or graph paper (optional)

Instructions:

1. **Forming a Binary Relation:**
   - Consider the set $A = \{1,2,3\}$.
   - List all possible ordered pairs in $A \times A$.
   - Choose a subset of these ordered pairs to define a relation RRR. Example: $R = \{(1,1), (1,2), (2,3), (3,3)\}$.
2. **Representation:**
   - Draw an arrow diagram showing the relation RRR.
   - Create a matrix representation for RRR using a 3×3 matrix with 1s and 0s.
3. **Analyzing Properties:**
   - Reflexive: Check if every element in A is related to itself (i.e., (a,a)∈R for all $a \in A$.
   - Symmetric: If $(a, b) \in R$, then check if $(b, a) \in R$ as well.
   - Transitive: If $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ and $(b, c) \in R$, then check if $(a, c) \in R$.
4. **Discussion Questions:**
   - Is the given relation reflexive, symmetric, and transitive? Justify your answer.
   - Can you modify the relation R to make it an equivalence relation?
   - Where do you see relations like this in real life (e.g., social networks, road maps)?

Expected Outcome:

By the end of this activity, students will be able to construct binary relations, represent them visually, and analyze their properties, strengthening their understanding of relations in mathematics.

NOTE

☑ Check Your Progress 10.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is a binary relation on a set?

.....

.....

.....

2. Every symmetric relation is also reflexive. True or False:

3. Which of the following properties must an equivalence relation satisfy?

a) Reflexive only

b) Reflexive and symmetric

c) Reflexive, symmetric, and transitive

d) Symmetric and transitive

4. A relation R is said to be _____ if for every $(a, b) \in R$, $(b, a) \in R$ also holds.

.....

10.6. SUMMARY

The unit Operations and Properties of Relations introduced the fundamental concepts of relations in mathematics and discrete structures, focusing on how relations can be combined, analyzed, and applied. The unit began with operations on relations, including the union and intersection of relations, which help combine or compare relationships between elements. Students also learned about the composition of relations, an important concept used to connect two relations in sequence, and the inversion or reversal of a relation, which swaps the order of related elements. These operations provide a strong foundation for understanding how relationships between objects are represented and manipulated in mathematics and computer science.

The unit further explored the important properties of relations such as reflexive, symmetric, transitive, and antisymmetric relations. Students learned how to identify these properties and understand their significance in defining special types of relations. Reflexive relations show self-connection among elements, symmetric relations represent mutual relationships, transitive relations help establish indirect connections, and antisymmetric relations are useful in ordering systems. Understanding these properties is essential for studying equivalence relations, partial ordering, graph theory, and database systems.

NOTE

Another major focus of the unit was the concept of binary relations on a set. Students studied the definition of binary relations and examined several examples to understand how elements within a set can be related to one another. This topic strengthened logical thinking and improved the ability to represent relationships using ordered pairs, tables, and diagrams.

The concepts covered in this unit are highly relevant in real-world applications such as computer programming, data organization, network analysis, artificial intelligence, and mathematical modeling. These ideas also form the basis for advanced topics in discrete mathematics and theoretical computer science. Students are encouraged to further explore equivalence relations, functions, graphs, and ordered sets to deepen their understanding and enhance their analytical and problem-solving skills.

10.7 GLOSSARY

- **Binary Relation** – A relation defined between elements of the same set, represented as a subset of the Cartesian product $A \times A$.
- **Reflexive Relation** – A relation where every element is related to itself, ensuring self-connection.
- **Symmetric Relation** – A relation where if one element is related to another, the reverse is also true.
- **Transitive Relation** – A relation where indirect connections between elements imply a direct connection.
- **Equivalence Relation** – A relation that is reflexive, symmetric, and transitive, forming equivalence classes.
- **Composition of Relations** – An operation that combines two relations to establish indirect relationships.
- **Inverse Relation** – A relation obtained by swapping the first and second elements in every ordered pair.
- **Relation Matrix** – A matrix representation of a relation where 1 indicates a connection, and 0 indicates no connection.
- **Graph Representation** – A directed graph used to visualize relations between elements in a set.
- **Applications of Relations** – Real-world uses include database structures, social networks, and logic systems.

10.8 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

NOTE

Check Your Progress 10.1

1. What is a binary relation on a set?

Answer: A binary relation on a set A is a subset of the Cartesian product $A \times A$.

It consists of ordered pairs (a, b) where $a, b \in A$.

In other words, a binary relation describes how elements of the set relate to each other.

2. Every symmetric relation is also reflexive. True or False:

Answer*:* False.

A symmetric relation means if $(a, b) \in R$

But reflexive means $(a, a) \in R$ for all $a \in A$, which is not guaranteed by symmetry alone.

3. Which of the following properties must an equivalence relation satisfy?

Answer: c) Reflexive, symmetric, and transitive

An equivalence relation must have all three properties:

- Reflexive: $(a, a) \in R$
- Symmetric: $(a, b) \in R \Rightarrow (b, a) \in R$
- Transitive: $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

4. A relation R is said to be _____ if for every (a,b)∈R, (b,a)∈R also holds.

Answer: Symmetric

This is the definition of a symmetric relation**.**

10.9 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1250	

NOTE

10.10 REFERENCE

- https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_relations.htm
- http://users.pja.edu.pl/~msyd/mad-lectures/relations.pdf
- 3. https://math24.net/operations-relations.html
- https://byjus.com/jee/properties-of-relations-in-set-theory/
- https://cse.buffalo.edu/~xinhe/cse191/Classnotes/note09-1x2.pdf
- https://www.geeksforgeeks.org/binary-relation/
- https://www.cuemath.com/algebra/reflexive-relations/
- https://www.cuemath.com/algebra/symmetric-relations/
- https://byjus.com/maths/antisymmetric-relation/
- https://www.cuemath.com/algebra/transitive-relations/
- https://en.wikipedia.org/wiki/Equivalence_relation
- https://www.geeksforgeeks.org/discrete-mathematics-representing-relations/
- https://math.stackexchange.com/questions/175344/composition-of-relations

10.11 EXERCISE

Part 1

A. Multiple choice question

1. **Which of the following is NOT an operation on relations?**

   a) Union

   b) Intersection

   c) Differentiation

   d) Composition

2. **If a relation R on set A satisfies $(a, a) \in R$ for all $a \in A$, then the relation is called:**

   a) Symmetric

   b) Reflexive

   c) Transitive

   d) Antisymmetric

3. **A relation R on a set A is called symmetric if:**

   a) $(a, b) \in R \Rightarrow (b, a) \in R$

   b) $(a, a) \in R$ for all $a \in A$

   c) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

   d) $(a, b) \in R \Rightarrow (b, a) \notin R$

NOTE

4. Which of the following is an example of an equivalence relation?

a) "Is greater than" relation

b) "Is less than" relation

c) "Is equal to" relation

d) "Is a subset of" relation

5. In a relation matrix, what does the entry at position $(i, j) = 1$ indicate?

a) The element i is not related to j

b) The relation is symmetric

c) There is a relation from element i to element j

d) The matrix represents a function

Answers with Explanations:

1. (c) Differentiation

Explanation: Operations on relations include union, intersection, difference, inverse, and composition. Differentiation is a concept from calculus, not related to operations on relations.

2. (b) Reflexive

Explanation: A relation is reflexive if every element is related to itself, meaning $(a, a) \in R$ for all $a \in A$

3. (a) $(a, b) \in R \Rightarrow (b, a) \in R$

Explanation: A relation is symmetric if whenever (a, b) is in R, the reverse pair (b, a) must also be in R.

4. (c) "Is equal to" relation

Explanation: An equivalence relation must be reflexive, symmetric, and transitive. The equality relation satisfies all these properties.

5 (c) There is a relation from element i to element j

Explanation: A relation matrix represents a relation using 1s and 0s. If the entry at (i, j) is 1, it means the ordered pair (i, j) is included in the relation.

B. Short answer type question

1. Define the composition of two relations.
2. Give an example of a reflexive relation.
3. What is an inverse relation?

NOTE

4. How can a relation be represented using a matrix?
5. What is an antisymmetric relation?

Part 2

C. Long answer type question (200 words)

1. Explain in detail the operations on relations such as union, intersection, and composition with examples.
2. What are the properties of relations? Discuss reflexivity, symmetry, and transitivity with suitable examples.
3. Define binary relation and give at least three real-life examples where binary relations are used.
4. Explain how relations can be represented using matrices and directed graphs.
5. Discuss the significance of equivalence relations and their applications in mathematics and real life.

Part 3

D. Long answer type question (300 words)

1. Explain in detail the concept of binary relations on a set. Discuss its significance, representation methods (arrow diagrams, matrices, and directed graphs), and provide at least three real-life applications of binary relations.
2. Define and elaborate on the different types of relations (reflexive, symmetric, antisymmetric, transitive, and equivalence relations) with appropriate examples. Also, explain how these properties impact the classification of relations in mathematics.
3. Discuss in depth the operations on relations, including union, intersection, difference, inverse, and composition. Provide step-by-step examples demonstrating each operation and explain their importance in set theory and discrete mathematics.
4. Explain the concept of equivalence relations. Prove that a given relation is an equivalence relation using mathematical reasoning. Also, explain how equivalence classes are formed and discuss their significance in partitioning a set.
5. Describe the role of binary relations in computer science, databases, and networking. Explain with examples how relations help in designing relational databases, structuring data, and optimizing search operations in large-scale systems.

UNIT 11
IMPORTANT TYPES OF RELATIONS

NOTE

UNIT STRUCTURE

11.1 Introduction
11.2 Objective
11.3 Equivalence Relation
- **11.3.1 Definition of Equivalence Relation**
- **11.3.2 Properties of Equivalence Relation**
- **11.3.3 Examples of Equivalence Relations**
- **11.3.4 Equivalence Classes and Partitioning of a Set**
- **11.3.5 Applications of Equivalence Relations**

11.4 Partial Order Relation
- **11.4.1 Definition of Partial Order Relation**
- **11.4.2 Properties of Partial Order Relation**
- **11.4.3 Examples of Partial Order Relations**
- **11.4.4 Hasse Diagrams and Representation of Partially Ordered Sets (Poset)**
- **11.4.5 Totally Ordered Set (or linear order).**
- **11.4.6 Applications of Partial Order Relations**

11.5 Summary
11.6 Glossary
11.7 Answer to check your progress
11.8 Suggested readings – OER
11.9 Reference
11.10 Exercise

11.1 INTRODUCTION

In the previous unit, you explored the basics of relations, including binary relations, their operations, and properties such as reflexivity, symmetry, and transitivity. These concepts helped in understanding how elements within a set interact. You also learned about different ways to represent relations using matrices and graphs. This knowledge serves as a foundation for more advanced types of relations.

Building on this knowledge, this unit introduces two significant types of relations: Equivalence Relations and Partial Order Relations. Equivalence relations help in grouping elements with similar properties, forming equivalence classes. Partial order relations establish a hierarchy among elements, which can be visually represented using

NOTE

Hasse diagrams. Understanding these relations is essential for mathematical reasoning and applications.

This unit bridges theoretical concepts with practical applications by explaining how equivalence relations partition sets and how partial order relations organize data. You will explore real-world uses of these relations in computer science, database management, logic, and decision-making systems. Mastering these concepts enhances problem-solving skills in various fields, including mathematics and computer science.

11.2 OBJECTIVE

After studying this unit, you will be able to:

- Define the concept of Equivalence Relations.
- Explain Equivalence Classes and Partitioning of a Set.
- Analyze Partial Order Relations.
- Interpret Hasse Diagrams as a graphical representation of partially ordered sets
- Explore real-world applications of Equivalence and Partial Order Relations.

11.3 EQUIVALENCE RELATION

11.3.1 Definition of Equivalence Relation

An equivalence relation on a set A is a binary relation $R \subseteq A \times A$ that satisfies the following three properties:

- **Reflexive**: Every element is related to itself

  $i.e.$, $(a, a) \in R$ or aRa for all $a \in A$.

- **Symmetric**: If an element is related to another, the relation holds in reverse

  $i.e.$, If $(a, b) \in R$, or aRb then $(b, a) \in R$. or bRa.

- **Transitive**: If an element is related to a second, and the second to a third, then the first is related to the third

  $i.e.$, If $(a, b) \in R \; or \; aRb$ and $(b, c) \in R \; or \; bRc$, then $(a, c) \in R \; or \; aRc$.

Example:

Consider the set S of all people, and define the relation "has the same birthday as":

$aRb \Leftrightarrow$Person a and Person b were born on the same day of the year.

NOTE

Checking the properties:

- **Reflexive**: Every person has the same birthday as themselves. (TRUE)
- **Symmetric**: If person a has the same birthday as person b, then b has the same birthday as a. (TRUE)
- **Transitive**: If person a has the same birthday as person b, and person b has the same birthday as person c, then person a has the same birthday as person c. (TRUE)

Since all three properties hold, "has the same birthday as" is an equivalence relation.

11.3.2 Properties of Equivalence Relation

A relation is called an equivalence relation if and only if it satisfies all three properties: Reflexivity, Symmetry, and Transitivity.

An equivalence relation on a set S is a relation that satisfies the following three fundamental properties:

1. **Reflexivity:** A relation R on S is reflexive if for every element $a \in S$, we have: aRa

   Example: The relation "has the same birthday as" is reflexive because every person has the same birthday as themselves.

2. **Symmetry:** A relation R on S is symmetric if for all elements $a, b \in S$, whenever aRb, then: bRa

   Example: The relation "is a sibling of" is symmetric because if A is a sibling of B, then B is also a sibling of A.

3. **Transitivity:** A relation R on S is transitive if for all elements $a, b, c \in S$, whenever aRb and bRc, then: aRc

   Example: In modular arithmetic, if $a \equiv b \, mod \, 3$ and $b \equiv c \, mod \, 3$, then $a \equiv c \, mod \, 3$, satisfying transitivity.

11.3.3 Examples of Equivalence Relation

Example 1: Equality on Integers

Let $A = \{1,2,3\}$. The relation R defined as $R = \{(1,1),(2,2),(3,3)\}$ is an equivalence relation because it satisfies reflexivity, symmetry, and transitivity.

Example 2: Congruence Modulo Relation

Let $A = Z$ (the set of integers). Define the relation $a \sim b$ if $a \equiv b \ (mod \ n)$, where $n > 0$

For example, if $n = 3$, $a \sim b$ if $a - b$ is divisible by 3.

This is an equivalence relation.

NOTE

Example 3: Divisibility Relation

Let $A = \{6,12,18\}$. Define $a \sim b$ if $gcd\ (a, b) = a$ (i.e., $a\ divides\ b$).

For instance, $6 \sim 12$ and $6 \sim 18$.

The relation is reflexive, symmetric, and transitive.

Example 4: Same Color in a Set of Objects

Let A represent a set of objects, and define $a \sim b$ if a and b have the same color.

This is an equivalence relation because it satisfies reflexivity, symmetry, and transitivity.

11.3.4 Equivalence Classes and Partitioning of a Set

Equivalence class: An equivalence class is a subset of a set S formed by grouping all elements that are equivalent to each other under an equivalence relation. If R is an equivalence relation on S, then the equivalence class of an element $a \in S$, denoted by $[a]$, is: $[a] = \{x \in S \mid xRa\}$

Partitioning of a Set: An equivalence relation divides a set into disjoint equivalence classes, forming a partition of the set.

This means:

1. Each element belongs to exactly one equivalence class.
2. The union of all equivalence classes covers the entire set.
3. No two equivalence classes have common elements.

Example:

Consider the set $S = \mathbb{Z}$ (set of all integers) with the equivalence relation:

$$aRb \Leftrightarrow a \equiv bmod\ 3$$

This means that two integers are related if they have the same remainder when divided by 3.

Equivalence Classes:

- **Class [0]**: All numbers divisible by 3 $\rightarrow \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$
- **Class [1]**: All numbers giving remainder 1 when divided by 3 $\rightarrow \{.., -5, -2, 1, 4, 7, ..\}$
- **Class [2]**: All numbers giving remainder 2 when divided by 3 $\rightarrow \{.., -4, -1, 2, 5, 8, ..\}$

These three classes partition $\mathbb{Z}$, meaning every integer belongs to exactly one class, and no two classes overlap.

NOTE

Conclusion:

- Equivalence classes group elements that are indistinguishable under a given equivalence relation.
- Every equivalence relation partitions the set into mutually exclusive equivalence classes.

11.3.5 Applications of Equivalence Relations

Equivalence relations play a crucial role in various fields of mathematics, computer science, and real-world applications. Equivalence relations help in classification, simplification, and organization of data across various domains.

Here are some key areas where equivalence relations are applied:

1. **Modular Arithmetic**
   - Used in cryptography, computer science, and clock arithmetic.

     Example: The relation $a \equiv b \bmod n$ groups integers based on their remainder when divided by n.

2. **Classification of Objects**
   - Equivalence relations are used to classify objects into similar categories.

     Example: In geometry, triangles are classified as congruent if they have the same shape and size.

3. **Graph Theory and Network Analysis**
   - Equivalence relations help in grouping nodes in networks that are structurally similar.

     Example: In social networks, people can be grouped based on the relation "is a friend of".

4. **Computer Science and Programming**
   - File system permissions use equivalence relations (same access rights = same class).

     Example: Compiler optimization groups equivalent expressions to optimize code execution.

5. **Partitioning Data in Databases**
   - Equivalence relations are used to partition data into meaningful subsets.

     Example: In relational databases, equivalence classes help in query optimization and data indexing.

NOTE

6. **Set Theory and Logic**
   - Equivalence relations help define quotient sets and partitions of sets.

     Example: The relation "is logically equivalent to" partitions propositional logic statements into classes of equivalent statements.

7. **Image Processing and Machine Learning**
   - Equivalence relations are used to group similar images or data points.

     Example: Face recognition systems classify images based on equivalence classes of features.

8. **Physics and Engineering**
   - In thermodynamics, equivalence relations are used to define equilibrium states.

     Example: In mechanical systems, parts with similar behavior are grouped into equivalence classes for analysis.

11.4 PARTIAL ORDER RELATION

11.4.1 Definition of Partial Order Relation

A **partial order relation** is a binary relation R on a set S that satisfies the following three properties:

1. **Reflexivity**: aRa for all $a \in S$ (every element is related to itself).
2. **Antisymmetric**: If aRb and bRa, then $a = b$.
3. **Transitivity**: If aRb and bRc, then aRc.

Thus, if a relation R satisfies these three properties, then (S, R) is called a partially ordered set (Poset).

Example:

Consider the set $S = \{1,2,3,4\}$ with the relation $\leq$.

Checking properties:

- **Reflexive**: Every number is less than or equal to itself, so $a \leq a$ for all $a \in S$ ✓
- **Antisymmetric**: If $a \leq b$ and $b \leq a$, then a=b. ✓
- **Transitive**: If $a \leq b$ and $b \leq c$, then $a \leq c$ ✓

Since all three conditions hold, $\leq$ is a **partial order relation** on S.

NOTE

Conclusion:

A partial order relation organizes elements in a structured way but does not necessarily mean that every pair of elements is comparable. If every pair is comparable, the relation is called a total order (or linear order).

11.4.2 Properties of Partial Order Relation

A partial order relation on a set A is a binary relation $R \subseteq A \times A$ that satisfies the following properties:

- **Reflexive**: $(a, a) \in R$ for all $a \in A$.
- **Antisymmetric**: If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.
- **Transitive**: If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

A partial order relation on a set S is a binary relation R that satisfies the following three properties:

1. **Reflexivity**:
   - Every element in S is related to itself.
   - Mathematically, for all $a \in S$: aRa

   Example: In the relation "less than or equal to" ($\leq$) on numbers, $a \leq a$ for all a.

2. **Antisymmetric**:
   - If two elements are related in both directions, they must be equal.
   - Mathematically, for all $a, b \in S$: aRb and $bRa \Rightarrow a = b$

   Example: In the subset relation ($\subseteq$) on a power set, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

3. **Transitivity**:
   - If one element is related to a second, and the second to a third, then the first must be related to the third.
   - Mathematically, for all $a, b, c \in S$: aRb and $bRc \Rightarrow aRc$.

   Example: In the "divisibility" relation (|), if a | b and b | c, then a | c.

NOTES:

- A set with a partial order relation is called a partially ordered set (Poset).
- If every pair of elements in a Poset is comparable, it is called a totally ordered set (or linear order).
- Partial order relations are often represented using **Hasse diagrams** for clarity.

NOTE

11.4.3 Examples of Partial Order Relation

Example 1: Divisibility

Let $A = \{1,2,4,8\}$. Define $a \leq b$ if a divides b.

For example, $(1,2), (2,4), (1,8) \in R$.

This is reflexive, antisymmetric, and transitive.

Example 2: Subset Relation

Let $A = \{\{1\}, \{1,2\}, \{1,2,3\}\}$. Define R as $A1 \subseteq A2$.

For example, $(\{1\}, \{1,2\}) \in R$, $(\{1\}, \{1,2,3\}) \in R$.

This is a partial order relation.

Example 3: Alphabetical Order

Let $A = \{a, b, c\}$, and define R as $a \leq b$ according to alphabetical order.

For example, $(a, b), (a, c), (b, c) \in R$.

This satisfies reflexivity, antisymmetric, and transitivity.

Example 4: Less Than or Equal Relation

Let $A = \{1,2,3\}$. Define $R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\}$.

This is a partial order relation.

11.4.4 Hasse Diagrams and Representation of Partially Ordered Sets (Posets)

Definition of Hasse Diagram: A Hasse diagram is a graphical representation of a partially ordered set (Poset) that shows the relationships between elements without displaying redundant information. It helps visualize the structure of a Poset by omitting:

1. **Reflexive relations** (i.e., loops like $a \leq a$).
2. **Transitive relations** (if $a \leq b$ and $b \leq c$, then $a \leq c$ is not drawn).
3. **Arrows** (edges are drawn in an upward direction instead of explicitly showing arrowheads).

Steps to Construct a Hasse Diagram:

1. **List the elements** of the Poset.
2. **Identify the partial order relation** between elements.
3. **Place the smallest elements at the bottom** and the largest at the top.
4. **Draw edges only for direct relations** (omit transitive edges).
5. **Connect elements with upward lines** to show ordering.

NOTE

Example 1: Divisibility Relation

Consider the set $S = \{1,2,4,8\}$ with the divisibility relation $|$

(i.e., $a \mid b$ means a divides b).

- **Partial Order Relation:** 1|2, 1|4, 1|8, 2|4, 4|8, and 2|8 (not drawn due to transitivity).
- **Hasse Diagram Representation:**

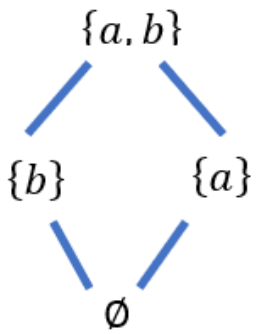


Here, 1 is the least element (bottom), and 8 is the greatest element (top).

Example 2: Power Set with Subset Relation

Consider $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ with the subset ($\subseteq$) relation.

- **Partial Order Relation:**
  - $\emptyset \subseteq \{a\},\ \emptyset \subseteq \{b\},\ \emptyset \subseteq \{a, b\}$
  - $\{a\} \subseteq \{a, b\},\ \{b\} \subseteq \{a, b\}$
- **Hasse Diagram Representation:**



Here, $\emptyset$ (empty set) is the minimum element, and $\{a, b\}$ is the maximum element.

Importance of Hasse Diagrams:

1. **Simplifies visualization** of complex partial order structures.
2. **Removes redundant information**, showing only essential relationships.

NOTE

3. **Helps in studying lattices, Poset and order relations.**

11.4.5 Totally Ordered Set (or Linear Order).

Definition: A totally ordered set (also called a linearly ordered set) is a partially ordered set (Poset) in which every pair of elements is comparable.

Formally, a totally ordered set is a set (S, R) where R is a partial order relation that satisfies:

1. **Reflexivity:** aRa for all $a \in S$.
2. **Antisymmetric:** If aRb and bRa, then $a = b$.
3. **Transitivity:** If aRb and bRc, then aRc.
4. **Comparability (Totality):** For all $a, b \in S$, either aRb or bRa must be true.

Example:

Consider the set $S = \{1,2,3,4\}$ with the relation $\leq$ (less than or equal to).

- **Reflexive:** $1 \leq 1$, $2 \leq 2$, etc. ✓
- **Antisymmetric:** If $a \leq b$ and $b \leq a$, then $a = b$. ✓
- **Transitive:** If $1 \leq 2$ and $2 \leq 3$, then $1 \leq 3$. ✓
- **Comparability:** For any two elements a, b in S, either $a \leq b$ or $b \leq a$ holds. ✓

Since all four properties are satisfied, $(S, \leq)$ is a totally ordered set.

Difference Between Partial and Total Order:		
Property	**Partial Order**	**Total Order**
Reflexivity	✓	✓
Antisymmetric	✓	✓
Transitivity	✓	✓
Comparability	✗ (not required)	✓ (must hold for all elements)

- **Example of Partial Order:** The subset relation $\subseteq$ on power sets. Not all sets are comparable.
- **Example of Total Order:** The numerical relation $\leq$ on real numbers $\mathbb{R}$. Every two numbers are comparable.

NOTE

11.4.6 Applications of Partial Order Relations

Partial order relations are widely used in various fields, including mathematics, computer science, decision-making, and real-world applications. Partial order relations are essential in structuring, ranking, and optimizing data in mathematics, computer science, project management.

Below are some key applications:

1. **Task Scheduling and Dependency Management**
   - **Application:** Used in project planning, dependency resolution, and scheduling tasks.
   - **Example:**
     - In job scheduling, some tasks must be completed before others

       (e.g., task A must finish before task B starts).
     - The dependencies between tasks can be represented using a Hasse diagram.
     - *Example:* In software development, compilation dependencies follow a partial order, where some files depend on others.
2. **Hierarchical Structures**
   - **Application:** Used in organizational charts, ranking systems, and taxonomy.
   - **Example:**
     - The management hierarchy in a company follows a partial order where employees report to a supervisor.
     - The biological classification of species follows a hierarchical structure.
3. **File System Permissions**
   - **Application:** Determines access levels for users in an operating system.
   - **Example:**
     - The relation "has more privileges than" follows a partial order.
     - A system administrator has more access rights than a regular user, but some users may have equal rights.
4. **Divisibility in Number Theory**
   - **Application:** Used to study divisibility relations among numbers.

NOTE

- **Example:**
  - The set $S = \{1,2,4,8\}$ with the divisibility relation (|) forms a partially ordered set:
  - $1 \mid 2, 1 \mid 4, 1 \mid 8, 2 \mid 4, and\ 4 \mid 8.$
  - This structure is represented using a Hasse diagram.

5. Boolean Algebra and Lattice Theory

- **Application:** Used in circuit design, logic gates, and database indexing.
- **Example:**
  - In Boolean algebra, the subset relation (⊆) defines a partially ordered set.
  - Lattice structures are used in database indexing and optimization.

6. Version Control Systems

- **Application:** Helps in tracking changes and maintaining versions in software development.
- **Example:**
  - Git version control follows a partial order where commits and branches have a dependency structure.
  - A commit is only valid if its parent commit exists.

7. Preference Ordering in Decision Making

- **Application:** Used in ranking choices, recommendation systems, and voting.
- **Example:**
  - In decision-making, preferences such as "A is better than B" form a partial order.
  - In multi-criteria decision analysis, different alternatives are ranked based on attributes.

8. Memory Consistency in Distributed Systems

- **Application:** Ensures proper execution of concurrent processes.
- **Example:**
  - In distributed databases, events are ordered using causal ordering (Lamport timestamps).
  - This prevents conflicts in transactions and maintains data consistency.

NOTE

Activity 11.1

Identifying and Classifying Relations

Objective: To help students understand and differentiate between Equivalence Relations and Partial Order Relations by analyzing given sets and relations.

Materials Required:

- Pen and paper
- Printed worksheets (or digital document) with relation sets
- Graph sheets for Hasse Diagram representation

Instructions:

1. **Understanding Equivalence Relations:**
   - Consider the set $A = \{1,2,3,4,5,6\}$ and the relation R defined by:
     $$R = \{(x, y) \mid x \equiv y mod\ 2\}$$
   - Check whether this relation satisfies reflexivity, symmetry, and transitivity.
   - Identify the equivalence classes formed under this relation.
2. **Understanding Partial Order Relations:**
   - Consider the set $B = \{1,2,3,4,6,12\}$ and the divisibility relation PPP defined by:
     $$P = \{(x, y) \mid x\ divides\ y\}$$
   - Verify if the relation is reflexive, antisymmetric, and transitive.
   - Construct the Hasse Diagram for the Poset (B, P).
3. **Classifying Relations:**
   - Analyze the given relations and classify them as either Equivalence Relations or Partial Order Relations based on their properties.
4. **Discussion & Conclusion:**
   - Discuss with your peers how equivalence relations partition a set, and how partial order relations structure elements hierarchically.
   - Write a short summary of your observations on how these relations apply to real-world problems (e.g., grouping students by grades vs. ranking employees by seniority).

Expected Outcome:

- Students will be able to identify and classify different types of relations.
- They will gain hands-on experience in constructing equivalence classes and Hasse diagrams.
- They will understand real-world applications of these mathematical concepts.

NOTE

☑ Check Your Progress 11.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define an equivalence relation with an example.

……

……

……

2. Every partial order relation is also an equivalence relation. True or False:

3. Which of the following is not a property of an equivalence relation?

A) Reflexivity B) Symmetry

C) Antisymmetric D) Transitivity

4. In a Hasse diagram, the edges represent the relation without showing _____ edges.

11.5 SUMMARY

This unit introduced two fundamental types of relations in discrete mathematics: equivalence relations and partial order relations. It explained the definition and essential properties of equivalence relations, namely reflexive, symmetric, and transitive properties. Various examples were discussed to help students understand how equivalence relations are formed and identified in mathematical sets and real-life situations. The concept of equivalence classes and partitioning of a set showed how elements can be grouped based on a common relationship, providing a structured way to classify objects. Applications of equivalence relations in computer science, coding theory, data classification, and modular arithmetic highlighted their practical importance.

The unit also explored partial order relations, which are defined using reflexive, antisymmetric, and transitive properties. Students learned how partially ordered sets (posets) represent relationships where some elements may not be directly comparable. Examples of partial order relations helped clarify their practical meaning and usage. The study of Hasse diagrams provided a simple graphical method to represent posets and visualize ordering relationships clearly. The concept of totally ordered sets, also known as linear orders, explained situations where every pair of elements can be compared directly. Applications of partial order relations in scheduling, hierarchy structures, database systems, and decision-making processes demonstrated their significance in real-world systems.

NOTE

By the end of the unit, students should be able to identify different types of relations, verify their properties, construct equivalence classes, analyze posets, and interpret Hasse diagrams effectively. These concepts form an important foundation for advanced topics in mathematics, computer science, logic, and data organization. Students are encouraged to further explore relation theory through problem-solving exercises, graphical representations, and practical applications in computing and information systems to strengthen their understanding and analytical skills.

11.6 GLOSSARY

- **Relation:** A set of ordered pairs defining a connection between elements of a set.
- **Equivalence Relation:** A relation that groups elements into classes based on shared properties.
- **Reflexivity:** A property where every element is related to itself (aRa).
- **Symmetry:** A property where if a is related to b, then b is related to a ($aRb \Rightarrow bRa$).
- **Transitivity:** A property where if aaa is related to b and b is related to ccc, then a is related to c (aRb and $bRc \Rightarrow aRc$).
- **Equivalence Class:** A subset of elements that are all related under an equivalence relation.
- **Partition:** A division of a set into disjoint, non-empty subsets based on an equivalence relation.
- **Partial Order:** A relation that establishes a hierarchy among elements based on order properties.
- **Hasse Diagram:** A visual representation of a Poset where redundant edges are removed for clarity.
- **Poset (Partially Ordered Set):** A set equipped with a partial order relation that structures elements hierarchically**.**

11.7 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 11.1

1. Define an equivalence relation with an example.

Answer: An equivalence relation on a set is a relation that is reflexive, symmetric, and transitive.

Example: Let $A = \{1,2,3\}$ and define a relation R on A such that

$$R = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$$

Here:

- Reflexive: Each element is related to itself.

NOTE

- Symmetric: If (1,3) ∈ R
- Transitive: (1,3) and (3,1) imply (1,1) ∈ R (already present).

Hence, R is an equivalence relation.

2. Every partial order relation is also an equivalence relation. True or False:

Answer: False.

A partial order is reflexive, antisymmetric, and transitive,

whereas an equivalence relation is reflexive, symmetric, and transitive.

So, the properties are different.

3. Which of the following is not a property of an equivalence relation?

Answer: C) Antisymmetric

Antisymmetric is a property of partial orders, not equivalence relations.

4. In a Hasse diagram, the edges represent the relation without showing _____ edges.

Answer: Reflexive edges.

In a Hasse diagram, reflexive and transitive edges are omitted for clarity.

But since the question asks for one, the answer is reflexive.

11.8 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1251	

11.9 REFERENCE

- https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_relations.htm

NOTE

- https://faculty.ksu.edu.sa/sites/default/files/4.2equivalence_relations_0.pdf
- https://www.cse.iitk.ac.in/users/rmittal/prev_course/f18/reports/4_posets.pdf
- https://testbook.com/maths/equivalence-classes
- https://testbook.com/maths/partition-of-a-set
- https://www.tpointtech.com/partial-ordering-relations
- https://en.wikipedia.org/wiki/Total_order
- https://www.cuemath.com/questions/what-do-you-mean-by-reflexive-transitive-and-symmetric-relation-explain-with-some-suitable-examples/
- https://www.slideshare.net/slideshow/hasse-diagram/45553181
- https://www.lkouniv.ac.in/site/writereaddata/siteContent/20200503134926 7328Hasse%20Diagram%20for%20BCA%204TH%20SEM.pdf

11.10 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following is an example of an equivalence relation?

a) "Is greater than" on real numbers

b) "Has the same birthday as" on people

c) "Is a subset of" on power set

d) "Divides" on integers

2. A relation that is reflexive, antisymmetric, and transitive is called a:

a) Function

b) Partial Order

c) Equivalence Relation

d) Total Order

3. In a Hasse diagram, the elements are arranged such that:

a) The lowest element is always the greatest element

b) The highest element is the least element

c) Higher elements are placed above lower elements based on the order relation

NOTE

d) No ordering rules exist

4. **Which property is always satisfied by both equivalence and partial order relations?**

   a) Reflexivity

   b) Symmetry

   c) Antisymmetric

   d) None of the above

5. **If a relation partitions a set into distinct subsets, it must be:**

   a) A function

   b) A partial order

   c) An equivalence relation

   d) A total order

Answers with Explanations:

1. **B) "Has the same birthday as" on people**

   Explanation:

   - **Reflexive:** A person has the same birthday as themselves.
   - **Symmetric:** If A has the same birthday as B, then B has the same birthday as A.
   - **Transitive:** If A has the same birthday as B and B has the same birthday as C, then A has the same birthday as C.

2. **B) Partial Order**

   Explanation: A partial order relation satisfies reflexivity, antisymmetric, and transitivity, but does not require symmetry.

3. **C) Higher elements are placed above lower elements based on the order relation**

   Explanation: In a Hasse diagram, an element a is drawn below element b if $a \leq b$ in the Poset.

4. **A) Reflexivity**

   Explanation: Both equivalence relations and partial order relations must be reflexive

   (aRa for all a). However, equivalence relations are symmetric and partial orders are antisymmetric.

5. **C) An equivalence relation**

   Explanation: An equivalence relation groups elements into equivalence classes, forming a partition of the set. A partial order

NOTE

structures elements hierarchically but does not necessarily form partitions.

B. Short answer type question

1. What are the three properties of an equivalence relation?
2. What does an equivalence relation do to a set?
3. What is a Hasse diagram used for?
4. Give an example of a partial order relation.
5. What is the main difference between a total order and a partial order?

Part 2

C. Long answer type question (200 words)

1. Define an equivalence relation and explain its properties with examples.
2. Explain equivalence classes and partitioning of a set with an example.
3. Discuss partial order relations with examples and applications.
4. Explain Hasse diagrams and show how they are used to represent posets with an example.
5. Compare and contrast equivalence relations and partial order relations with suitable examples.

Part 3

D. Long answer type question (300 words)

1. Discuss the concept of equivalence relations in detail. Explain how equivalence classes form partitions of a set with a real-world application.
2. Explain partial order relations with an example. Describe their properties and discuss how they differ from equivalence relations.
3. Describe Hasse diagrams in depth. Explain their construction, representation, and significance in ordering relations. Provide an example.
4. Explain the applications of partial order relations in computer science, scheduling, and mathematics. Provide real-world use cases.

NOTE

5. Analyze the role of equivalence and partial order relations in various mathematical and practical fields. Provide detailed examples and use cases.

UNIT 12
LATTICES

NOTE

UNIT STRUCTURE

12.1 Introduction
12.2 Objective
12.3 Lattices
12.3.1 Definition of Lattices
12.3.2 Properties of Lattices
13.3.3 Examples of Lattices
12.4 Types of Lattices
12.4.1. Bounded lattice
12.4.2. Distributive lattice
12.4.3 Complemented lattice
12.5. Lattice theory in mathematical structures
12.5.1 Boolean algebra and logic circuits
12.5.2 Order theory and Poset
12.5.3 Graph theory and lattice structures
12.5.4 Lattice theory in data science and machine learning
12.5.5 Applications in economics and optimization
12.6 Real-world applications of bounded, distributive, and complemented latticesSummary
12.7 Summary
12.8 Glossary
12.9 Answer to check your progress
12.10 Suggested readings – OER
12.11 Reference
12.12 Exercise

12.1 INTRODUCTION

In the previous unit, you explored the basics of relations and partially ordered sets, which form the foundation of lattice theory. Lattices extend the concept of order relations by introducing structured operations like meet and join. Understanding lattices helps in various applications, including algebra, logic, and computer science. This unit introduces different types of lattices and their properties.

Building on this knowledge, we now delve into the concept of lattices, a fundamental structure in discrete mathematics. A lattice is a set equipped with two binary operations—meet ($\wedge$) and join ($\vee$)—that satisfy specific properties. Different types of lattices, such as bounded,

NOTE

distributive, and complemented lattices, play crucial roles in logic, algebra, and optimization. This unit explores these concepts with definitions, properties, and examples.

This unit bridges the understanding of relations and order structures with the study of lattices, which are essential in mathematical modeling and computer science. Lattices provide a systematic way to analyze ordered structures with operations that model logical and algebraic systems. By the end of this unit, you will understand the properties of lattices and their real-world applications in computation, decision-making, and circuit design.

12.2 OBJECTIVE

After studying this unit, you will be able to:

- Define the Concept of Lattices with examples.
- Explain the Properties of Lattices.
- Classify Different Types of Lattices.
- Apply Lattice Theory in Mathematical Structures.
- Interpret and Represent Lattices Graphically.

12.3 LATTICES

12.3.1 Definition of Lattices

In abstract algebra, a lattice is a partially ordered set (POSET) $(L, \leq)$ in which every pair of elements has:

1. A least upper bound (supremum) or join $(a \vee b)$.
2. A greatest lower bound (infimum) or meet $(a \wedge b)$.

Mathematically, A Lattice Satisfies:

$$\forall a, b \in L, \quad a \vee b \ \ and \ \ a \wedge b \ exist.$$

Example:

Consider the set $L = \{1,2,3,6\}$ with the divisibility relation $(|)$:

- Join $(a \vee b)$ is the least common multiple (LCM).
- Meet $(a \wedge b)$ is the greatest common divisor (GCD).

For example:

- $2 \vee 3 = 6$ $(LCM \ of \ 2 \ and \ 3)$.
- $2 \wedge 3 = 1 \ = \ 1$ $(GCD \ of \ 2 \ and \ 3)$.

Thus, $(L, |)$ forms a lattice.

12.3.2 Properties of Lattices

- A lattice is bounded if it has a greatest element (1, top) and a least element

  (0, bottom).

NOTE

- A lattice is distributive if the join and meet operations distribute over each other:

  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *and* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

A lattice satisfies the following fundamental properties:

1. Idempotent Property

- $a \vee a = a$ *and* $a \wedge a = a$

2. Commutative Property

- $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

3. Associative Property

- $(a \vee b) \vee c = a \vee (b \vee c)$
- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

4. Absorption Property

- $a \vee (a \wedge b) = a$
- $a \wedge (a \vee b) = a$

12.3.3 Examples of Lattices

Example 1: Power Set of a Set

- Let $A = \{x, y\}$, and consider its power set $P(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Define $\leq$ as the subset relation ($\subseteq$).

Solution:

- The meet ($A \wedge B$) is the intersection ($A \cap B$).
- The join ($A \vee B$) is the union ($A \cup B$).

For example:

- $\{x\} \wedge \{x, y\} = \{x\}, \quad \{x\} \vee \{x, y\} = \{x, y\}$.
- $\{x\} \wedge \{y\} = \emptyset, \ \{x\} \vee \{y\} = \{x, y\}$.

This forms a lattice because every pair of subsets has a **meet** and a **join**.

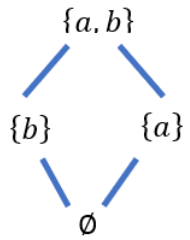
- Consider the power set $P(S)$ of a set $S = \{a, b\}$ with the subset relation ($\subseteq$):

$$P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

- Join ($\vee$) = Union ($\cup$) of sets.
- Meet ($\wedge$) = Intersection ($\cap$) of sets.

NOTE

Hasse Diagram Representation:



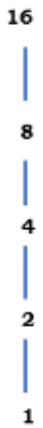
Thus, $(P(S), \subseteq)$ forms a lattice.

Example 2: Integer Lattice with Divisibility

Consider the set $L = \{1,2,4,8,16\}$ with divisibility relation (|):

- Join ($\vee$) = LCM (Least Common Multiple).
- Meet ($\wedge$) = GCD (Greatest Common Divisor).

Hasse Diagram Representation:



Thus, $(L, |)$ forms a **lattice**.

Example 1: Divisibility Lattice

Let $L = \{1,2,4,8\}$, and define $a \leq b$ if $a \mid b$ (divides).

Solution:

- The meet ($a \wedge b$) is the greatest common divisor ($gcd(a, b)$).
- The join ($a \vee b$) is the least common multiple ($lcm\ (a, b)$).

For example:

- $2 \wedge 4 = 2, 2 \vee 4 = 4$.
- $4 \wedge 8 = 4, 4 \vee 8 = 8$.

This forms a lattice since every pair has a **meet** and a **join**.

Example 3: Boolean Lattice

Let $L = \{0,1\}n$ (binary strings of length n), where $n = 2$ and the ordering is by component wise $\leq$.

For example, $00 \leq 01 \leq 11$.

NOTE

Solution:

- The meet $(a \wedge b)$ is the bitwise AND.
- The join $(a \vee b)$ is the bitwise OR.

For example:

- $01 \wedge 11 = 01, 01 \vee 11 = 11$
- $10 \wedge 11 = 10, 10 \vee 11 = 11$

This forms a lattice.

Example 4: Integer Intervals

Let $L = \{[a,b] \mid a,b \in Z, a \leq b\}$, and define $[a,b] \leq [c,d]$ if $a \geq c$ and $b \leq d$.

Solution:

- The meet $([a,b] \wedge [c,d])$ is $[max(a,c), min(b,d)]$ (intersection of intervals).
- The join $([a,b] \vee [c,d])$ is $[min(a,c), max(b,d)]$ (union of intervals).

For example:

- $[1,4] \wedge [2,6] = [2,4]$, $[1,4] \vee [2,6] = [1,6]$
- $[3,5] \wedge [4,7] = [4,5]$, $[3,5] \vee [4,7] = [3,7]$

This is a lattice because every pair of intervals has a meet and a join.

12.4 TYPES OF LATTICES

Lattices can be classified into different types (Bounded, Distributive, and Complemented Lattices)

12.4.1 Bounded Lattice

A lattice $(L,\vee,\wedge)$ is called bounded if it has:

- A greatest element (denoted as 1 or top element $\top$).
- A least element (denoted as 0 or bottom element $\perp$).

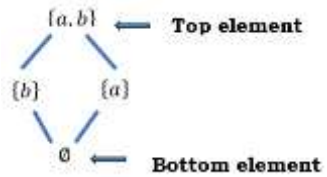
Example:

Consider the power set lattice $P(S)$ of $S = \{a,b\}$ with the subset relation $\subseteq$:

$$P(S) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

- The least element is $\emptyset$ (bottom element 0).
- The greatest element is $\{a,b\}$ (top element 1).

Hasse Diagram:



This is a bounded lattice because it has a top and bottom element.

12.4.2. Distributive Lattice

NOTE

A lattice is distributive if it satisfies the distributive laws:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Example:

Consider the set L={0,1,2,3,6} with the divisibility relation |.

- $Meet\ (\wedge) = GCD$
- $Join\ (\vee) = LCM$

$$2 \vee (3 \wedge 6) = (2 \vee 3) \wedge (2 \vee 6)$$

$$2 \wedge (3 \vee 6) = (2 \wedge 3) \vee (2 \wedge 6)$$

Since the distributive laws hold, this is a distributive lattice.

12.4.3. Complemented Lattice

A lattice is complemented if:

- It is bounded (has a top 1 and bottom 0).
- Every element aa has a complement a′, such that:

$$a \vee a' = 1 \quad and \quad a \wedge a' = 0\ .$$

Example: Boolean Algebra Lattice

Consider the Boolean algebra $B = \{0,1\}$ with:

- $Meet\ (\wedge) = AND$ operation
- $Join\ (\vee) = OR$ operation
- Complement (a') = NOT operation

For any element:

- $0' = 1$ and $1' = 0$.
- $0 \vee 1 = 1,\ 0 \wedge 1 = 0$.

Since every element has a complement, this is a complemented lattice.

Thus,

Lattice Type	Condition	Example
Bounded	Has a **least element (0)** and **greatest element (1)**	Power Set Lattice
Distributive	Follows distributive laws	Divisibility Lattice
Complemented	Has complements for all elements	Boolean Algebra

12.5 LATTICE THEORY IN MATHEMATICAL STRUCTURES

NOTE

Lattice theory is a fundamental branch of discrete mathematics that studies algebraic structures known as lattices. Lattices are widely used in mathematical structures such as Boolean algebra, order theory, topology, and abstract algebra.

Lattice is a set equipped with two binary operations:

- **Meet** (∧): Represents the greatest lower bound (GLB) or intersection of two elements.
- **Join** (∨): Represents the least upper bound (LUB) or union of two elements.

Below are some key areas where lattice theory plays an important role:

12.5.1 Boolean Algebra and Logic Circuits

- Lattices form the foundation of Boolean algebra, which is essential for designing logic gates and digital circuits.
- In Boolean lattices, the elements are 0 (False) and 1 (True), with operations corresponding to AND (∧), OR (∨), and NOT (Complement).

Example: In Boolean algebra, the complemented lattice follows:

- $A \vee A' = 1$ (OR operation)
- $A \wedge A' = 0$ (AND operation)

This is applied in computer processors and digital logic design.

12.5.2 Order Theory and Posets

- Partially ordered sets (Poset) are a generalization of lattices where elements follow a hierarchical relationship.
- Every lattice is a Poset but not every Poset is a lattice.
- Hasse diagrams are used to represent lattices in order theory.

Example: The divisibility lattice on {1,2,4,8} under the relation "divides" (|) forms a lattice where:

- Meet (∧) is the greatest common divisor (GCD).
- Join (∨) is the least common multiple (LCM).

12.5.3 Graph Theory and Lattice Structures

- Graph theory uses lattice representations for shortest paths, spanning trees, and flow networks.
- Lattices help in organizing hierarchical structures like network routing and database indexing.

NOTE

Example: The subset lattice of a set $S = \{a, b, c\}$ has elements:

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Where meet ($\wedge$) is intersection and join ($\vee$) is union.

12.5.4. Lattice Theory in Data Science and Machine Learning

- Lattices are used in concept lattices, which help in knowledge representation and classification.
- Formal concept analysis (FCA) builds hierarchical structures based on data relationships.

Example: In machine learning, hierarchical clustering uses lattice-based structures for decision trees and neural networks.

12.5.5 Applications in Economics and Optimization

Lattice theory serves as a bridge between algebra, logic, and computer science. It provides essential tools for digital systems, optimization, data structures, and artificial intelligence. Understanding lattices helps in solving problems related to hierarchical structures, decision-making, and information retrieval.

- Lattice theory is used in economic models, decision-making, and resource allocation.
- In optimization problems, lattices help in finding the best possible outcome within constraints.

Example: In supply chain management, a distributive lattice can be used to find optimal product distribution paths.

12.6 REAL-WORLD APPLICATIONS OF BOUNDED, DISTRIBUTIVE, AND COMPLEMENTED LATTICES

Lattices are widely used in various fields, including computer science, mathematics, physics, and social sciences. Below are some real-world applications of bounded, distributive, and complemented lattices.

12.6.1. Applications of Bounded Lattices

A bounded lattice has both a least element (0) and a greatest element (1). These lattices are used in:

(a) Digital Logic and Boolean Algebra

- **Application:** Used in designing digital circuits, logic gates, and computer programming.
- **Example:** The Boolean lattice $B = \{0,1\}$ follows:
  - 0 (False) as the least element.
  - 1 (True) as the greatest element.
- Logical operations (AND, OR, NOT) follow lattice properties.

NOTE

(b) File System Permissions

- **Application:** Used in operating systems for user access control.
- **Example:** Linux file permissions:
  - Read (R), Write (W), Execute (X) form a bounded lattice.
  - Least element (0): No access.
  - Greatest element (1): Full access (R + W + X).

(c) Hierarchical Organizational Structures

- **Application:** Used in companies, governments, and classification systems.
- **Example:**
  - Top (CEO or President) – Greatest element.
  - Lowest-level employee – Least element.
  - Intermediate positi
  - ons form a lattice structure.

2. Applications of Distributive Lattices

A distributive lattice follows the distributive laws and is used in:

(a) Database Theory (Query Optimization)

- **Application:** Used in relational databases for query optimization.
- **Example:**
  - Consider queries using AND and OR conditions.
  - $(A\ OR\ (B\ AND\ C)) = ((A\ OR\ B)\ AND\ (A\ OR\ C))$ helps in query restructuring for efficiency.

(b) Artificial Intelligence and Machine Learning

- **Application:** Used in decision trees and rule-based systems.
- ***Example:*** In AI, rules like

$(Rain \vee Snow) \wedge Umbrella = (Rain \wedge Umbrella) \vee (Snow \wedge Umbrella)$

help in logical reasoning and decision-making.

(c) Task Scheduling in Operating Systems

- Application: Helps in scheduling tasks based on dependencies.
- *Example:*
  - Tasks that require multiple resources are structured as a distributive lattice.
  - Example: Process dependencies in cloud computing**.**

3. Applications of Complemented Lattices

NOTE

A complemented lattice has a complement for every element and is used in:

(a) Cryptography and Cybersecurity

- Application: Used in encryption-decryption processes and secure communications.
- *Example:*
  - Bitwise complement operation (NOT gate in digital logic).
  - Boolean lattice used in cryptographic algorithms.

(b) Internet Search Engines (Boolean Retrieval Model)

- Application: Used in web search algorithms.
- *Example:*
  - If a user searches for (Python AND NOT Java), the complement operation excludes "Java" results.
  - The search engine applies Boolean lattice principles.

(c) Electrical Circuit Design

- Application: Used in circuit minimization.
- *Example:* Karnaugh Maps (K-maps) simplify Boolean expressions in circuit design using complemented lattice properties.

Thus,

Lattice Type	Real-World Applications
Bounded Lattice	Digital logic, file system permissions, organizational structures
Distributive Lattice	Database query optimization, AI decision trees, task scheduling
Complemented Lattice	Cryptography, search engines, electrical circuit design



Activity 12.1

Exploring Lattices and Their Properties

Objective:

To help students understand the concept of lattices, their properties, and different types of lattices through hands-on problem-solving and graphical representation.

Instructions:

1. **Understanding Lattices:**
   - Consider the set $S = \{2,4,8,16\}$ with the divisibility relation (|).

NOTE

- Construct the Hasse diagram of the lattice formed by this set.
- Identify the meet (∧) and join (∨) of any two elements in this lattice.

2. Identifying Properties:

- Verify if the given lattice satisfies commutativity, associativity, absorption, and idempotency.
- Provide examples for each property.

3. Classifying Lattices:

- Determine if the lattice is bounded by identifying its greatest (1) and least (0) elements.
- Check if the lattice is distributive using the distributive property:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

- Identify whether it is a complemented lattice by checking if each element has a complement.

4. Real-World Connection:

- Research and present one real-life example where lattices are applied (e.g., computer science, decision-making, or database structures).

5. Group Discussion & Reflection:

- Discuss your findings with your peers and answer:
  - How do lattice properties help in organizing structured information?
  - What are some challenges in identifying different types of lattices?

Expected Outcome:

- A correctly drawn Hasse diagram representing the given lattice.
- A clear understanding of lattice properties through verification.
- Identification of whether the given lattice is bounded, distributive, or complemented.

A real-world example showing how lattices are useful in practical applications.

☑ Check Your Progress 12.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is a lattice?

……

……

……

NOTE

2. Every lattice is a Poset, but every Poset is not necessarily a lattice. True/False:

……

3. Which of the following is an example of a distributive lattice?

(A) Boolean Algebra

(B) Divisibility Lattice of Natural Numbers

(C) Power Set Ordered by Inclusion

(D) All of the above

……

4. A lattice is called a _____ lattice if it has both a greatest element and a least element.

……

12.7 SUMMARY

This unit introduced the concept of Lattice Theory and explained its importance in mathematical structures and computational systems. The unit began with the definition of lattices, where students learned that a lattice is a special algebraic structure in which every pair of elements has a unique least upper bound and greatest lower bound. The properties of lattices, including commutative, associative, absorption, and idempotent laws, were discussed to help students understand how lattice operations function. Several examples of lattices were also explored to illustrate the practical representation of ordered structures. The unit further explained different types of lattices such as bounded lattices, distributive lattices, and complemented lattices, highlighting their unique characteristics and mathematical significance. Students also learned how these structures are connected to important concepts in Boolean Algebra, logic circuits, order theory, and partially ordered sets (Posets). Applications of lattice theory in graph theory, data science, machine learning, economics, and optimization demonstrated the broad relevance of lattices in modern scientific and technological fields. Real-world applications showed how bounded, distributive, and complemented lattices support decision-making systems, database organization, artificial intelligence, and optimization problems. By the end of the unit, students should be able to identify different lattice structures, understand their properties, and apply lattice concepts to mathematical and computational problems. This unit also encourages further exploration of advanced topics such as abstract algebra, formal logic, artificial intelligence, and optimization techniques, where lattice theory continues to play a significant role in research and practical applications.

12.8 GLOSSARY

NOTE

- **Lattice** – A set with two binary operations (meet ∧ and join ∨) that satisfy ordering properties.
- **Meet (∧)** – The greatest lower bound (GLB) of two elements in a lattice. It represents the intersection in set theory.
- **Join (∨)** – The least upper bound (LUB) of two elements in a lattice. It represents the union in set theory.
- **Partially Ordered Set (Poset)** – A set with a binary relation (≤) that is reflexive, antisymmetric, and transitive. Lattices are a special type of Poset
- **Bounded Lattice** – A lattice with a greatest element (1) and a least element (0).
- **Distributive Lattice** – A lattice in which meet and join operations distribute over each other.
- **Complemented Lattice** – A lattice where every element has a complement that interacts with meet and join to form the identity elements 0 and 1.
- **Boolean Algebra** – A mathematical structure based on lattices, used in logic gates, set operations, and computer science.
- **Hasse Diagram** – A graphical representation of a Poset or lattice, where edges indicate ordering relations.
- **Application of Lattices** – Lattices are used in data analysis, digital logic design, decision-making models, and artificial intelligence.

12.9 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 12.1

1. **Answer:** A lattice is a partially ordered set (Poset) in which every pair of elements has both:
   - A least upper bound (also called join), and
   - A greatest lower bound (also called meet).

   In other words, for any two elements a and b in a lattice, both $a \vee b$ (join) and $a \wedge b$ (meet) exist.
2. **Answer:** True.

   Every lattice is a Poset with additional properties (join and meet for every pair),

NOTE

but a Poset might not have join or meet for all pairs, so it may not be a lattice.

3. **Answer:** (D) All of the above

   Explanation:

   - Boolean Algebra is a distributive lattice.
   - Divisibility Lattice of Natural Numbers (under gcd and lcm) is distributive.
   - Power Set ordered by inclusion is also distributive.

4. **Answer:** Bounded lattice.

   A bounded lattice has:

   - At least element (usually denoted as 0), and
   - A greatest element (usually denoted as 1).

12.10 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1252	

12.11. REFERENCE

- https://www.scribd.com/document/437034972/Lattices
- https://cse.poriyaan.in/topic/properties-of-lattices---lattices-as-algebraic-systems---sublattices---direct-product-and-homomorphism-some-special-lattices-50669/
- https://mathworld.wolfram.com/BoundedLattice.html
- https://ncatlab.org/nlab/show/distributive+lattice
- https://www.scribd.com/document/363639579/complemented-lattice-pdf
- https://math.libretexts.org/Bookshelves/Abstract_and_Geometric_Algebra/Abstract_Algebra%3A_Theory_and_Applications_(Judson)/19%3A_Lattices_and_Boolean_Algebras
- https://www.lkouniv.ac.in/site/writereaddata/siteContent/20200503134926732 8Hasse%20Diagram%20for%20BCA%204TH%20SEM.pdf

12.12 EXERCISE

NOTE

Part 1

A. Multiple choice question

1. Which of the following properties does every lattice satisfy?

a) Associativity

b) Idempotency

b) Absorption

d) All of the above

2. What is the greatest lower bound (GLB) of two elements in a lattice called?

a) Join

b) Meet

c) Complement

d) None of the above

3. A lattice is said to be distributive if:

a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

b) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

c) Both (A) and (B)

d) None of the above

4. Which of the following is NOT a type of lattice?

a) Complemented Lattice

b) Modular Lattice

c) Integral Lattice

d) Distributive Lattice

5. In a bounded lattice, the greatest and least elements are called:

a) Top and Bottom

b) 1 and 0

c) Maximum and Minimum

d) Both (A) and (B)

Answers with Explanations:

1. d) All of the above

Explanation: Every lattice follows associativity, idempotency, absorption, and commutativity properties.

NOTE

2. b) Meet

Explanation: The meet (∧) of two elements is the largest element that is less than or equal to both.

3. c) Both (A) and (B)

Explanation: A distributive lattice satisfies both distributive laws.

4 c) Integral Lattice

Explanation: Integral lattice is not a standard classification in lattice theory.

5. d) Both (A) and (B)

Explanation: In a bounded lattice, the greatest element is called Top (1) and the least element is called Bottom (0).

B. Short answer type question

1. What are the two main operations in a lattice?
2. What is the least upper bound (LUB) in a lattice?
3. Give an example of a bounded lattice.

Part 2

C. Long answer type question (200 words)

1. Explain the fundamental properties of lattices with examples.
2. Describe the difference between a bounded, distributive, and complemented lattice.
3. Explain how lattices are represented using Hasse diagrams with an example.

Part 3

D. Long answer type question (300 words)

1. Define a lattice. Explain meet and join operations with suitable examples.
2. What is a distributive lattice? Give a formal definition and prove that every Boolean algebra forms a distributive lattice.
3. Explain complemented lattices with examples. How does it relate to Boolean algebra?

UNIT 13
FUNCTION

NOTE

UNIT STRUCTURE

13.1 Introduction
13.2 Objective
13.3 Function
  13.3.1 Definition of a Function
  13.3.2 Properties of Function
  13.3.3 Examples of Function
13.4 Mapping
  13.4.1 Definition of Mapping (Function)
  13.4.2 Types of Mapping
  13.4.3 Examples of Mapping
13.5 Types of Functions
  13.5.1 Types of Functions
13.6 Cardinality of a Set
  13.6.1 Definition of Cardinality of a Set
  13.6.2 Types of Cardinalities
  13.6.3 Examples of Cardinality
13.7 Composition of Relation and Function
  13.7.1 Definition Composition of Relation
    13.7.1.1 Examples of Composition of Relation
  13.7.2 Definition Composition of Functions
    13.7.2.1 Examples of Composition of Function
13.8 Composition of Function
  13.8.1 Definition of Composition of Function
  13.8.2 Examples of Composition of Function
13.9 Existence of Inverse Function (Mapping)
  13.9.1 Definition of Existence of Inverse Function (Mapping)
  13.9.2 Condition for the Existence of Inverse Function
  13.9.3 Examples of Existence of Inverse Function
13.10 Set Image and Preimage of a Function
  13.10.1 Set Image and Preimage of a Function
  13.10.2 Examples of Set Image and Preimage of a Function
13.11 Summary
13.12 Glossary
13.13 Answer to check your progress
13.14 Suggested readings - OER
13.15 Reference
13.16 Exercise

NOTE

13.17 Feedback form

13.1 INTRODUCTION

Building on your understanding of sets, relations, and operations, this unit introduces functions and mappings as special types of relations that establish unique connections between elements of two sets. The concept of cardinality deepens your knowledge of set sizes, while composition of relations and functions extends your ability to combine and analyze these mappings. Exploring inverse functions and set images further enhances your ability to model and solve problems involving structured relationships.

This unit focuses on the study of functions and mappings, which are key tools for understanding relationships between sets. You will explore different types of functions, their compositions, and how they can be combined or reversed to solve problems. The concepts of cardinality and set images provide insights into the size and behavior of functions. Mastering these topics is essential for applications in mathematics, computer science, and data analysis.

The concepts in this unit can be related to real-life scenarios, such as tracking relationships in databases, where mappings connect data points uniquely, or understanding transformations in geometry, like scaling or rotating shapes. The cardinality of sets helps in counting items efficiently, while inverse functions find applications in reversing processes, like decoding encrypted messages. These practical connections will help you comprehend how functions and mappings are used to model and solve real-world problems.

13.2 OBJECTIVE

After studying this unit, you will be able to:

- Define the concept of functions and mappings and their role in connecting elements between sets.
- Identify and differentiate between various types of functions, including one-to-one, onto, and bijective functions.
- Determine the cardinality of sets and relate it to the behavior of functions and mappings.
- Perform and analyze the composition of relations and functions to solve complex problems.
- Explain and verify the existence of inverse functions and their applications in reversing processes.
- Determine and interpret the set image and preimage of functions in mathematical and real-world contexts.

13.3 FUNCTION

NOTE

13.3.1 Definition of a Function

A function is a mapping $f: A \to B$ from a set A (domain) to a set B (codomain) that assigns exactly one element $b \in B$ to each element $a \in A$.

13.3.2 Properties of Function

Injective (One-to-One): Each element of B is mapped by at most one element of A.

Surjective (Onto): Every element of B is mapped by at least one element of A.

Bijective (One-to-One and onto): A function that is both injective and surjective, establishing a one-to-one correspondence between A and B.

13.3.3 Examples of Function

Example 1: Linear Function

Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x + 1$.

Solution:

- **Injective**: If $f(a) = f(b)$, then $2a + 1 = 2b + 1$ implies $a = b$. Hence, f is injective.
- **Surjective**: For any $y \in \mathbb{R}$, there exists $x = \frac{y-1}{2} \in \mathbb{R}$ such that $f(x) = y$.

  Hence, f is surjective.
- **Bijective**: Since f is both injective and surjective, it is bijective.

Example 2: Constant Function

Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 5$.

Solution:

- **Injective**: Not injective because $f(a) = f(b)$ for all $a, b \in \mathbb{R}$,

  regardless of whether $a \neq b$.
- **Surjective**: Not surjective because not all elements in $\mathbb{R}$ are mapped

  (only 5 is mapped).
- **Bijective**: Not bijective as it fails both injective and surjective.

Example 3: Modular Arithmetic Function

Let $f: \mathbb{Z} \to \mathbb{Z}_n$ be defined as $f(x) = x \bmod n$, where $\mathbb{Z}_n = \{0,1,2, \dots, n-1\}$.

Solution:

NOTE

- **Injective**: Not injective because $f(a) = f(b)$ if $a \equiv b \ mod \ n$, meaning multiple integers map to the same residue class.
- **Surjective**: f is surjective because every element of $\mathbb{Z}_n$ is reached by some $x \in \mathbb{Z}$
- **Bijective**: Not bijective as it is not injective.

Example 4: Exponential Function on Natural Number

Let $f: \mathbb{N} \to \mathbb{N}$ be defined as $f(x) = 2^x$.

Solution:

- **Injective**: If $f(a) = f(b)$, then $2^a = 2^b$ implies $a = b$. Hence, f is injective.
- **Surjective**: Not surjective because not every $y \in \mathbb{N}$ can be written as 2^x.

  (e.g., $y = 3$).
- **Bijective**: Not bijective as it is not surjective.

13.4. MAPPING

13.4.1 Definition of Mapping (Function)

A **mapping** (or function) is a specific type of relation that associates each element of a set A (domain) to exactly one element of another set B (codomain).

A mapping $f: A \to B$ is denoted as $f(a) = b$, where $a \in A$ and $b \in B$.

13.4.2 Types of Mapping

1. **Injective (One-to-One)**: Distinct elements in A map to distinct elements in B.
2. **Surjective (Onto)**: Every element of B is the image of at least one element of A.
3. **Bijective (One-to-One and onto)**: A function that is both injective and surjective.

13.4.3. Examples of Mapping

Example 1: Injective Mapping

Let $f: \{1,2,3\} \to \{a, b, c, d\}$, defined as: $f(1) = a, f(2) = b, f(3) = c$.

Solution:

- **Injective**: f is injective because each element in the domain maps to a distinct element in the codomain.
- **Surjective**: f is not surjective because $d \in B$ is not mapped to by any element of A.

NOTE

- **Bijective**: Not bijective because it is not surjective.

Example 2: Surjective Mapping

Let $g: \{1,2,3\} \to \{a, b\}$, defined as: $g(1) = a, g(2) = b, g(3) = a$.

Solution:

- **Injective**: g is not injective because $g(1) = g(3) = a$.
- **Surjective**: g is surjective because every element in B (a and b) is mapped to by at least one element of A.
- **Bijective**: Not bijective because it is not injective.

Example 3: Bijective Mapping

Let $h: \{1,2,3\} \to \{a, b, c\}$, defined as:$h(1) = a, h(2) = b, h(3) = c$.

Solution:

- **Injective**: h is injective because each element in A map to a unique element in B.
- **Surjective**: h is surjective because every element in B is mapped to by an element of A.
- **Bijective**: h is bijective because it is both injective and surjective.

Example 4: Constant Mapping

Let $k: \{1,2,3\} \to \{a\}$, defined as: $k(1) = a, k(2) = a, k(3) = a$.

Solution:

- **Injective**: k is not injective because multiple elements in A map to the same element $a \in B$.
- **Surjective**: k is surjective because $a \in B$ is mapped by all elements of A.
- **Bijective**: Not bijective because it is not injective.

13.5. TYPES OF FUNCTION

13.5.1 Types of Function

In abstract algebra, functions (or mappings) are used to describe relationships between sets. Functions can be classified based on their properties into types such as **Injective**, **Surjective**, and **Bijective**. These functions play a key role in understanding algebraic structures like groups, rings, and fields.

1. **Injective Function (One-to-One)**

   A function $f: A \to B$ is **injective** if different elements in A map to different elements in B.

NOTE

- If $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Example: $f(x) = 2x$ for $x \in \mathbb{R}$.

Solution: Injective because $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, so $x_1 = x_2$.

2. Surjective Function (Onto)

A function $f: A \to B$ is **surjective** if every element in B is mapped by at least one element in A.

- For all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Example: $f(x) = x^2$ for $x \in \mathbb{R}$ and $f: \mathbb{R} \to [0, \infty)$.

Solution: Surjective because every $y \geq 0$ has a preimage $x \in \mathbb{R}$ (either $x = \sqrt{y}$ *or* $x = -\sqrt{y}$).

3. Bijective Function (One-to-One and onto)

A function $f: A \to B$ is **bijective** if it is both injective and surjective.

- Each element of A maps to a unique element of B, and every $b \in B$ has a preimage in A.

Example: $f(x) = 2x + 3$ for $x \in \mathbb{R}$

Solution:

- **Injective:** If $f(x_1) = f(x_2)$, then $2x_1 + 3 = 2x_2 + 3$, so $x_1 = x_2$.
- **Surjective:** For any $y \in R, x = \frac{y-3}{2} \in R$.

  Therefore, f is bijective.

4. Constant Function

A function $f: A \to B$ is **constant** if all elements of A map to the same single element in B.

- For all $a_1, a_2 \in A, f(a_1) = f(a_2)$.

Example: $f(x) = 5$ or $x \in \mathbb{R}$.

Solution:

- Not injective because $f(x_1) = f(x_2) = 5$ for all x_1, x_2.
- Surjective if the codomain is restricted to {5}; otherwise, not surjective.
- Not bijective unless the codomain is {5}.

13.6 CARDINALITY OF SET

13.6.1 Definition of Cardinality of a Set

The **cardinality** of a set A, denoted $|A|$, is the number of distinct elements in the set. If a set is finite, its cardinality is simply the count

NOTE

of its elements. For infinite sets, the cardinality refers to the size of the set in terms of the types of infinity (e.g., countably infinite, or uncountably infinite).

13.6.2 Types of Cardinalities

1. **Finite Cardinality**: A set has a finite number of elements.
2. **Infinite Cardinality**: A set has an infinite number of elements. It can be further classified as:
   - **Countably Infinite**: The set has the same cardinality as the set of natural numbers $\mathbb{N}$.
   - **Uncountably Infinite**: The set has a larger cardinality than $\mathbb{N}$, such as the set of real numbers $\mathbb{R}$.

13.6.3 Examples of Cardinality

Example 1: Finite Set

Let $A = \{1,2,3,4,5\}$.

Solution:

- The set A has 5 elements.
- Therefore, $| A | = 5$.

Example 2: Finite Set

Let $B = \{a, b, c\}$.

Solution:

- The set B has 3 elements.
- Therefore, $| B | = 3$

Example 3: Countably Infinite Set

Let $C = \mathbb{N} = \{1,2,3,4,5, \dots\}$ (the set of natural numbers).

Solution:

- The set C is countably infinite because we can establish a one-to-one correspondence with the set of natural numbers.
- Therefore, $| C | = \aleph_0$, where $\aleph_0$ denotes the cardinality of countably infinite sets.

Example 4: Uncountably Infinite Set

Let $D = \mathbb{R}$ (the set of real numbers).

Solution:

- The set D is uncountably infinite because it has a larger cardinality than $\mathbb{N}$.

NOTE

- Therefore, $|D|$ is said to have the cardinality of the continuum, denoted by c, and is strictly larger than $\aleph_0$.

13.7 COMPOSITION OF RELATION AND FUNCTION

13.7.1. Definition of Composition of Relation

The **composition of relations** involves combining two relations to form a new relation. If R and S are relations on sets A and B, and B and C, respectively, then the **composition of R and S**, denoted S∘R, is a relation from A to C.

Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, the composition of R and S, denoted as $S \circ R$, is defined as:

$S \circ R = \{(a, c) \in A \times C \mid$ there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S\}$.

13.7.1.1. Examples of composition of relation

Example 1: Composition of Relations (Basic Example)

Let $A = \{1,2\}, B = \{a, b\}, C = \{x, y\}$, and define the relations:

- $R = \{(1, a), (2, b)\}$ (from A to B)
- $S = \{(a, x), (b, y)\}$ (from B to C)

Now, compute the composition $S \circ R$.

Solution:

- From $(1, a) \in R$ and $(a, x) \in S$, we get $(1, x) \in S \circ R$.
- From $(2, b) \in R$ and $(b, y) \in S$, we get $(2, y) \in S \circ R$.

Thus, the composition $S \circ R = \{(1, x), (2, y)\}$.

Example 2: Composition of Relations with Different Sets

Let $A = \{1,2,3\}, B = \{a, b, c\}, C = \{x, y\}$, and define the relations:

- $R = \{(1, a), (2, b), (3, c)\}$ (from A to B)
- $S = \{(a, x), (b, y)\}$(from B to C)

Now, compute the composition $S \circ R$.

Solution:

- From $(1, a) \in R$ and $(a, x) \in S$, we get $(1, x) \in S \circ R$.
- From $(2, b) \in R$ and $(b, y) \in S$, we get $(2, y) \in S \circ R$.
- The pair $(3, c) \in R$ has no corresponding pair in S, so no relation for 3.

Thus, the composition $S \circ R = \{(1, x), (2, y)\}$

NOTE

13.7.2. Composition of Function

The **composition of functions** is a process where two functions are combined to form a new function. If $f: A \to B$ and $g: B \to C$, then the **composition of functions**

$g \circ f: A \to C$ is defined as: $(g \circ f)(a) = g(f(a))$

13.7.2.1. Examples of Composition of Function Examples

Example 1: Composition of Functions (Basic Example)

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$

Now, compute the composition $g \circ f$.

Solution:

$(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2$

Thus, $(g \circ f)(x) = (2x + 1)^2$

Example 2: Composition of Function with Different Domain

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 3$, and let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x$.

Now, compute the composition $g \circ f$.

Solution:

$(g \circ f)(x) = g(f(x)) = g(x + 3) = 2(x + 3) = 2x + 6$

Thus, $(g \circ f)(x) = 2x + 6$

13.8 COMPOSITION OF FUNCTION

13.8.1. Definition of Composition of Function

The composition of two functions f and g, denoted as $(f \circ g)(x)$, is defined as:

$(f \circ g)(x) = f(g(x))$

This means you first apply $g(x)$ and then apply f to the result of $g(x)$.

13.8.2. Examples of Composition of Function

Example 1:

Let $f(x) = 2x + 3$ and $g(x) = x^2 - 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

1. $(\boldsymbol{f \circ g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$

   Substituting $g(x) = x^2 - 1$ into $f(x) = 2x + 3$:

NOTE

$f(g(x)) = f(x^2 - 1) = 2(x^2 - 1) + 3 = 2x^2 - 2 + 3 = 2x^2 + 1$

So, $(f \circ g)(x) = 2x^2 + 1$

2. $(\boldsymbol{g \circ f})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$

Substituting $f(x) = 2x + 3$ into $g(x) = x^2 - 1$:

$g(f(x)) = g(2x + 3) = (2x + 3)^2 - 1 = 4x^2 + 12x + 9 - 1$
$= 4x^2 + 12x + 8$

So, $(g \circ f)(x) = 4x^2 + 12x + 8$

Example 2:

Let $f(x) = x + 5$ and $g(x) = 3x$. Find $(f \circ g)(x)$ and evaluate $(f \circ g)(2)$.

Solution:

1. $(\boldsymbol{f \circ g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$:

Substituting $g(x) = 3x$ into $f(x) = x + 5$:

$f(g(x)) = f(3x) = 3x + 5$

So, $(f \circ g)(x) = 3x + 5$

2. Evaluate $(\boldsymbol{f \circ g})(\mathbf{2})$:

Substitute $x = 2x = 2$ into $3x + 5$:

$(f \circ g)(2) = 3(2) + 5 = 6 + 5 = 11$

Example 3:

Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

1. $(\boldsymbol{f \circ g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$:

Substituting $g(x) = \sqrt{x}$ into $f(x) = x^2$

$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

So, $(f \circ g)(x) = x$.

2. $(\boldsymbol{g \circ f})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$:

Substituting $f(x) = x^2$ *i*nto $g(x) = \sqrt{x}$:

$g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$ (absolute value because of square root)

So, $(g \circ f)(x) = |x|$.

Example 4:

Let $f(x) = sin(x)$ and $g(x) = x^2 + 1$. Find $(f \circ g)(x)$ and evaluate $(f \circ g)(\pi)$.

NOTE

Solution:

1. $(f \circ g)(x) = f(g(x))$:

   Substituting $g(x) = x^2 + 1$ into $f(x) = sin\,(x)$:

   $f(g(x)) = f(x^2 + 1) = sin\,(x^2 + 1)$

   So, $(f \circ g)(x) = sin\,(x^2 + 1)$

2. **Evaluate** $(f \circ g)(\pi)$:

   Substitute $x = \pi$ into $sin(\pi^2 + 1)$:

   $(f \circ g)(\pi) = sin\,(\pi^2 + 1)$

   Approximate: $\pi^2 \approx 9.87$, so $\pi^2 + 1 \approx 10.87$

   $sin\,(10.87) \approx -0.999$

   Thus, $(f \circ g)(\pi) \approx -0.999$

13.9 EXISTENCE OF INVERSE FUNCTION (MAPPING)

13.9.1. Definition of Existence of Inverse Function (Mapping)

An **inverse function** $f^{-1}$is a function that reverses the effect of the original function f. Specifically, if $f: A \to B$ is a function, then the inverse function $f^{-1}: B \to A$ is defined such that:

$f^{-1}(f(x)) = x$ for all $x \in A$ and $f(f^{-1}(y)) = y$ for all $y \in B$.

In simpler terms, f^{-1} undoes the mapping of f

13.9.2. Condition for The Existence of Inverse Function

For a function $f: A \to B$ to have an inverse, it must satisfy two key properties:

1. **Injectivity (One-to-One)**: f is injective if different elements in A map to different elements in B, i.e., $f(a_1) = f(a_2) \implies a_1 = a_2$
2. **Surjective (Onto)**: f is surjective if every element in B is the image of some element in A, i.e., for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$

When a function is **both injective and surjective**, it is called **bijective**, and only then does an inverse function exist.

Thus, a function $f: A \to B$ has an inverse function if and only if it is **bijective**.

13.9.3. Examples of Existence of Inverse Function

Example 1: Bijective Function

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$

NOTE

Solution:

- **Injectivity**: Assume $f(x_1) = f(x_2)$.

  Then, $2x_1 + 3 = 2x_2 + 3$, which simplifies to $x_1 = x_2$, so f is injective.

- **Surjective**: For any $y \in \mathbb{R}$, solve $y = 2x + 3$ for x, which gives $x = \frac{y-3}{2}$.

  Thus, f is surjective.

- Since f is both injective and surjective, it is bijective, and an inverse exists.

- **Inverse Function**: The inverse of $f(x) = 2x + 3$ is given by solving $y = 2x + 3$ for x: $f^{-1}(y) = \frac{y-3}{2}$.

  Thus, $f^{-1}(y) = \frac{y-3}{2}$ is the inverse function.

Example 2: Non-Bijective Function (No Inverse)

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

Solution:

- **Injectivity**: The function $f(x) = x^2$ is **not injective** because $f(-x) = f(x)$,

  i.e., $f(2) = f(-2) = 4$, but $2 \neq -2$.

  Therefore, f is not injective.

  Since f is not injective, there is no inverse function.

Example 3: Bijective Function on a Restricted Domain

Let $f: [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2$

Solution:

- **Injectivity**: The function $f(x) = x^2$ is **injective** on the domain $[0, \infty)$ because

  if $f(x_1) = f(x_2)$, then $x_1 = x_2$ (since both x_1 and x_2 must be non-negative).

- **Surjective**: For any $y \in \mathbb{R}$, there is no solution for negative y.

  However, if we restrict the range to $[0, \infty)$, then $f(x)$ is surjective.

  Since f is both injective and surjective on $[0, \infty)$, it is bijective.

- **Inverse Function**: The inverse of $f(x) = x^2$ on the domain $[0, \infty)$ is:

  $f^{-1}(y) = \sqrt{y}, y \geq 0$.

  Thus, $f^{-1}(y) = \sqrt{y}$ is the inverse function.

Example 4: Non-Surjective Function (No Inverse)

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$

NOTE

Solution:

- **Injectivity**: The function $f(x) = x + 1$ is **injective** because

  if $f(x_1) = f(x_2)$, then $x_1 + 1 = x_2 + 1$, implying $x_1 = x_2$

- **Surjective**: $f(x) = x + 1$ is surjective because for any $y \in \mathbb{R}$,

  there exists an $x = y - 1$ such that $f(x) = y$.

  Since f is both injective and surjective, it is bijective, and an inverse exists.

- **Inverse Function**: The inverse of $f(x) = x + 1$ is given by: $f^{-1}(y) = y - 1$.

  Thus, $f^{-1}(y) = y - 1$ is the inverse function.

13.10 SET IMAGE/PRESET IMAGE OF FUNCTION

13.10.1. Set Image and Preimage of a Function:

1. Image of a Function

The **image** (or set image) of a function $f: A \to B$ refers to the set of all elements in B that are mapped to by at least one element in A.

Definition:

If $f: A \to B$ is a function, the **image of f**, denoted as $Im(f)$, is:

$Im(f) = \{f(a) \mid a \in A\} \subseteq B$.

In simpler terms, the image is the subset of B that contains all outputs of f.

2. Preimage of a Function

The **preimage** (or inverse image) of a function $f: A \to B$ refers to the set of all elements in A that map to a given subset of B.

Definition:

If $f: A \to B$ is a function and $S \subseteq B$, the **preimage of S** under f, denoted as $f^{-1}(S)$, is:

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}$$

In simpler terms, the preimage is the set of inputs in A that produce outputs in

$S \subseteq B$.

13.10.2. Examples of Set Image and Preimage of a Function

Example 1: Image of a Function

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

- **Domain**: $A = \mathbb{R}$

NOTE

- **Codomain**: $B = \mathbb{R}$

Solution: The image of f is:

$Im(f) = \{f(x) \mid x \in \mathbb{R}\} = \{x^2 \mid x \in \mathbb{R}\}$

Since $x^2 \geq 0$ for all $x \in \mathbb{R}$:

$$Im(f) = [0, \infty)$$

Example 2: Preimage of a Function

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, and consider the subset $S = [4,9] \subseteq \mathbb{R}$

Solution: The preimage of S is:

$f^{-1}(S) = \{x \in \mathbb{R} \mid f(x) \in [4,9]\} = \{x \in \mathbb{R} \mid x^2 \in [4,9]\}$.

This gives:

$$f^{-1}(S) = \{x \mid 4 \leq x^2 \leq 9\}$$

Thus:

$$f^{-1}(S) = [-3, -2] \cup [2,3]$$

Example 3: Image of a Function with Restricted Domain

Let $f: \{1,2,3\} \to \{a, b, c\}$ defined by:

$$f(1) = a, f(2) = b, f(3) = a$$

Solution: The image of f is:

$$Im(f) = \{f(1), f(2), f(3)\} = \{a, b\}.$$

Example 4: Preimage of a Single Element

Let $f: \{1,2,3\} \to \{a, b, c\}$ be defined by:

$$f(1) = a, f(2) = b, f(3) = a$$

Find the preimage of $S = \{a\}$

Solution: The preimage is:

$$f^{-1}(\{a\}) = \{x \in \{1,2,3\} \mid f(x) = a\}$$

Thus:

$$f^{-1}(\{a\}) = \{1,3\}$$

Activity 13.1

Problem1: Ty pes of Functions Activity:

Given the function $f: A \to B$, where $A = \{1,2,3\}$ and $B = \{a, b, c\}$ with the mapping

$f(1) = a, f(2) = b, f(3) = c$:

- Determine if the function is injective (one-to-one) and surjective (onto).
- If it is not injective or surjective, explain why.

Material needed:

Pen, paper

NOTE

Activity 13.2

Problem1: Composition of Functions Activity:

Given two functions:

$$f(x) = 2x + 1 \; and \; g(x) = x^2$$

- Find the composition $f(g(x))$.
- Verify the composition by substituting a value for x (e. g., $x = 3$).

Material needed:

Pen, paper

☑ Check Your Progress 13.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Find the inverse of the function $f(x) = x + 5$

.....

.....

.....

2. A function is surjective if every element in the codomain has a preimage in the domain.

True or False:

3. Which of the following is true for a bijective function?

A) It is only one-to-one (injective).

B) It is only onto (surjective).

C) It is both one-to-one and onto.

D) It has no inverse.

4. The cardinality of a set refers to the number of _____ in the set.

13.11 SUMMARY

This unit introduced the fundamental concept of a function and its importance in mathematics and computer science. Students learned that a function is a special type of relation where each input is associated with exactly one output. The unit explained the definition, properties, and examples of functions, helping learners understand how functions describe relationships between sets and variables. Mapping concepts were also discussed in detail, including the definition of mapping and various types such as one-one, many-one, onto, and into

NOTE

mappings. These concepts provided a clear understanding of how elements of one set correspond to elements of another set.

The unit further explored different types of functions and their characteristics, enabling students to identify and classify functions based on their behavior. The concept of cardinality of a set was introduced to explain the size or number of elements in a set, along with different types of cardinalities and practical examples. Students also studied the composition of relations and functions, learning how two relations or functions can be combined to form a new relation or function. The existence of inverse functions was another important topic, where students understood the conditions required for a function to have an inverse, particularly the need for one-one and onto properties.

In addition, the concepts of set image and preimage of a function were explained through definitions and examples, helping students analyze outputs and trace corresponding inputs. Overall, this unit strengthened students' logical and analytical thinking skills and provided a foundation for advanced topics in algebra, discrete mathematics, computer programming, and data analysis. Understanding functions and mappings is highly useful in real-life applications such as database management, machine learning, economics, engineering, and scientific modeling.

Students are encouraged to practice more problems, explore graphical representations of functions, and study advanced mathematical applications to gain deeper conceptual clarity and problem-solving abilities.

13.12 GLOSSARY

- **Function (Mapping)**:

  A rule or correspondence $f: A \to B$ that assigns each element in the domain A exactly one element in the codomain B. Functions is also called mappings in abstract algebra.

- **Types of Functions**:

  - **Injective (One-to-One)**: No two elements in A map to the same element in B.
  - **Surjective (Onto)**: Every element in B is the image of at least one element in A.
  - **Bijective**: A function that is both injective and surjective, ensuring a one-to-one correspondence between A and B.

- **Cardinality of a Set**:

NOTE

The number of elements in a set. For finite sets, it is a natural number (e.g., $| A |= 4$). For infinite sets, it distinguishes between countable (e.g., integers) and uncountable sets (e.g., real numbers).

- **Composition of a Relation and Function**:

  A process combining a relation $R \subseteq A \times B$ and a function $f: B \to C$, resulting in a new relation $f \circ R \subseteq A \times C$, connecting A to C via R and f.

- **Composition of Functions**:

  Combining two functions $f: A \to B$ and $g: B \to C$ to create a new function

  $g \circ f: A \to C$, where $(g \circ f)(x) = g(f(x))$.

  This operation is associative but not always commutative.

- **Existence of Inverse Function (Mapping)**:
  A function $f: A \to B$ has an inverse $f^{-1}: B \to A$ if and only if it is bijective. The inverse satisfies $f^{-1}(f(x)) = x$ for all $x \in A$, and $f(f^{-1}(y)) = y$ for all $y \in B$.
- **Image of a Function**:
  The set of all outputs of a function $f: A \to B$, denoted $Im(f) = \{f(a) \mid a \in A\}$ It is a subset of the codomain B.
- **Preimage of a Function**:

  For a subset $S \subseteq B$, the preimage under $f: A \to B$, denoted $f^{-1}(S)$, is the set of all elements in A that map to S, i.e., $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$

- **Domain and Codomain**:
  - **Domain**: The set of all possible inputs to a function, denoted as A for $f: A \to B$
  - **Codomain**: The set in which outputs of the function are contained, denoted as B for $f: A \to B$.

- **Properties of Functions**:

  Characteristics of functions such as linearity, monotonicity, symmetry, and periodicity.

13.13 ANSWER TO CHECK YOUR PROGRESS – POSSIBLE ONE

Check Your Progress 13.1

1. Find the inverse of the function $f(x) = x + 5$

Answer: To find the inverse, follow these steps:

1. Start with the function: $y = f(x) = x + 5$

NOTE

2. Swap x and y: $x = y + 5$
3. Solve for y: $y = x - 5$

So, the inverse function is: $f^{-1}(x) = x - 5$

2. A function is surjective if every element in the codomain has a preimage in the domain. True or False:

Answer: True

This is the correct definition of a surjective (onto) function.

Every element in the codomain must be mapped by some element from the domain.

3. Which of the following is true for a bijective function?

Answer: C) It is both one-to-one and onto.

A bijective function is:

- Injective (one-to-one): No two inputs map to the same output.
- Surjective (onto): Every element of the codomain is mapped.

Hence, bijective means both injective and surjective.

4. The cardinality of a set refers to the number of _____ in the set.

Elements

Answer: The cardinality of a set is the count of distinct elements in that set.

13.14 SUGGESTED READINGS OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1253	

13.15 REFERENCE

- https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_functions.htm
- https://www.geeksforgeeks.org/functions/
- https://www.mathspanda.com/A2Ma/Lessons/Mappings_and_functions_LESSON.pdf

NOTE

- https://www.toppr.com/guides/maths/relations-and-functions/types-of-functions/
- https://byjus.com/question-answer/what-is-cardinality-of-set/
- https://www.cuemath.com/algebra/cardinality/
- https://www.geeksforgeeks.org/composition-of-relations/
- https://byjus.com/maths/composition-of-functions/
- https://www.cuemath.com/algebra/inverse-functions/
- https://math.libretexts.org/Courses/Monroe_Community_College/MTH_220_Discrete_Math/5%3A_Functions/5.4%3A_Onto_Functions_and_Images%2F%2FPreimages_of_Sets

13.16 EXERCISE

Part 1

A. Multiple choice question

1. What is a mapping in the context of sets and functions?

a) A relation between any two sets

b) A function that assigns each element of one set to exactly one element of another set

c) A subset of a Cartesian product

d) A function that is one-to-one but not onto

2. Which of the following is NOT a type of function?

a) One-to-One Function

b) Onto Function

c) Bijective Function

d) Complementary Function

3. What is the cardinality of the set $A = \{2, 4, 6, 8, 10\}$?

a) 4

b) 5

c) 10

d) Infinite

4. Which condition must be satisfied for a function to have an inverse?

a) It must be reflexive and transitive

b) It must be symmetric

NOTE

c) It must be bijective

d) It must be one-to-many

5. If $f(x) = 2x + 3$ and $g(x) = x^2$, what is $(f \circ g)(x)$?

a) $2x^2 + 3$

b) $2x + 3$

c) $x^2 + 6$

d) $2(x^2) + 3$

Answers with Explanations:

1. b) A function that assigns each element of one set to exactly one element of another set

Explanation: A mapping is a function that assigns each element of the domain (first set) to exactly one element in the codomain (second set). This ensures the function is well-defined and adheres to the principle of determinism.

2. d) Complementary Function

Explanation: The first three options describe standard types of functions:

One-to-One (Injective): Each element in the codomain is mapped by at most one element in the domain. Onto (Surjective): Every element of the codomain is mapped by some element in the domain. Bijective: Both one-to-one and onto. "Complementary function" is unrelated to functions and mappings in set theory.

3. b) 5

Explanation: The cardinality of a set refers to the number of elements in the set. Set $A = \{2,4,6,8,10\}$ has 5 elements, so the cardinality is 5.

4. c) It must be bijective

Explanation: A function must be bijective (both one-to-one and onto) to have an inverse. This ensures every element in the codomain is uniquely mapped to one element in the domain, making the inverse function well-defined.

5. d) $2(x^2) + 3$

Explanation: The composition of functions $(f \circ g)(x)$ means applying $g(x)$ first and then $f(x)$ Substituting $g(x)$ into $f(x)$: $f(g(x)) = f(x^2) = 2(x^2) + 3$

NOTE

B. Short answer type question

1. What is the difference between a function and a mapping?
2. Define cardinality of a set with an example.
3. What is a bijective function?
4. Explain the composition of functions with an example.
5. What is the inverse of a function, and when does it exist?

Part 2

C. Long answer type question (200 words)

1. Explain the Cartesian (Cross) Product of Sets in detail and with examples.
2. Define a binary relation on a set. Discuss its types and representation methods with examples.
3. Discuss the operations on relations with detailed examples.
4. Explain the concept of partial ordered relations and their significance with examples.
5. Define lattices. Explain their properties with examples.

Part 3

D. Long answer type question (300 words)

1. Explain in detail the Cartesian Product of Sets, its role in defining relations, and its practical applications in mathematics. Provide relevant examples
2. Discuss the concept of binary relations on a set. Describe the different properties (reflexive, symmetric, transitive, antisymmetric) of relations with examples, and explain their relevance in mathematics and applications.
3. Describe in detail the operations on relations (union, intersection, complement, difference, and composition). Provide their definitions, properties, and examples,
4. Define and explain partial ordered relations. Discuss their significance and visualization using Hasse diagrams, where partial orders are applied.

DR. C.V. RAMAN UNIVERSITY

Chhattisgarh, Bilaspur AN A/SECT GROUP UNIVERSITY

Approved by : PCI | AICTE | NCTE | BCI | Member of : AIU | Joint Committee : (UGC | DEB | AICTE) | Recognized by : MSC | A NAAC Accredited University

DR. C. V. RAMAN UNIVERSITY (C.G.)

SLM FEEDBACK FORM

PROGRAMME : _____

COURSE : _____

BLOCK : _____

UNIT : _____

Dear Student,

While studying the units of this block, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill up and send us the following questionnaire, which pertains to this block. If you find the space provided insufficient, kindly use a separate sheet.

Please mail to aisectcontent@aisect.org/ for online submission please scan QR code:

Questionnaire

1. **Name** : _____

2. **Enrolment No.** : | | | | | | | | | | | |

3. **How many hours did you need for studying the units?** : | |

4. **Please give your reactions to the following items based on your reading of the block:**

Items	Excellent	Very Good	Good	Poor	Give specific Examples (If poor)
Presentation Quality					
Language and Style					
Illustrations					
Used (diagrams, tables, etc.)					
Conceptual Clarity					
Check Your Progress Questions					
Suggested Readings - OER					

5. **Any other comments:**

_____

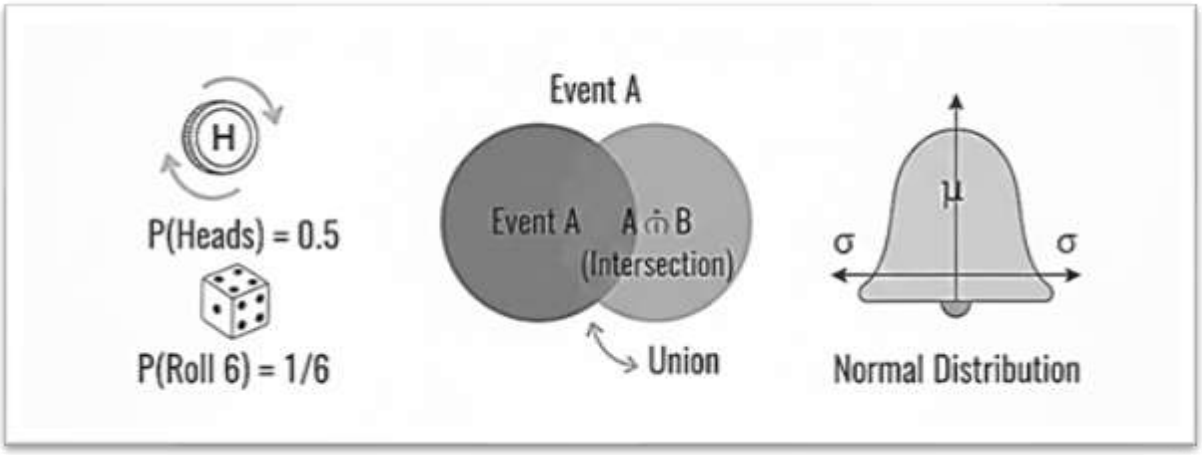
_____

_____

_____

***

BLOCK IV	GRAPH THEORY	241-320
Unit 14	Fundamental of Graph Theory	243
Unit 15	Isomorphism and Subgraphs	271
Unit 16	Graph Operations and Walks	287
Unit 17	Connectivity, Eulerian Graphs and Planarity	303



It introduces the powerful framework of Graph Theory, widely used in networks, algorithms, and real-world modelling. It begins with the fundamentals of graph theory, covering vertices, edges, and basic terminology. Learners then move to isomorphism and subgraphs, gaining insight into graph equivalence and structural relationships. The study continues with graph operations and walks, focusing on combining graphs, traversing through nodes, and understanding graph behaviour. The block concludes with connectivity, Eulerian graphs, and planarity, highlighting essential properties that determine whether a graph is continuous, traceable, or drawable without edge intersections.

UNIT 14
FUNDAMENTAL OF GRAPH THEORY

NOTE

UNIT STRUCTURE

14.1 Introduction

14.2 Objective

14.3 Graph

- **14.3.1 Definition of Graph**
- **14.3.1 Applications of Graph**

14.4 Directed Graph

- **14.4.1 Definition of Directed Graph (Digraph)**
- **14.4.2 Terminology used in Graph**
- **14.4.3 Simple Graph, Multigraph, and Pseudograph**

14.5 Applications of Directed Graphs

14.6 Summary

14.7 Glossary

14.8 Answer to check your progress

14.9 Suggested readings – OER

14.10 Reference

14.11 Exercise

14.1 INTRODUCTION

In the previous unit, you explored the basics of set theory and relations, which form the foundation for graph theory. You learned about different types of relations and their properties. Additionally, you studied how mathematical structures help in modeling real-world problems. Now, we will delve into the fundamental concepts of graph theory.

Building on this knowledge, graph theory, including its basic definitions and properties. We also studied different types of graphs and their representations. Additionally, we now offer a detailed study of directed graphs, simple graphs, multigraphs, and pseudographs. We will also explore the concepts of loops, multiple edges, and parallel edges. Understanding these topics will help in solving complex problems in network analysis and computer science.

This unit bridges abstract mathematical concepts with real-world applications. Graphs are widely used in computer networks, transportation systems, and social media analysis. Directed graphs play

NOTE

a crucial role in representing relationships, dependencies, and flow structures in various fields.

14.2 OBJECTIVE

After studying this unit, you will be able to:

- Define and differentiate various types of graphs.
- Explain the concept of directed graphs and their applications.
- Identify loops, multiple edges, and parallel edges in graphs.
- Assist and apply graph theory in real-world problems.
- Analyze the significance of different types of graphs in computer science and networking.

14.3 GRAPH

14.3.1 Definition of Graph

Graph theory is a branch of discrete mathematics that studies **graphs**. It has numerous applications in computer science, engineering, social sciences, and biology. Graph: A graph with the formula $G = (V, E)$ has a set of vertices (V) and a set of edges (E), each of which joins two vertices.

Vertices: The points or nodes in a graph (e.g.,a, b, c).

Edges: Connections between vertices (e.g., $a - b, b - c$).

Example:

In Fig 14.1 (a), V $= \{a, b, c\}$ and E $= \{(a, b), (a, b), (b, c)\}$ the member or vertices and edges are $| v(G) | = 3$ and $|E(G)| = 3$. In this graph, the vertices a *and* b, a *and* c are adjacent vertices.

In Fig 14.1 (b). V $= \{u, v, w, x\}$ *and* $E = \{(u, v), (v, w), (w, x)\}$

Here vertices u *and* v, v *and* w, w *and* x *are* adjacent, whereas u and w, u and x and v and x are non-adjacent.

The number of vertices and edges are $|v(G)| = 4$ and $| E(G)| = 3$.

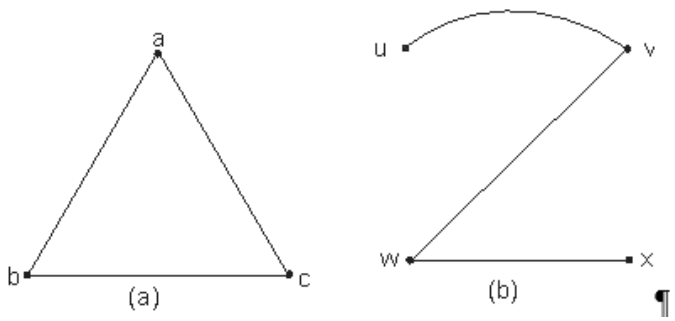


Figure No.14.1 (a) & (b): Graph

NOTE

14.3.2 Applications of Graph

In mathematics and computer science, a graph is a mathematical structure that consists of two main components: vertices (or nodes) and edges. The study of these graphs in various contexts is called graph theory.

There are various applications of graph theory in real life such as in computer graphics and networks, biology, and many other fields as well. In this article, we will discuss real-life applications of graph theory in various fields like Computer Science, Biology, Sociology etc.

1. Computer Science and Technology: Graph theory has numerous applications in Computer Science and Technology, ranging from network design to artificial intelligence. Graph theory plays a fundamental role in computer science, helping solve complex problems efficiently.

Here are some key areas where graph theory is applied:

a) Computer Networks and Communication

- Routing Algorithms: Graphs help design optimal routing in networks (e.g., Dijkstra's Algorithm for shortest paths).
- Internet and Web Graphs: The World Wide Web is modeled as a directed graph where websites are nodes, and hyperlinks are edges.

b) Data Structures and Algorithms

- Graph Traversal Algorithms: BFS (Breadth-First Search) and DFS (Depth-First Search) are essential in search operations.
- Minimum Spanning Tree: Used in network design (e.g., Kruskal's and Prim's algorithms).

c) Artificial Intelligence and Machine Learning

- Neural Networks: Modeled using graph structures, where neurons are nodes and connections are edges.
- Knowledge Representation: Graphs are used in semantic networks and knowledge graphs (e.g., Google Knowledge Graph).

d) Database Management Systems

- Graph Databases: Used in NoSQL databases (e.g., Neo4j) for efficient relationship management.
- Dependency Graphs: Help optimize query execution and detect deadlocks in databases.

e) Social Network Analysis

- Friendship Graphs: Used by platforms like Facebook and Twitter to suggest connections.

NOTE

- Community Detection: Algorithms identify clusters of users with similar interests.

f) Cryptography and Cybersecurity

- Graph-based Cryptographic Algorithms: Used in secure communication protocols.
- Attack Graphs: Model vulnerabilities in computer networks to improve security.

g) Image Processing and Computer Vision

- Graph Cuts: Used in image segmentation.
- Feature Extraction: Graphs help recognize patterns in images and videos.

h) Compiler Design

- Control Flow Graph (CFG): Helps in program optimization and debugging.
- Syntax Trees: Used in parsing source code.

i) Robotics and Path Planning

- Graph-based Navigation: Robots use graphs for pathfinding in an environment.
- Motion Planning: Uses algorithms like A* to determine efficient movement.

j) Software Engineering

- Module Dependency Graphs: Help in software testing and debugging.
- UML Diagrams: Used for system design representation.

2. Mathematics and Science: Graph theory has widespread applications in Mathematics and Science, helping model relationships, optimize processes, and solve complex problems. Graph theory provides powerful tools for solving complex problems in mathematics and science

Below are some key applications:

1. Mathematics Applications of Graph Theory

a. Combinatorics and Optimization

- **Graph Coloring:** Used in scheduling problems, like exam timetables and register allocation in compilers.
- **Hamiltonian and Eulerian Paths:** Used in solving routing problems such as the Seven Bridges of Königsberg.
- **Shortest Path Problems:** Algorithms like Dijkstra's Algorithm and Bellman-Ford Algorithm solve path optimization problems.

NOTE

b. Linear Algebra and Matrix Representation

- **Adjacency Matrix and Incidence Matrix:** Help represent graphs mathematically for efficient computation.
- **Spectral Graph Theory:** Uses eigenvalues and eigenvectors to analyze graph properties.

c. Number Theory and Cryptography

- **Graph-based Cryptography:** Used in cryptographic systems like Zero-Knowledge Proofs.
- **Prime Graphs:** Study of prime numbers through graph structures.

d. Game Theory

- **Graph-Based Strategies:** Used in decision-making models in competitive and cooperative games.
- **Nash Equilibrium Representation:** Can be visualized using graphs in economic and strategic planning.

2. Science Applications of Graph Theory

a. Physics

- **Quantum Mechanics:** Graphs model quantum states and particle interactions.
- **Electric Circuits:** Kirchhoff's laws use graphs to analyze electrical networks.

b. Chemistry

- **Molecular Structures:** Chemical compounds are represented using graphs (e.g., benzene rings as cycles in graphs).
- **Reaction Networks:** Graphs model chemical reactions, helping predict product formation.

c. Biology and Genetics

- **Protein-Protein Interaction Networks:** Graphs analyze how proteins interact in biological systems.
- **Gene Regulation Networks:** Used in genetics to study gene expression patterns.

d. Medicine and Epidemiology

- **Spread of Diseases:** Epidemic models (like SIR models) use graphs to track disease transmission.
- **Neural Networks in Brain Research:** The human brain's neuron connections are studied as graphs.

NOTE

e. Environmental Science

- **Food Webs:** Graphs represent ecological relationships between species.
- **Climate Networks:** Used in modeling global climate systems.

f. Astronomy and Space Science

- **Galactic Networks:** Graphs represent galaxy distributions and celestial body interactions.
- **Satellite Communication:** Optimizes satellite placement using graph-based algorithms.

3. Education: Graph theory plays a significant role in education, helping in various domains such as curriculum design, student performance analysis, e-learning platforms, and administrative management. Graph theory enhances education by optimizing learning structures, improving student experiences, and streamlining administrative processes.

Below are some key applications**:**

a. Curriculum Design and Course Prerequisites

- Graph Representation of Courses:
  - University courses can be modeled as a directed graph, where nodes represent courses, and edges represent prerequisite relationships.
  - Example: If Course A is a prerequisite for Course B, then there is a directed edge from A to B.
- Dependency Graphs help in designing structured learning paths for students.

b. Student Performance Analysis

- Social Learning Networks:
  - Graphs can analyze student interactions in classrooms, group projects, and discussion forums.
- Knowledge Graphs:
  - Used to identify gaps in a student's understanding of a subject by mapping their knowledge to an ideal knowledge network.

c. E-Learning and Personalized Learning

- Adaptive Learning Platforms (like Coursera, Udemy, and Khan Academy):
  - Use graph-based recommendation algorithms to suggest courses based on student progress.

NOTE

- Concept Maps:
  - A graph-based structure that helps students visualize relationships between different topics.

d. Examination Scheduling and Classroom Management

- Graph Coloring in Exam Timetabling:
  - Ensures no two exams of common students overlap.
- Bipartite Graphs in Classroom Allocation:
  - Matches classrooms with student groups efficiently.

e. Educational Data Mining and Analytics

- Graph-Based AI Models:
  - Used to track student engagement, learning patterns, and performance prediction.
- Collaborative Learning Graphs:
  - Identify how students collaborate on assignments and projects to improve teamwork strategies.

f. School Transportation and Resource Management

- Graph Theory in Bus Route Optimization:
  - Helps design shortest and most efficient routes for school buses.
- Resource Allocation in Schools:
  - Bipartite graphs can be used to match teachers to courses and students to extracurricular activities efficiently.

4. Transportation: Graph theory is widely used in transportation systems for route optimization, traffic management, logistics, and network design.

Below are some key applications:

a. Route Optimization and Navigation

- Shortest Path Algorithms:
  - Dijkstra's Algorithm: Used in GPS and Google Maps to find the shortest path between two locations.
  - Floyd-Warshall Algorithm: Finds shortest paths between all pairs of nodes in a transportation network.
- Applications:
  - Airline route planning.
  - Public transport (bus, metro, railway) optimization.

NOTE

b. Traffic Flow and Management

- Traffic Congestion Analysis:
  - Graph-based models help identify high-traffic areas and suggest alternative routes.
- Urban Road Networks:
  - Represented as weighted graphs, where edges have weights based on travel time, distance, or congestion levels.
- One-Way and Two-Way Traffic Problems:
  - Directed Graphs (Digraphs) model road networks with one-way and two-way streets.

c. Logistics and Supply Chain Optimization

- Vehicle Routing Problem (VRP):
  - Graphs help in determining optimal delivery routes for logistics companies like Amazon, FedEx, and UPS.
- Transportation Networks in Warehousing:
  - Spanning Tree Algorithms (Kruskal's, Prim's) minimize travel costs within warehouse distribution networks.

d. Railway and Metro Network Design

- Graph Representation:
  - Stations are represented as nodes, and rail lines are edges.
- Applications:
  - Optimizing train schedules.
  - Finding the best interchange stations.
  - Avoiding deadlocks in railway signaling systems.

e. Airline Network and Flight Scheduling

- Graph Representation:
  - Airports = Nodes, Flights = Edges.
- Applications:
  - Minimizing layover time using shortest path algorithms.
  - Finding alternative routes in case of flight delays.
  - Hub-and-Spoke network design for airline companies.

f. Maritime Transportation and Shipping Routes

- Graph-based Optimization:
  - Used to determine optimal sea routes while avoiding congested ports and unfavorable weather conditions.
- Container Loading and Unloading:

NOTE

- Graph algorithms help in scheduling and optimizing port operations.

g. Smart Transportation and Autonomous Vehicles

- Self-Driving Cars:
  - Use graph-based AI models to navigate roads and avoid obstacles.
- Intelligent Traffic Systems:
  - Predict and manage real-time traffic using sensor networks modeled as graphs.

5. Economics and Business: Graph theory is widely used in economics and business for modeling relationships, optimizing networks, and improving decision-making. Graph theory plays a vital role in modern economics and business by improving efficiency, predicting market trends, and optimizing decision-making

Below are some key applications:

a. Economic Networks and Trade Analysis

- International Trade Networks:
  - Countries are represented as nodes, and trade relationships as edges.
  - Helps analyze global trade patterns, dependencies, and economic stability.
- Input-Output Models:
  - Represent industries as nodes and their economic interactions as edges.
  - Used to study the impact of changes in one sector on others.

b. Business Supply Chain and Logistics

- Supply Chain Optimization:
  - Graphs help optimize delivery routes for reducing costs and improving efficiency.
  - Example: Amazon and Walmart use graph algorithms for warehouse management and delivery planning.
- Transportation and Distribution Networks:
  - Spanning Tree Algorithms (Kruskal's, Prim's) minimize costs in supply chain distribution.

c. Market Analysis and Competition

- Game Theory and Strategic Decision Making:
  - Graph-based game theory helps in analyzing market competition and pricing strategies.

NOTE

- Used by companies like Google and Facebook for ad bidding models.

- Product Recommendation Systems:
  - Companies like Netflix and Amazon use graph-based recommendation engines to analyze customer behavior.

d. Stock Market and Financial Networks

- Stock Market Analysis:
  - Graphs model the relationships between stock prices and financial indicators.
  - Centrality Measures identify influential stocks in a market network.
- Fraud Detection:
  - Banks and financial institutions use graph-based AI models to detect fraudulent transactions and money laundering activities.

e. Organizational Management and HR Analytics

- Employee Collaboration Networks:
  - Graphs analyze communication and teamwork efficiency in organizations.
  - Helps HR teams improve productivity and leadership development.
- Decision-Making and Influence Graphs:
  - Used to study organizational hierarchies and optimize workflow.

f. Business Intelligence and Data Mining

- Social Network Analysis for Marketing:
  - Businesses use graph-based models to analyze customer connections and improve targeted advertising.
  - Example: Facebook's Ad Network uses graph algorithms to identify influential users.
- Big Data and Graph Databases:

g. Economic Growth and Development Models

- Domar Macro Model:
  - Uses graph structures to analyze investment, savings, and economic growth relationships.
- Samuelson's Investment Model:

NOTE

- Graph-based economic modeling of investment cycles and business fluctuations.

6. Medicine and Healthcare: Graph theory plays a crucial role in medicine and healthcare, helping in disease modeling, medical imaging, drug discovery, and hospital management. Graph theory enhances healthcare efficiency, disease control, and medical research.

Below are some key applications:

a. Disease Spread and Epidemiology

- Epidemic Modeling (SIR Model):
  - Nodes represent individuals, and edges represent disease transmission paths.
  - Helps predict and control outbreaks like COVID-19, influenza, and Ebola.
- Contact Tracing Networks:
  - Identifies high-risk individuals by analyzing connections between infected and healthy people.

b. Medical Imaging and Diagnostics

- Graph-based Image Processing:
  - Used in MRI, CT scans, and X-rays to detect abnormalities (e.g., tumors, fractures).
- Brain Network Analysis:
  - Functional Connectivity Graphs study brain disorders like Alzheimer's and Parkinson's disease.
- Pattern Recognition in Pathology:
  - Helps classify cancerous vs. non-cancerous cells in biopsy samples.

c. Drug Discovery and Pharmacology

- Protein-Protein Interaction Networks (PPINs):
  - Identify drug targets by analyzing how proteins interact in the human body.
- Chemical Compound Graphs:
  - Represent molecules as nodes (atoms) and edges (chemical bonds) to study drug behavior.
- Disease-Gene Association:
  - Graph algorithms help find genetic links to diseases and possible treatments.

NOTE

d. Hospital and Healthcare Management

- Hospital Resource Allocation:
  - Optimizes the distribution of beds, medical staff, and equipment using bipartite graphs.
- Patient Flow Optimization:
  - Reduces hospital wait times by modeling patient movement as a graph.
- Medical Supply Chain Management:
  - Uses graph algorithms for efficient distribution of medicines and vaccines.

e. Biomedical Research and Genetics

- Genetic Networks:
  - Help understand hereditary diseases by modeling genes and their interactions as graphs.
- DNA and RNA Structure Analysis:
  - Uses tree-based graphs to study genetic sequences and mutations.

f. AI in Healthcare and Disease Prediction

- Graph Neural Networks (GNNs):
  - Used in AI-powered healthcare applications for disease prediction and diagnosis.
- Healthcare Chatbots:
  - Use graph-based algorithms to improve doctor-patient interactions.

7. Social Sciences: Graph theory is widely used in social sciences to analyze relationships, study human behavior, and model social networks. Graph theory helps social scientists understand complex human interactions, political trends, crime patterns, and media influence.

Below are some key applications:

a. Social Network Analysis (SNA)

- Graph Representation of Social Networks:
  - Nodes represent individuals, and edges represent relationships (friendship, communication, influence).
  - Example: Facebook, Twitter, LinkedIn use graph algorithms to suggest friends/connections.

NOTE

- Centrality Measures:
  - Identify influential people in a network (leaders, influencers).
  - Degree Centrality: Number of direct connections.
  - Betweenness Centrality: Key connectors in a network.
  - Closeness Centrality: Nodes with quick access to others.

b. Political Science and Governance

- Elections and Voting Patterns:
  - Graphs model relationships between voters, political parties, and candidates.
  - Used to analyze voting trends and campaign strategies.
- Terrorism and Security Networks:
  - Helps track criminal organizations and predict attacks by analyzing hidden social structures.
- Policy Impact Analysis:
  - Graph-based models study the effect of government policies on society.

c. Psychology and Human Behavior

- Cognitive and Neural Networks:
  - Used to study brain connections and mental disorders (e.g., schizophrenia, depression).
- Decision-Making and Group Behavior:
  - Graph-based decision trees model human choices in social and economic contexts.

d. Sociology and Community Analysis

- Group Dynamics and Social Cohesion:
  - Cluster Analysis identifies closely connected groups in society.
  - Used to study urban planning and community building.
- Diffusion of Information and Innovation:
  - Graphs help track how ideas, trends, and innovations spread in a society (e.g., viral content on social media).

e. Linguistics and Language Networks

- Word Co-occurrence Graphs:
  - Analyze how words are related in texts (used in AI-driven translation tools).

NOTE

- Social Communication Networks:
  - Study language evolution and dialect variations through connected communities.

f. Criminology and Law Enforcement

- Crime Network Analysis:
  - Graphs help identify criminal organizations, drug cartels, and fraud networks.
- Judicial Case Networks:
  - Courts use graphs to analyze past cases and legal precedents for faster decision-making.

g. Media, Journalism, and Information Flow

- News Propagation Networks:
  - Analyze how news spreads across media platforms.
  - Helps detect fake news and misinformation.
- Influencer Analysis in Marketing:
  - Companies use graph algorithms to identify key social media influencers for product promotions.

8. Games and Entertainment: Graph theory plays a crucial role in games, movies, sports, and digital entertainment by enabling efficient design, optimization, and analysis. Graph theory enhances game design, sports analytics, movies, music, and virtual experiences by improving efficiency, realism, and user engagement.

Below are key applications:

a. Game Development and Level Design

- Graph Representation of Game Worlds:
  - Games like Chess, Checkers, and Tic-Tac-Toe are modeled using graphs, where:
    - Nodes = game states or board positions.
    - Edges = possible moves.
- Pathfinding Algorithms:
  - Used for character movement in open-world games like Minecraft, Assassin's Creed, GTA.
  - Algorithms like A* and Dijkstra's help NPCs navigate efficiently.
- Graph-based AI in Gaming:
  - Helps create intelligent enemies and NPCs that make realistic decisions.

NOTE

b. Puzzle and Strategy Games

- Sudoku, Maze, and Crossword Generation:
  - Represented as constraint satisfaction problems using graphs.
- Graph-Based Board Games:
  - Games like Go, Monopoly, and Risk use graph structures for board movement and strategies.

c. Sports Analytics and Tournament Scheduling

- Graph-Based League Scheduling:
  - Sports tournaments (like FIFA, IPL, NBA) are modeled as graphs where:
    - Teams = Nodes.
    - Matches = Edges.
  - Helps schedule fair matches and avoid conflicts.
- Player and Team Performance Analysis:
  - Uses social network analysis to track player connections and teamwork efficiency.

d. Animation and Computer Graphics

- Graph-Based 3D Modeling:
  - Nodes represent points in space, and edges form object structures.
  - Used in movies, VR, and CGI effects (e.g., Pixar, Disney).
- Motion Capture and Character Rigging:
  - Graphs help animate human movements in realistic ways.

e. Movie Recommendation Systems

- Graph Databases in Streaming Services:
  - Platforms like Netflix, Amazon Prime, and YouTube use graph-based AI to:
    - Recommend movies based on user preferences.
    - Analyze how viewers are connected through similar interests.

f. Music and Sound Processing

- Music Recommendation Systems:
  - Apps like Spotify and Apple Music use graphs to suggest songs based on user behavior.

NOTE

- Graph-Based Sound Analysis:
  - Used in speech recognition, auto-tune, and AI-generated music.

g. Virtual Reality (VR) and Augmented Reality (AR)

- Graph-Based Scene Navigation:
  - VR games use graph traversal algorithms to optimize 3D movement and interactions.
- Graph-Based AR Object Placement:
  - Helps position virtual objects in the real world for AR applications (e.g., Pokémon GO).

h. Online Multiplayer Game Networking

- Server and Player Connection Optimization:
  - Online multiplayer games like Fortnite, Call of Duty, and PUBG use graphs to:
    - Connect players in matchmaking.
    - Optimize latency and server load balancing.
- Cheat Detection in Online Games:
  - Graphs track suspicious player activities to detect cheating behavior.

14.4 DIRECTED GRAPH

14.4.1 Definition of Directed Graph (Digraph)

A directed graph (or digraph) G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices, whereas a graph where each edge has a no direction is an undirected graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

This digraph represents a partial ordering or flow of tasks, where some tasks can be completed independently while others depend on previous tasks.

Thus, a directed graph (or digraph) is a graph where each edge has a direction, represented by an arrow. It is defined as $G = (V, E)$, where:

- V is the set of vertices (nodes).
- E is the set of ordered pairs of vertices (directed edges).

Example 1: Each directed edge (u, v) goes from vertex u (tail) to vertex v (head).

NOTE

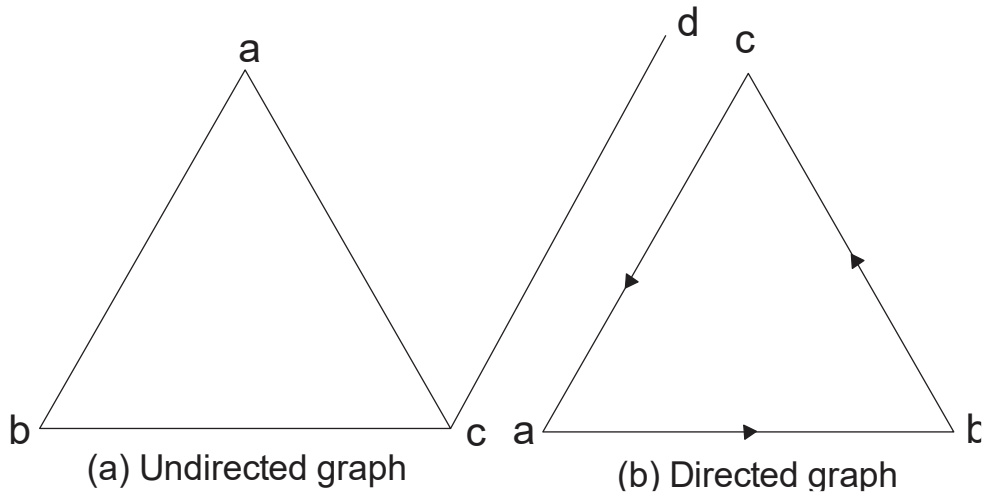


Figure No.14.2: (a) Undirected Graph and (b) Directed Graph

Fig.14.2 (a) is an example of an undirected graph we can refer to an edge joining the

vertex pair i and j as either (i,j) or (j,i). Fig. 14.2 (b) is an example of a directed graph.

Suppose $e = (u,v)$ is a directed edge in a digraph, then

(i) u is called the initial vertex of e and v is the terminal vertex of e.

(ii) e is said to be incident from u and to be incident to v.

(iii) u is adjacent to be v and v is adjacent from u.

Example 2 Explain Graph:

- **Vertices**: a, b, c
- **Edges**: $a \to b, b \to c, c \to a$

Solution: Represent the directed edges with arrows as fig no.5.1(b):

- $a \to b$: An arrow from $a\ to\ b$.
- $b \to c$: An arrow from $b\ to\ c$.
- $c \to a$: An arrow from $c\ to\ a$.

The graph forms a **directed cycle**.

14.4.2 Terminology Used In Graph

- **Loop or Self-Loop:** An edge of a graph that joins a vertex to itself is called a loop or self-loop (Fig No.14.3: (a)) i.e., a loop is an edge $(v_i, v_j)\ where\ v_i = v_j$.

  Example: Vertex V contain 2 self-loops.

NOTE

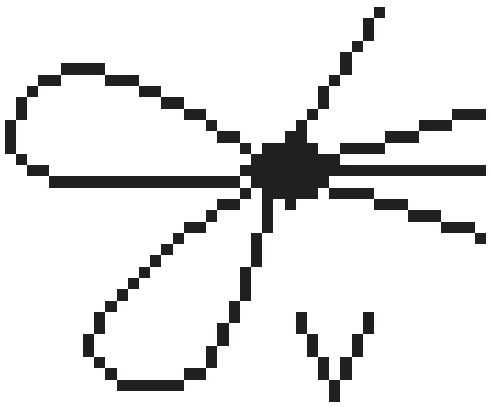


Figure No.14.3: (a) Loop or Self-loop

- **Multiple or Parallel Edges**: In some directed as well as undirected graphs, we may contain pair of vertices joined by more than one edges, such edges are called multiple or parallel edges (Fig No.14.3: (b)). Two edges (v_i, v_j) and $(v_f, v_r)$are parallel edges if $v_i = v_f$ and $v_j = v_r$.

  Example: Vertex u and w contain two parallel edges whereas vertex V and W contain three multiple or parallel edges.

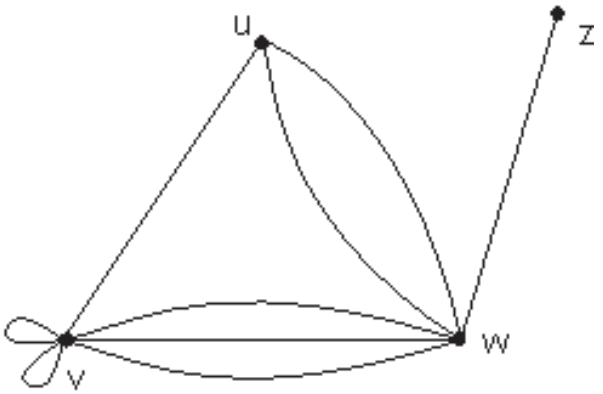


Figure No.14.3: (b) Multiple or Parallel Edges:

14.4.3 Simple Graph, Multigraph and Pseudograph

- **Simple Graph:**

A graph which has neither loops nor multiple edges i.e., where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a simple graph as in Fig 14.4 (b) represents simple undirected graph and Fig No.14.4: (a) (c) represents undirected Multiple graphs because the graphs contain multiple edges.

Example 1. State which of the following graphs are simple graph?

NOTE

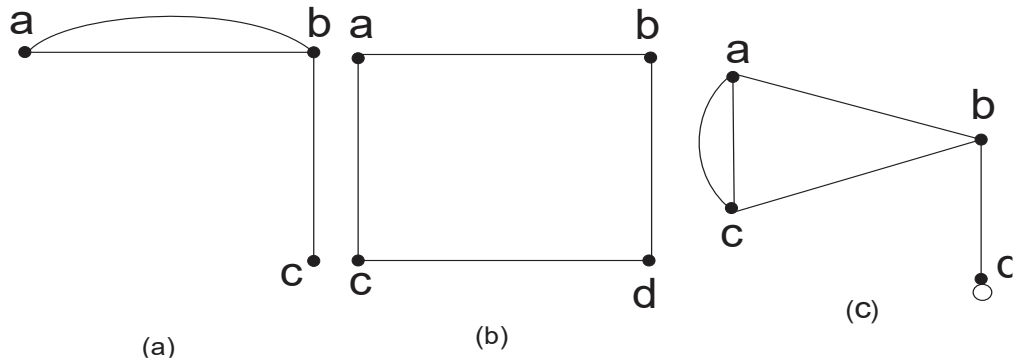


Figure No.14.4: (b) Simple graph & (a), (c) Multiple graph

Solution. Fig No.14.4: (a) The graph is not a simple graph, since it contains parallel edge between two vertices a and b.Also, the graph is a simple graph; it does not contain loop and parallel edge. Fig No.14.4: (a) (c) The graph is not a simple graph but a Multiple graph since it contains parallel edge

- **Multiple Graph:** Any graph which contains some multiple edge is called a multigraph. In a multigraph, no loops are allowed. Graph theory classifies graphs into various types based on their properties. One such classification is multiple graphs, which allow multiple edges between the same pair of vertices. These graphs can be either undirected or directed.

(A) Undirected Multiple Graph

An undirected multiple graph (also called a multigraph) is a graph where:

- Multiple edges (parallel edges) are allowed between two vertices.
- Edges do not have a direction (i.e., if an edge exists between vertices A and B, then movement is possible in both directions).
- Loops (edges connecting a vertex to itself) may or may not be present.

- **Example of an Undirected Multiple Graph**

  Consider a transportation network, where multiple roads connect two cities.

  - Cities represent Vertices
  - Roads represents Edges
  - If two cities are connected by multiple roads, it forms a multiple edge.

(B) Directed Multiple Graph

A directed multiple graph (or directed multigraph) is a graph where:

- Multiple directed edges (arcs) are allowed between two vertices.

NOTE

- Each edge has a direction, meaning it moves from one vertex to another in a specific direction.
- Loops may or may not be present.

o **Example of a Directed Multiple Graph:**

Consider a **one-way road system** between intersections.

- Intersections = **Vertices**
- One-way roads = **Directed Edges**

(C) Example by Fig Representation:

In $Fig\ no.14.5\ (a)$ there are two parallel edges associated with $v_2\ and\ v_3$ also $v_1\ and\ v_2$ known as multiple edges of an undirected multiple graphs. In $Fig\ no.14.5\ (b)$ there are three parallel edges associated with $v_2\ and\ v_3$ also $v_1\ and\ v_2$ with two parallel edges known as multiple edges of a directed multiple graphs.

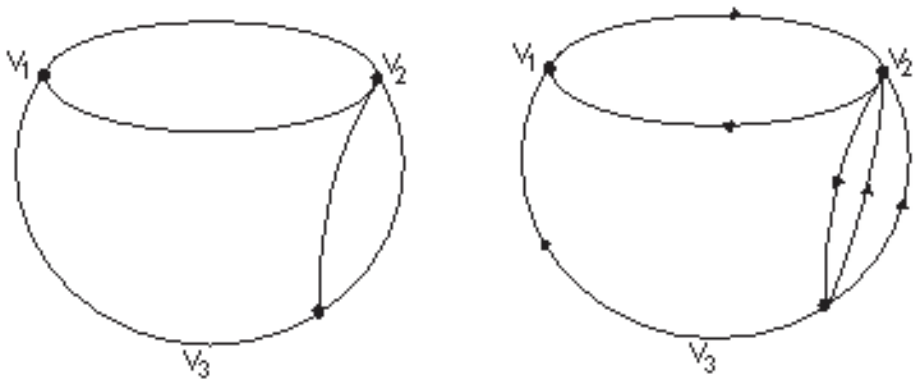


Figure No.14.5: (a) Undirected multiple graphs and (b) Directed multiple graph

o **Pseudograph:**

A graph with self-loops and multiple edges is called a **Pseudograph**.

A pseudograph is a type of graph in which:

1. Multiple edges (parallel edges) between two vertices are allowed.
2. Loops (self-loops) are allowed, meaning an edge can start and end at the same vertex.
3. Edges can be directed or undirected (depending on the definition).

(a) Example of A Pseudograph

Consider a transportation network, where:

- A city (vertex) has internal roads (loops).
- Multiple roads connect two cities (parallel edges).

NOTE

(b) Figure Representation Pseudograp:

Thus fig no. 14.6 is a Pseudograph The number of vertices and edges are $|v(G)| = 4$ and $|E(G)| = 9$. At vertex v there are two self-loops, three multiple edges with v *and* w and two multiple edges with u *and* w.

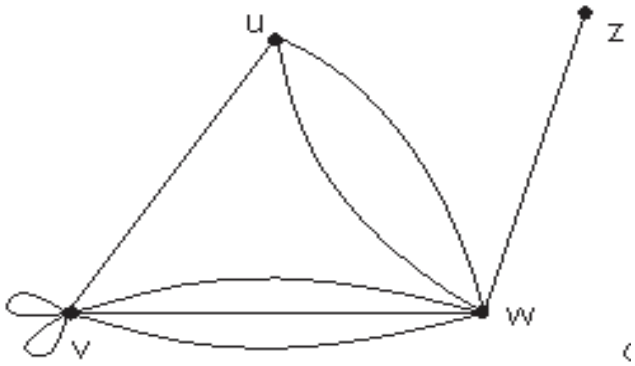


Figure No.14.6: Pseudograph

14.4.4 Applications of Directed Graphs

1. **Web Navigation**: Directed graphs are used to model hyperlinks between web pages.
2. **Task Scheduling**: Representing dependencies between tasks in project planning.
3. **Network Flow**: Modeling traffic in communication or transportation networks.

Activity 14.1

Identifying Different Types of Graphs

Objective:

To help students identify and differentiate between various types of graphs, including simple graphs, multigraphs, pseudographs, directed graphs, and graphs with loops or multiple edges.

Instructions:

1. Observe the given set of graphs (provide diagrams of different graphs).
2. Identify the type of each graph (Simple, Multigraph, Pseudograph, Directed, etc.).
3. Justify your classification with reasoning.
4. Discuss with your peers and compare your answers.

NOTE

Example Exercise:

Consider the following adjacency representations and determine the type of graph:

- Graph 1: A set of vertices connected without loops or multiple edges.
- Graph 2: A set of vertices with multiple edges between some pairs.
- Graph 3: A graph containing at least one self-loop.

Expected Learning Outcome:
Students will be able to recognize different types of graphs and justify their classifications based on graph theory definitions.

☑ Check Your Progress 14.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a simple graph.

……

……

……

2. A pseudograph can contain loops and multiple edges. (True or False).

3. In which type of graph can an edge connect a vertex to itself?

a) Simple Graph b) Multigraph

c) Pseudograph d) None of these

4. A graph that has directed edges is called a _____.

14.5 SUMMARY

The unit Fundamental of Graph Theory introduced the basic concepts and structures used to represent relationships between objects through graphs. Students learned the definition of a graph as a mathematical structure consisting of vertices (nodes) and edges (connections) that help model various real-life systems and networks. The unit also explained the importance and applications of graphs in areas such as computer networks, transportation systems, social media analysis, communication systems, and data organization. Understanding graph structures helps students visualize and solve complex relational problems efficiently.

NOTE

The unit further explored directed graphs, also known as digraphs, where edges have a specific direction indicating one-way relationships between vertices. Students became familiar with important graph terminology including vertices, edges, degree, path, cycle, adjacent vertices, and connectivity. Different types of graphs such as simple graphs, multigraphs, and pseudographs were also studied. A simple graph contains no loops or multiple edges, a multigraph allows multiple edges between the same pair of vertices, and a pseudograph permits both loops and multiple edges. These classifications help in selecting appropriate graph models for different practical situations.

Another major focus of the unit was the application of directed graphs in real-world scenarios. Directed graphs are widely used in traffic routing, web page linking, project scheduling, flowcharts, dependency management, and communication networks. By studying these concepts, students developed analytical and problem-solving skills that are essential in mathematics, computer science, engineering, and operations research. The unit encourages learners to further explore advanced topics such as graph traversal, shortest path algorithms, network optimization, and graph coloring. Continuous practice with graph representations and applications will strengthen understanding and prepare students for higher-level studies and practical problem-solving in technology and scientific research.

14.6 GLOSSARY

- **Graph -** A structure consisting of vertices and edges.
- **Directed Graph (Digraph) -** A graph where edges have a direction.
- **Vertex (Node) -** A fundamental unit of a graph.
- **Edge (Arc) -** A connection between two vertices.
- **Simple Graph -** A graph without loops or multiple edges.
- **Multigraph -** A graph that allows multiple edges between the same vertices.
- **Pseudograph -** A graph that allows both multiple edges and loops.
- **Loop (Self-loop) -** An edge that connects a vertex to itself.
- **Parallel Edges (Multiple Edges) -** More than one edge connecting the same pair of vertices.
- **Application of Graphs -** Graphs are used in social networks, road networks, and computer science.

NOTE

14.7 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 14.1

1. **Define a simple graph.**

   Answer: A simple graph is an unweighted, undirected graph that has no loops (edges connected at both ends to the same vertex) and no more than one edge between any two different vertices. It consists only of distinct vertices connected by at most one edge between any pair.

1. **A pseudograph can contain loops and multiple edges. (True or False)**

   Answer: True

   A pseudograph is a type of graph that may include both loops (edges connecting a vertex to itself) and multiple edges between the same pair of vertices.

2. **In which type of graph can an edge connect a vertex to itself?**

   Answer: c) Pseudograph

   Explanation: Only pseudographs allow loops, which are edges that connect a vertex to itself.

3. **A graph that has directed edges is called a _____**

   Answer: Directed Graph or Digraph

   Explanation: A directed graph (or digraph) is a graph in which the edges have a direction, going from one vertex to another.

14.8 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1254	

14.9 REFERENCE

- https://en.wikipedia.org/wiki/Graph_theory

NOTE

- https://www.geeksforgeeks.org/what-is-directed-graph-directed-graph-meaning/
- https://byjus.com/maths/graph-theory/
- https://www.tutorialspoint.com/edges-and-vertices-of-graph
- https://courses.grainger.illinois.edu/cs173/fa2020/Lectures/Notes/GraphsAndTreesNotes.pdf
- https://www.cip.ifi.lmu.de/~grinberg/t/22s/lec7.pdf
- https://testbook.com/maths/pseudograph
- https://www.kwcsangli.in/uploads/Graph-theory.pdf
- https://en.wikipedia.org/wiki/Multiple_edges#:~:text=In%20graph%20theory%2C%20multiple%20edges,and%20the%20same%20head%20vertex.

14.10 EXERCISE

Part 1

A. Multiple choice question

1. **Which of the following is NOT a type of graph?**
   a) Simple Graph
   b) Directed Graph
   c) Weighted Graph
   d) Undefined Graph
2. **A graph with multiple edges between two vertices is called a:**
   a) Simple Graph
   b) Multigraph
   c) Directed Graph
   d) Tree
3. **Which type of graph allows both loops and multiple edges?**
   a) Simple Graph
   b) Multigraph
   c) Pseudograph
   d) Tree
4. **Which graph type is best for modeling web page links?**
   a) Simple Graph
   b) Multigraph

NOTE

c) Directed Graph

d) Weighted Graph

5. If a graph contains vertices and directed edges with weights, it is known as a:

a) Simple Graph

b) Directed Weighted Graph

c) Tree

d) Multigraph

Answers with Explanations:

1. d) Undefined Graph

Explanation: There is no such concept as an "undefined graph" in graph theory.

2. b) Multigraph

Explanation: A multigraph allows multiple edges between the same pair of vertices.

3. c) Pseudograph

Explanation: A pseudograph is a graph that permits both loops and multiple edges.

4. c) Directed Graph

Explanation: Websites link to each other in a directed manner, forming a directed graph.

5. b) Directed Weighted Graph

Explanation: A directed weighted graph has edges with direction and associated weights.

B. Short answer type question

1. Define a directed graph.
2. What is a loop in a graph?
3. Give one real-world application of graph theory.
4. What is a multigraph?
5. What is the difference between a simple graph and a pseudograph?

NOTE

Part 2

C. Long answer type question (200 words)

1. Explain the properties and characteristics of directed graphs.
2. Compare and contrast a simple graph, multigraph, and pseudograph.
3. Discuss the applications of graph theory in social networks.
4. Define and explain loops and multiple edges with examples.
5. Explain how graphs are used in transportation and computer networks.

Part 3

D. Long answer type question (300 words)

1. Explain in detail the different types of graphs with suitable diagrams and examples.
2. Describe the role of directed graphs in real-life applications such as GPS navigation and web search engines.
3. Discuss the significance of graph theory in the field of artificial intelligence and machine learning.
4. Explain the importance of multiple edges and loops in network flow problems.
5. How are different types of graphs used in data structure and algorithms?

UNIT 15
ISOMORPHISM AND SUBGRAPHS

NOTE

UNIT STRUCTURE

15.1 Introduction
15.2 Objective
15.3 Isomorphism
- **15.3.1 Degree of Vertex**
- **15.3.2 Definition of Graph Isomorphism**
- **15.3.3 Some Examples of Isomorphic Graph**
- **15.3.4 Applications of Isomorphic Graph**

15.4 Subgraph
- **15.4.1 Definition of Subgraph**

15.5 Different Types of Subgraphs
15.6 Characteristics of Subgraph
15.7 Applications of Subgraph
15.8 Summary
15.9 Glossary
15.10 Answer to check your progress
15.11 Suggested readings – OER
15.12 Reference
15.13 Exercise

15.1 INTRODUCTION

In the previous unit, you explored fundamental concepts of graph theory, including different types of graphs and their properties. You learned about simple graphs, multigraphs, pseudographs, loops, and multiple edges. Additionally, you studied applications of graphs in real-world problems, preparing you for more advanced topics in graph theory.

Building on this knowledge, we now introduce graph isomorphism and subgraphs. You will learn how to determine whether two graphs are isomorphic, explore their properties, and understand their applications. Additionally, we will study subgraphs, their characteristics, and their significance in graph theory.

This unit connects abstract mathematical concepts with practical applications in computer science, network analysis, and data organization. Understanding graph isomorphism helps in pattern recognition, while subgraphs play a crucial role in social network analysis, circuit design, and database optimization.

NOTE

15.2 OBJECTIVE

After studying this unit, you will be able to:

- Define graph isomorphism and identify isomorphic graphs.
- Analyse the degree of a vertex and its significance in graphs.
- Differentiate between various types of subgraphs and their characteristics.
- Explore real-world applications of isomorphic graphs and subgraphs.
- Apply graph theory concepts to solve computational and network-related problems.

15.3 ISOMORPHISM

15.3.1 Degree of Vertex

The degree of a vertex of an undirected graph is the number of edge incident with it, except that loop at a vertex contributes twice to the degree of the vertex. The degree of the vertex v in a graph G may be denoted by $deg_G(v)$.

The degree of vertices in the graph G and H are given below:

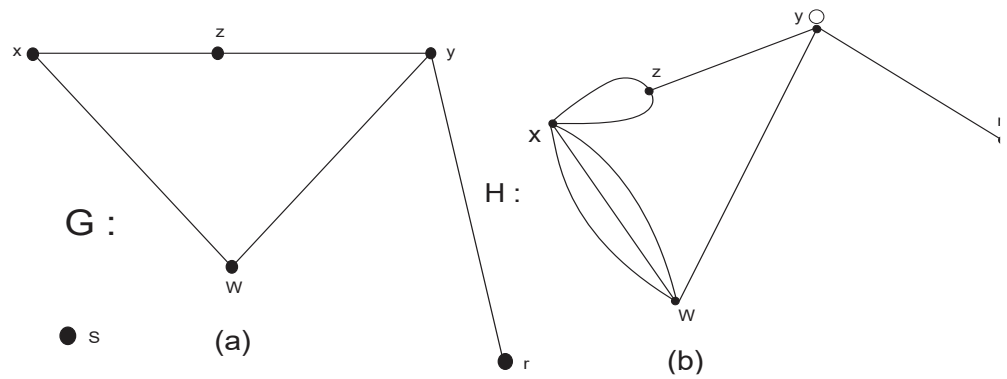


Figure No.15.1: (a) Simple graph and, (b) Multiple graph

In G as shown in Fig 15.1 (a)

$$deg_G(x) = 2 = deg_G(z) = deg_G(w), deg_G(y) = 3, deg_G(r) = 1 \text{ and } deg_G(s) = 0 .$$

And in H as shown in Fig 15.1 (b)

$$deg_H(x) = 5, \quad deg_H(z) = 3 ,$$
$$deg_H(y) = 5, deg_H(w) = 4, \text{and } deg_H(r) = 1 .$$

A Vertex of degree 0 is called isolated vertex. A vertex is pendant if and only if it has a degree 1. Vertex s in the graph G is isolated and vertex r is pendant. A vertex of a graph is called odd vertex or even vertex depending on whether its degree is odd or even.

NOTE

15.3.2 Definition of Graph Isomorphism

Two graphs are **isomorphic** if there exists a one-to-one correspondence between their vertices.

Following Steps to Verify Graph Isomorphism

1. **Check the number of vertices**: Both graphs must have the same number of vertices.
2. **Check the number of edges**: Both graphs must have the same number of edges.
3. **Check the degree of vertices**: The degree (number of edges connected) of each vertex in one graph must match the degree of the corresponding vertex in the other graph.
4. **Check adjacency**: Ensure the edges in one graph correspond to the edges in the other graph.

If any of these quantities differ in two graphs, they cannot be isomorphic.

15.3.3 Some Examples of Isomorphic Graph

Example1. Isomorphic graph.

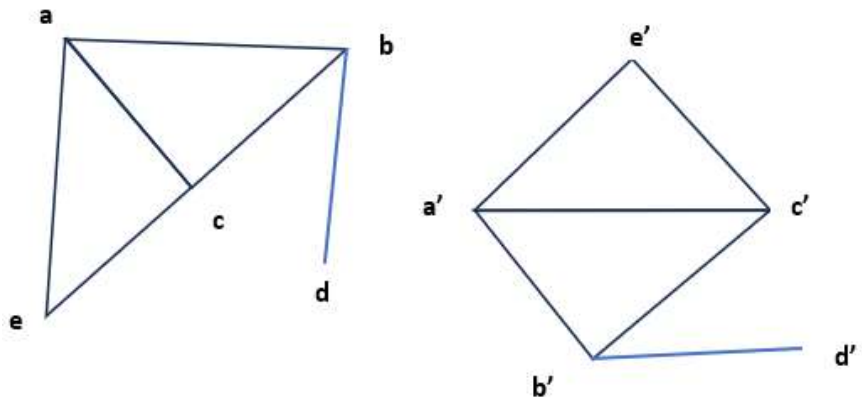


Figure No.15.2(a): Both graphs are Isomorphic graph

Example 2. Not an isomorphic graph.

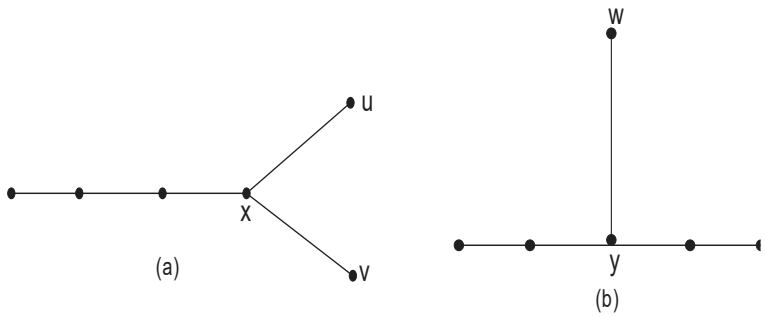


Figure No.15.2(b): Both graphs are not Isomorphic graph

NOTE

Example 3 Show that the given pair of graphs are isomorphic.

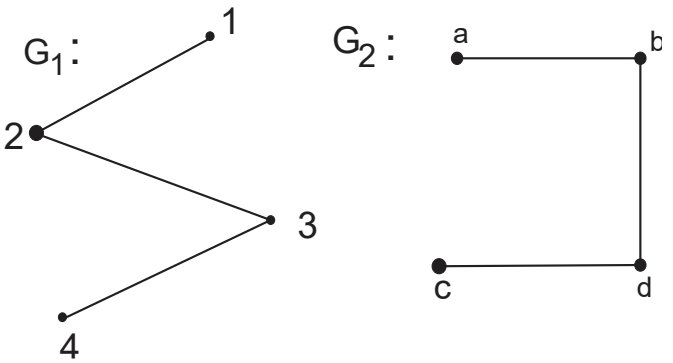


Figure No.15.3: Both graphs are Isomorphic graph

Solution. Here, $V(G_1) = \{1,2,3,4\},\ V\ (G_2) = \{a, b, c, d\}$.

$E\ (G_1) = \{\{1,2\},\{2,3\},\{3,4\}\}$ and $E\ (G_2) = \{\{a, b\},\{b, d\},\{d, c\}\}$.

$Hence\ |V(G_1)| = |V(G_2|\ and\ E(G_1)| = |E(G_2)|$.

The vertices of degree 1 in G_1 are $\{1,4\}$ and in G_2 are $\{a, c\}$.

The vertices of degree 2 in G_1 are $\{2,3\}$ and in G_2 are $\{b, d\}$.

Define a function $: V\ (G_1) \rightarrow V(G_2)\ as\ f(1) = a, f(2) = b,\ f(3) = d\ and\ f(4) = c$.

$f\ is$ clearly one – one and onto.

$Further,$

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E\ (G_2)$

$\{2,3\} \in E(G_1\ )$ and $\{f(2), f(3)\} = \{b, d\} \in E\ (G_2)$

$\{3,4\} \in E(G_1\ )$ and $\{f(3), f(4)\} = \{d, c\} \in E\ (G_2)$

Hence f preserves adjacency of the vertices.

Therefore, $G_1\ and\ G_2\ are\ isomorphic.$

Example 4. Show that the given pair of graphs of isomorphic.

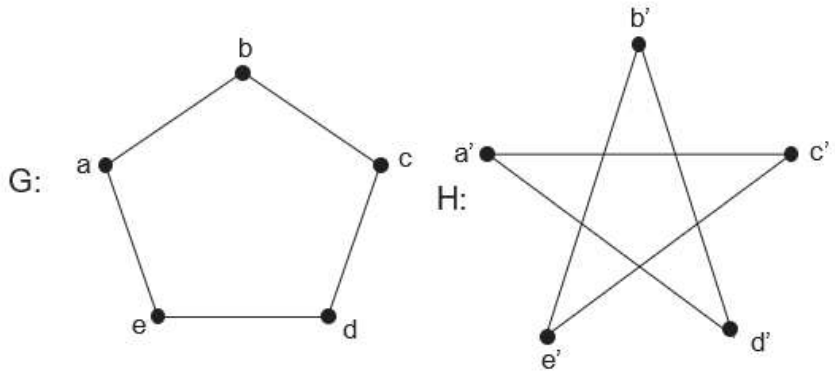


Figure No.15.4: Both graphs are Isomorphic graph

NOTE

Solution. Here $V(G) = \{a, b, c, d, e\}$, $V(H) = \{a', b', c', d', e'\}$, so $|V(G)| = |V(H)|$

Also, $|E(G)| = |E(H)|$

The vertices of degree 2 in G are $\{a, b, c, d, e\}$ and the vertices of degree 2 in H are $\{a', b', c', d', e'\}$.

Define a function $f : V(G) \to V(H)$ as $f(a) = a', f(b) = c', f(c) = e' f(d) = b'$ and $f(e) = d'$

f is clearly one – one and onto.

Further,

$\{a, b\} \in E(G)$ and $\{f(a), f(b)\} = \{a', c'\} \in E(H)$

$\{a, e\} \in E(G)$ and $\{f(a), f(e)\} = \{a', d'\} \in E(H)$

$\{b, c\} \in E(G)$ and $\{f(b), f(c)\} = \{c', e'\} \in E(H)$

$\{c, d\} \in E(G)$ and $\{f(c), f(d)\} = \{e', b'\} \in E(H)$

$\{d, e\} \in E(G)$ and $\{f(d), f(e)\} = \{b', d'\} \in E(H)$

Hence f preserves adjacency of the vertices.

Therefore, G and H are isomorphic.

Example 5. Determine whether the following graphs are isomorphic.

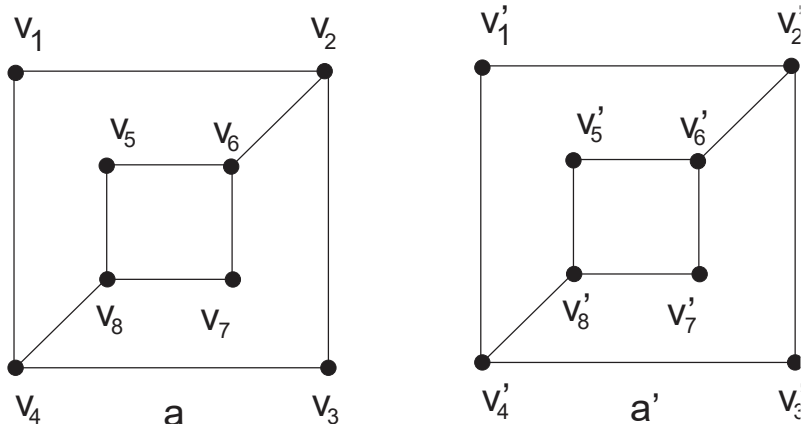


Figure No.15.5: Both graphs are not Isomorphic graph

Solution. The graphs G and G' both have 8 vertices and 10 edges.

They both have 4 vertices each of degree 3 and 4 vertices each of degree 2.

Now consider degree $(v_1) = 2$ in G. Then v_1 must correspond to either v_2', v_3', v_4', v_5'since these are vertices of degree 2 in G'.

However, each of these vertices in G' is adjacent to another vertex of degree 2 in G' viz.v'_2 is adjacent to v'_3 , v'_6 is adjacent to v'_7 but v_1 is adjoint to v_2 and v_4 in G which are of degree 3. Thus, the preservation of adjacency of the vertices is not maintained.

$\therefore$ G and G' are not isomorphic graphs.

NOTE

Example 6. Examine G and H for isomorphism.

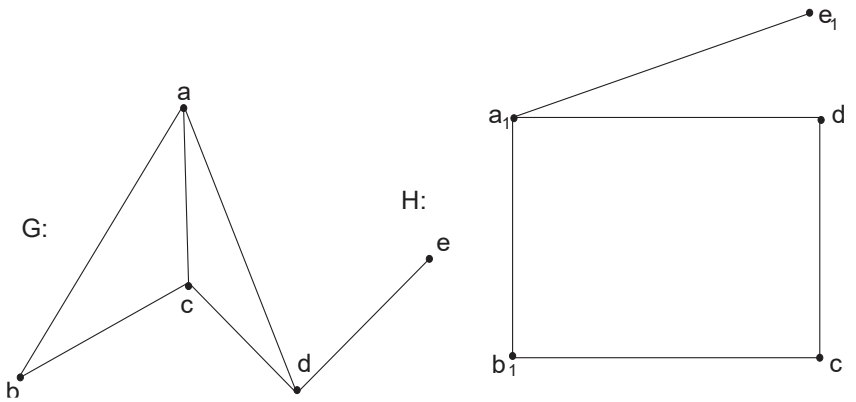


Figure No.15.6: Both graphs are not Isomorphic graph

Solution. The graph G has 5 vertices and 6 edges. The graph H has 5 vertices and 5 edges. So $|E(G)| \neq |E(H)|$

i.e., The number of edges in G is not equal the edges in H.

$\therefore$ G and H are not isomorphic.

15.3.4 Applications of Isomorphic Graph

1. **Network Analysis**: Comparing structures of communication or transportation networks.
2. **Chemistry**: Identifying molecules with identical bonding structures.
3. **Pattern Recognition**: Matching patterns in machine learning and computer vision.

15.4 SUBGRAPH

15.4.1. Definition of Subgraph

A **subgraph** is a graph formed by selecting a subset of the vertices and edges of a larger graph, called the parent graph. A subgraph must maintain the connectivity of the chosen vertices and edges from the parent graph. That is some graphs are concerned with only a part of entire graph. Such a graph is called a sub graph of the original graph.

Consider a graph $G = (V, E)$. A graph $H = (V', E')$ is called a sub graph of G if the vertices and edges of H are contained in the vertices and edges of G

that is if $V' \subseteq V$ *and* $E' \subseteq E$.

So, if H is a sub graph of G then

(i) All the vertices of H are in G.

(ii) All the edges of H are in G and

(iii) Each edge of H has the same end points in H as in G.

NOTE

Any sub graph of a graph G can be obtained by removing certain vertices and edges from G, it is understood that the removal of an edge leaves its points in place, whereas the removal of a vertex necessitates the removal of any edges with that vertex as an end point.

15.5 DIFFERENT TYPES OF SUBGRAPHS

(i) If V' $\subseteq V$ *and* $E' \subseteq E$. then H is called a **proper subgraph** of G.

(ii) A sub graph H of G, is called a **spanning sub graph** of G if and only if

$V(H) = V(G)$. i.e., H contains all vertices G but not necessary all edges of G.

(iii) A sub graph $H(V', E')$ of $G(V, E)$ is called the **induced subgraph** of G if $V' \subseteq V$ and its edge set E' contains all those edges in G that joins the vertices of set V' in G.

(iv) If a subset U of V and all the edges incident on the elements of U are deleted from a graph $G(V, E)$ then the resulting subgraph is called a **vertex deleted subgraph** of $G(V, E)$.

(v) If a subset S of E from a graph $G(V, E)$ is deleted, then the resulting subgraph is called an **edge deleted subgraph** of G.

Example 1. Consider the graph shown in Fig No: 15.7. Find the different subgraphs of this graph

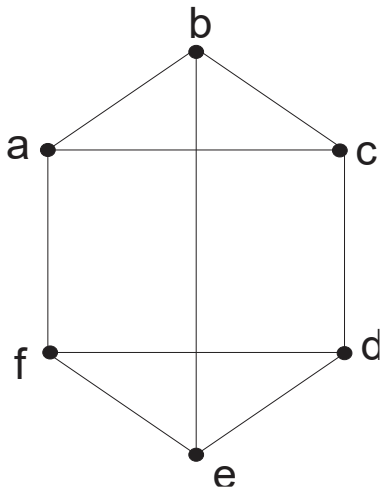


Figure No.15.7: Graph

Solution The following are subgraph of the above graph.

Note: That the subgraph does not have to be drawn the same way they appear in the presentation of G.

NOTE

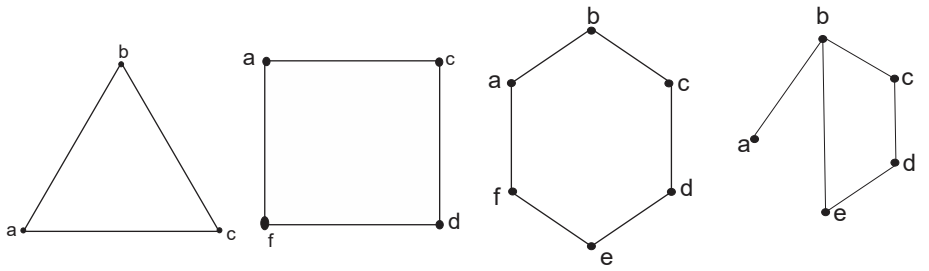


Figure No.15.8: Subgraph

Example 2. Construct edge deleted subgraphs and vertex deleted subgraphs of a graph G shown below.

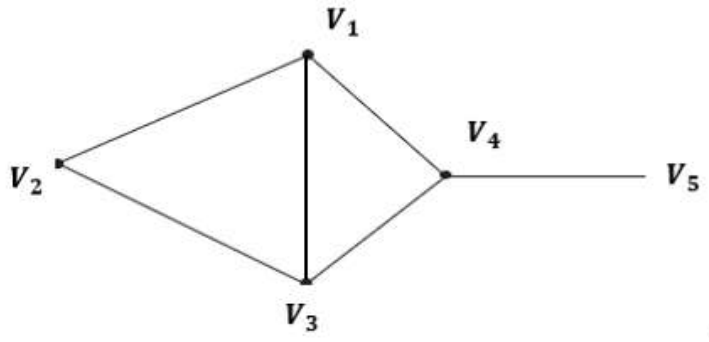


Figure No.15.9: Graph G

Solution. The graph H_1 *and* H_2 in fig No.15.10: are edge deleted subgraph of G (Fig No.15.9)

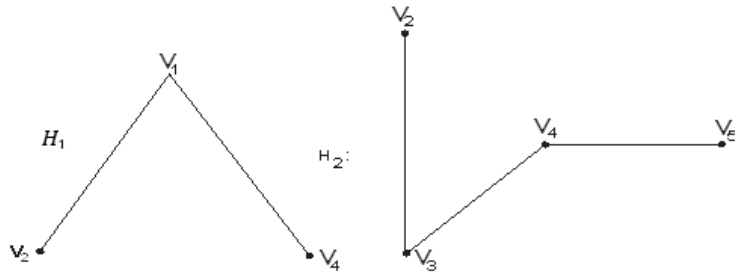


Figure No.15.10: Edge deleted subgraph of G graph H_1 and H_2

The graphs H_3 *and* H_4 in fig No.15.11 are vertex deleted subgraphs of G which are also edge deleted subgraphs of G Fig No.15.9.

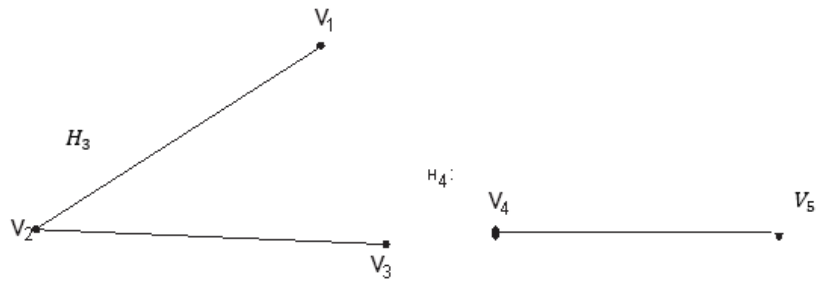


Figure No.15.11: Edge deleted subgraph of G graph H_3 and H_4

NOTE

Example 3. In Fig No.15.12: G_1 is a subgraph of G but not an induced subgraph where as G_2 is an induced subgraph and induced by $\{a, b, e, d\}$.

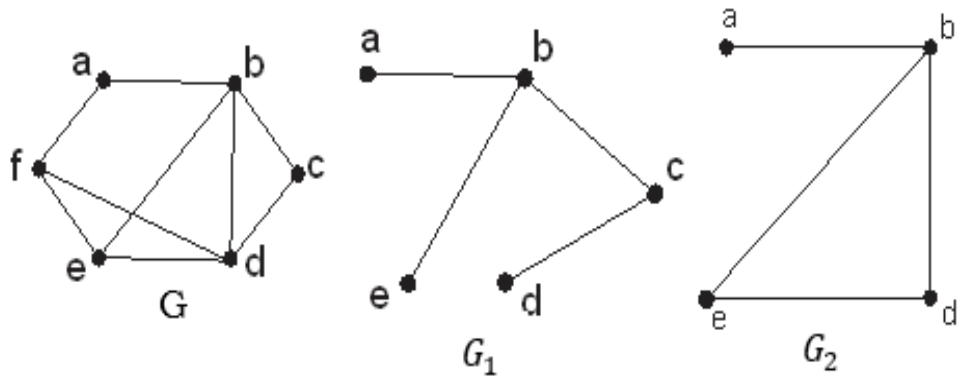


Figure No.15.12: Only G_2 is an induced subgraph of G

Example 4. In Fig 15.13: G_2 is a spanning subgraph of G but G_1 is not, whereas G_1 is an induced subgraph induced by $\{a, b, c, e\}$ but G_2 is not.

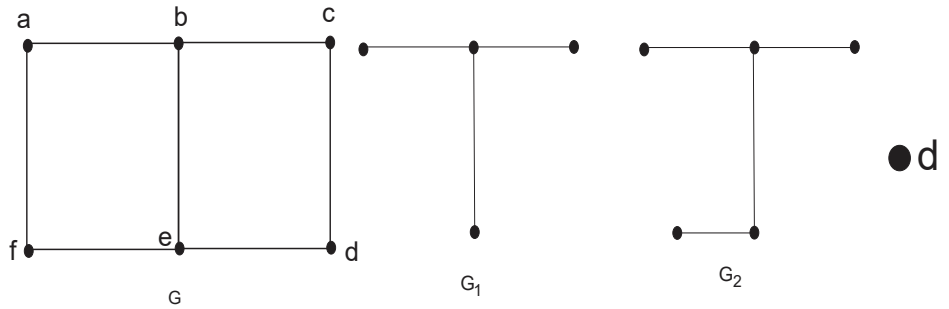


Figure No.15.13: Only G_2 is an induced subgraph of G

15.6 CHARACTERISTIC OF SUBGRAPH

1. A subgraph must only include edges that exist in the parent graph.
2. Vertices in the subgraph can be a subset of the parent graph, but they should align with the included edges.

15.7 APPLICATIONS OF SUBGRAPH

1. **Network Analysis**: Studying specific parts of a network (e.g., sub-networks within a communication system). Extracting specific routes in transportation networks.
2. **Clustering**: Identifying tightly connected components in a larger graph.
3. **Pathfinding**: Extracting sections of a graph to analyze paths or cycles within that part.

NOTE



Activity 15.1

Identifying Isomorphic Graphs and Subgraphs

Objective:

To help students understand graph isomorphism and subgraphs by analyzing and comparing given graphs.

Materials Required:

- Graph paper or digital graphing tools
- Pencil and ruler (if drawing manually)
- A set of pre-drawn graphs (provided by the instructor)

Instructions:

1. **Identify Isomorphic Graphs:**
    - Observe the given set of graphs and determine which pairs are isomorphic.
    - Justify your answer by comparing the number of vertices, edges, and degree sequences.
    - If two graphs are isomorphic, find a vertex correspondence between them.
2. **Exploring Subgraphs:**
    - Identify subgraphs within the given graphs.
    - Highlight induced subgraphs and spanning subgraphs.
    - Explain how the identified subgraphs are related to the original graphs.
3. **Real-World Application:**
    - Think of a real-world situation where isomorphic graphs or subgraphs are useful (e.g., social networks, molecular structures, road networks).
    - Discuss how understanding these concepts can help in problem-solving.

Expected Outcome:

By the end of the activity, students should be able to:

- Correctly identify isomorphic graphs and explain their properties.

NOTE

- Recognize different types of subgraphs and understand their applications.
- Relate isomorphism and subgraphs to practical problems in technology and science.

☑ Check Your Progress 15.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is the degree of a vertex in a graph?

.....

.....

.....

2. If two graphs have the same number of vertices and edges, they must be isomorphic. (True or False).

3. Which of the following is NOT a method to check graph isomorphism?

A) Checking degree sequence

B) Checking adjacency relationships

C) Counting number of edges only

D) Checking connectivity

4. A subgraph that includes all the vertices of the original graph but only a subset of edges is called a _____ subgraph.

.....

15.8 SUMMARY

This unit introduced the important concepts of graph isomorphism and subgraphs in graph theory. It explained how graphs can represent relationships between objects and how different graphs may actually describe the same structure even when drawn differently. The unit began with the concept of the degree of a vertex, which refers to the number of edges connected to a vertex and helps in analyzing graph structures. It then discussed graph isomorphism, where two graphs are considered isomorphic if there exists a one-to-one correspondence between their vertices and edges while preserving adjacency relationships. Several examples of isomorphic graphs were studied to help learners identify structural similarities between graphs despite

NOTE

differences in appearance. The applications of isomorphic graphs were also explored, particularly in computer science, chemistry, communication networks, and pattern recognition, where identifying equivalent structures is highly useful.

The unit further explained the concept of a subgraph, which is formed by selecting a subset of vertices and edges from a larger graph. Different types of subgraphs, including spanning subgraphs and induced subgraphs, were introduced along with their unique properties. Learners also studied the characteristics of subgraphs, such as connectivity, edge relations, and vertex inclusion, which help in understanding complex graph structures more effectively. The applications of subgraphs were highlighted in network analysis, transportation systems, circuit design, and social network studies, where smaller graph structures are used to simplify and solve larger problems.

By the end of the unit, students should be able to identify isomorphic graphs, determine vertex degrees, construct and analyze subgraphs, and understand their practical significance. These concepts form the foundation for advanced topics in graph theory, algorithms, and network analysis. Students are encouraged to further explore graph-based applications in data science, artificial intelligence, and computer networks to strengthen their understanding and analytical skills.

15.9 GLOSSARY

- **Graph Isomorphism:** A property where two graphs have the same structure but different vertex labeling.
- **Degree of a Vertex:** The number of edges connected to a vertex.
- **Adjacency Matrix:** A square matrix used to represent a graph, showing which vertices are connected.
- **Subgraph**: A graph formed by a subset of vertices and edges from a larger graph.
- **Induced Subgraph:** A subgraph formed by selecting a subset of vertices and keeping all edges between them.
- **Spanning Subgraph:** A subgraph containing all the vertices of the original graph but only some edges.
- **Connected Graph:** A graph where there is a path between every pair of vertices.
- **Cycle in Graph:** A closed path where the start and end vertex are the same.

NOTE

- **Bipartite Graph:** A graph whose vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent.
- **Complement of a Graph:** A graph obtained by connecting all pairs of vertices that were not adjacent in the original graph.

15.10 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 15.1

1. **Answer:** The degree of a vertex in a graph is the number of edges that are connected to it. In an undirected graph, it counts all the edges incident to the vertex. In a directed graph, it is divided into in-degree (edges coming in) and out-degree (edges going out).

2. **Answer:** False

   Explanation: Two graphs can have the same number of vertices and edges but different structures or connections. Therefore, they may **not** be isomorphic.

3. **Answer:** C) Counting number of edges only

   Explanation: Counting edges alone is **not sufficient** to determine isomorphism. Isomorphism requires checking structure, degree sequences, adjacency, and connectivity.

4. **Answer:** Spanning Subgraph

   Explanation: A spanning subgraph includes all vertices of the original graph and some or all its edges.

15.11 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1255	

NOTE

15.12 REFERENCE

- https://elearn.daffodilvarsity.edu.bd/pluginfile.php/783889/mod_label/intro/s9.3_graph_isomorphism.pdf
- https://ocw.mit.edu/courses/6-042j-mathematics-for-computer-science-spring-2015/mit6_042js15_session19.pdf
- https://w3.cs.jmu.edu/spragunr/CS228_S14/lectures/isomorphism/isomorphism.pdf
- https://www.geeksforgeeks.org/adjacency-matrix/
- https://ciscwww.cs.queensu.ca/courses/cisc203/Record/20200321.3%20Subgraphs.pdf
- https://testbook.com/maths/types-of-graphs-in-graph-theory
- https://math.stackexchange.com/questions/1013143/difference-between-a-sub-graph-and-induced-sub-graph
- https://www.quora.com/What-is-a-spanning-subgraph
- https://en.wikipedia.org/wiki/Connectivity_(graph_theory)
- https://en.wikipedia.org/wiki/Cycle_(graph_theory)
- https://www.geeksforgeeks.org/what-is-bipartite-graph/
- https://www.geeksforgeeks.org/complement-of-graph/

15.13 EXERCISE

Part 1

A. Multiple choice question

1. Two graphs are said to be isomorphic if they have:

a) The same number of edges and vertices

b) The same number of vertices only

c) The same adjacency relationship

d) The same degree sequence

2. The degree of a vertex in a simple graph is:

a) The number of edges it is connected to

b) The number of loops it has

c) The total number of edges in the graph

d) Always even

3. Which of the following is an example of a subgraph?

a) A graph with fewer vertices and edges from the original graph

b) A completely different graph

NOTE

c) A graph with more vertices than the original graph

d) A graph with the same number of vertices but additional edges

4. The process of checking if two graphs are isomorphic involves:

a) Checking only the number of vertices

b) Comparing adjacency matrices

c) Removing loops from the graph

d) Adding new edges to make them identical

5. Which type of subgraph contains all the vertices of the original graph?

a) Induced subgraph

b) Complement of a graph

c) Spanning subgraph

d) Bipartite subgraph

Answers with Explanations:

1. c) The same adjacency relationship

Explanation: Isomorphic graphs must preserve adjacency relations, not just the number of vertices and edges.

2. a) The number of edges it is connected to

Explanation: The degree of a vertex is defined as the count of edges attached to it.

3. a) A graph with fewer vertices and edges from the original graph

Explanation: A subgraph is formed by selecting some vertices and edges from the original graph.

4. b) Comparing adjacency matrices

Explanation: If two graphs have the same adjacency matrix under a relabeling of vertices, they are isomorphic.

5. c) Spanning subgraph

Explanation: A spanning subgraph includes all vertices but only some of the edges of the original graph.

B. Short answer type question

1. Define isomorphic graphs.
2. What is a spanning subgraph?

NOTE

3. How do you check if two graphs are isomorphic?
4. Give an example of an application of graph isomorphism.
5. What is an induced subgraph?

Part 2

C. Long answer type question (200 words)

1. Explain the characteristics of isomorphic graphs and provide an example.
2. What are the different types of subgraphs? Explain with diagrams.
3. Describe the process of determining whether two graphs are isomorphic.
4. Explain the concept of degree sequence and its role in isomorphism.
5. Discuss the importance of graph isomorphism in real-world applications.

Part 3

D. Long answer type question (300 words)

1. Prove that two graphs with the same degree sequence are not necessarily isomorphic. Give an example.
2. Explain different methods used to determine graph isomorphism in detail.
3. Discuss in depth the applications of subgraphs in real-world problems.
4. Compare and contrast spanning subgraphs and induced subgraphs with examples.
5. Provide a detailed analysis of the role of graph theory in network optimization, referencing subgraphs and isomorphism.

NOTE

UNIT 16
GRAPH OPERATIONS AND WALKS

UNIT STRUCTURE

16.1 Introduction
16.2 Objective
16.3 Operations on Graphs
- **16.3.1 Definition of Operation on Graph**
- **16.3.2 Applications of Graph Operation**

16.4 Walk and Their Classification
- **16.4.1 Definition of Walk in Graph**
- **16.4.2 Special Types of Walks**

16.5 Classification of Walk
- **16.5.1 Some More Examples on Walk**

16.6 Applications of Walk
16.7 Summary
16.8 Glossary
16.9 Answer to check your progress
16.10 Suggested readings – OER
16.11 Reference
16.12 Exercise

16.1 INTRODUCTION

In the previous unit, you explored the fundamental concepts of graphs, including their structure and different types such as simple graphs, multigraphs, and pseudographs. You also studied graph isomorphism, subgraphs, and their applications. These concepts helped in understanding how graphs are used in various domains. Now, we extend this knowledge by exploring operations on graphs and different types of walks.

Building on this knowledge, we now offer an in-depth study of operations on graphs, such as union, intersection, and Cartesian products. Additionally, we will examine different types of walks, including paths, trails, and circuits. Understanding these concepts is essential for analyzing graph traversal techniques and their applications in computer networks, transportation systems, and optimization problems.

This unit bridges abstract algebra with real-life applications by exploring operations on graphs and various types of walks. Graph

NOTE

operations provide essential tools for modifying and analyzing network structures, while walks help in modeling real-world scenarios like GPS navigation, road networks, and social connections. The classification of walks enhances our understanding of connectivity, optimization, and flow problems in applied mathematics and computer science.

16.2 OBJECTIVE

After studying this unit, you will be able to:

- Define the operations on graphs and their significance in graph theory.
- Define and explain different graph operations such as union, intersection, complement, and Cartesian product.
- Describe the concept of walks in graphs and classify them into different types such as trails, paths, and circuits.
- Identify and analyze special types of walks, including Eulerian and Hamiltonian paths and circuits.
- Explore real-world applications of graph operations and walks in various fields such as network analysis, transportation, and computer science.

16.3. OPERATIONS ON GRAPHS

16.3.1 Definition of Operation on Graph

Graph operations allow us to construct new graphs from existing ones, providing useful tools for solving problems in discrete mathematics, computer science, and network theory.

The major four types of graph operations include:

1. **Union of Graphs** ($G \cup H$): The vertex and edge sets of two graphs are combined to form a new graph. Used in network merging and social media analysis.
2. **Intersection of Graphs** ($G \cap H$): The new graph contains only the vertices and edges that are common to both graphs. Applied in data integration and common pattern detection.
3. **Graph Complement** (G'): A complement graph contains the same vertices as the original graph, but edges are present only where they were absent in the original. Used in network security and error detection.
4. **Cartesian Product of Graphs** ($G \times H$): A graph is created by pairing vertices from two graphs, useful in parallel computing and optimization.

NOTE

1. Union of Graph

Definition: The union of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is a new graph

$G \cup H = (V_1 \cup V_2, E_1 \cup E_2)$. It contains all vertices and edges from both graphs.

Example 1:

Let G have vertices $V(G) = \{A, B, C\}$ and edges $E(G) = \{AB, BC\}$

Let H have vertices $V(H) = \{C, D, E\}$ and edges $E(H) = \{CD, DE\}$.

Then, $G \cup H$ will have:

- **Vertices**: $\{A, B, C, D, E\}$
- **Edges**: $\{AB, BC, CD, DE\}$

Example 2:

As in Fig No.16.1: Combines two graphs G_1 and G_2 by taking their vertex and edge sets together.

If $G_1 = (V_1, E_2)$ and $G_2 = (V_2, E_2)$, $G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

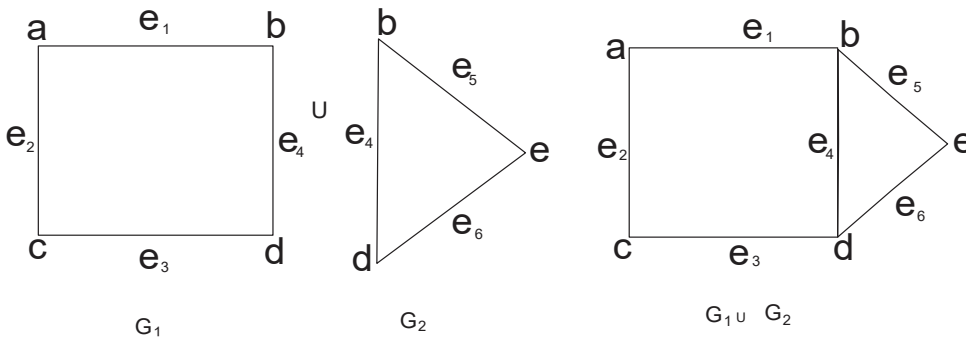


Figure No.16.1: Union of two graphs G_1 and G_2 is $G_1 \cup G_2$

Example 3:

- $G_1: V_1 = \{1,2\}, E_1 = \{(1,2)\}$
- $G_2: V_2 = \{3,4\}, E_2 = \{(3,4)\}$
- $G = G_1 \cup G_2: V = \{1,2,3,4\}, E = \{(1,2), (3,4)\}$.

Application: Social networks—merging two friend groups in a network.

2. Intersection of Graph

Definition: The intersection of two graphs G and H contains only the vertices and edges that are common to both.

Example 1: Using the graphs above:

- $G: \{A, B, C\}, \{AB, BC\}$

NOTE

- H: $\{C, D, E\}, \{CD, DE\}$
- $G \cap H$ contains only common elements $\rightarrow V(G \cap H) = \{C\}, E(G \cap H) = \emptyset$

(No common edges).

Example 2: Give two graphs G_1 and G_2 *with* at least one vertex in common than their intersection will be a graph such that.

Fig No.16.2: If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ *then* $G = G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$

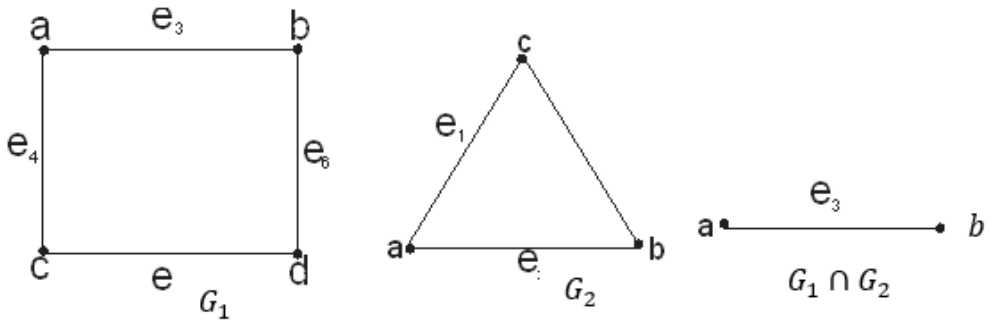


Figure No.16.2: Union of two graphs G_1 and G_2 is $G_1 \cap G_2$

Example 3:

- G_1: $V_1 = \{1,2,3\}$, $E_1 = \{(1,2), (2,3)\}$
- G_2: $V_2 = \{2,3,4\}$, $E_2 = \{(2,3), (3,4)\}$
- $G = G_1 \cap G_2$: $V = \{2,3\}, E = \{(2,3)\}$.

Application: Common friend detection in a social network.

3. Cartesian Product of Graph

Definition: In the Cartesian product $G \times H$, each vertex of G is paired with each vertex of H, forming edges between corresponding pairs.

Example 1:

Let G be a path graph with vertices {A, B} and H be a cycle graph with vertices $\{1, 2\}$.

- G: A—B
- H: 1—2—1
- G × H:
  - (A,1) — (B,1)
  - (A,2) — (B,2)
  - (A,1) — (A,2)
  - (B,1) — (B,2)

NOTE

Example 2:

In Fig No. 16.3: Creates a graph by combining vertices and edges of two graphs. To define the product $G_1 \times G_2$ of two graphs consider any two points

$u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$

whenever $[u_1 = v_1$ and u_2 *and* u_2 *adj* $v_2]$ or $[u_2 = v_1$ *and* u_1 *adj* $v_1]$.

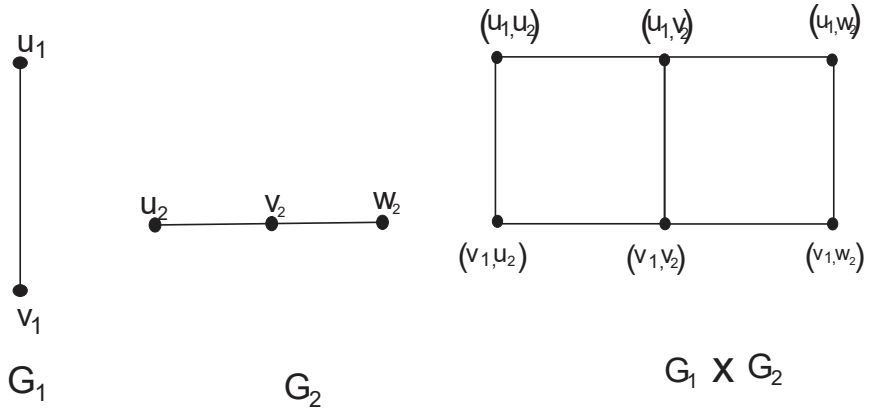


Figure No.16.3: product of two graphs G_1 and G_2 is $G_1 \times G_2$

Example 3:

- $G_1: V_1 = \{1,2\},\ E_1 = \{(1,2)\}$
- $G_2: V_2 = \{a, b\},\ E_2 = \{(a, b)\}$
- $G = G1 \times G2:\ V = \{(1, a), (1, b), (2, a), (2, b)\}$

Application: Parallel processing systems, where multiple processes run on different servers.

4. Complement of a Graph

Definition: The complement of a graph G' has the same vertices as G, but an edge exists in G'if and only if it does not exist in G.

Example 1: Consider Graph G:

- $V(G) = \{A, B, C, D\}$
- $E(G) = \{AB, BC\}$

The complement graph G'

- $V(G') = \{A, B, C, D\}$
- $E(G')$ will have all edges missing in $G \rightarrow \{AC, AD, BD, CD\}$.

Example 2: Fig No.16.4: Creates a graph where edges exist only between vertices that were not connected in the original graph.

If $G = (V, E)$ the complement G' of G is defined as a simple graph with the same vertex set as G and where two vertices u and v are adjacent

NOTE

only when they are not adjacent in G. Complement G': V remains the same, and E' contains all edges not in E.

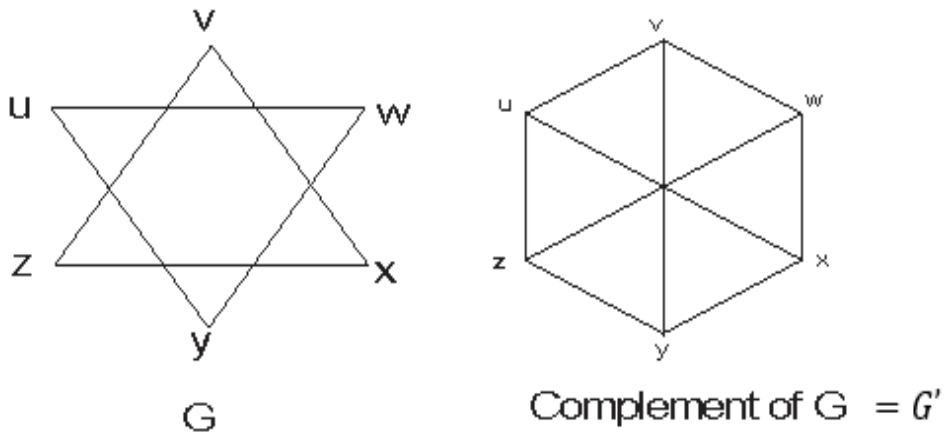


Figure No.16.4: Complement G' of a graph G

Example 1:

- G: $V = \{1,2,3\}, E = \{(1,2)\}$
- G': $V = \{1,2,3\}, E' = \{(1,3),(2,3)\}$.

Application: Network failure analysis—identifying missing links in a communication network.

16.3.2 Applications of Graph Operation:

1. **Networking:** Building efficient communication protocols.
2. **Data Science:** Representing relationships between entities in knowledge graphs.
3. **Scheduling:** Optimizing task dependencies.
4. **Biology:** Modeling protein interaction networks.
5. **Transportation:** Analyzing traffic flow or public transport routes.

16.4 WALK AND THEIR CLASSIFICATION

16.4.1 Definition of Walk in A Graph

A Walk in a graph G is a finite alternating sequence of vertices and edges of the graph

$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 \dots\dots\dots\dots\dots\dots\dots\dots\dots - e_n - v_n$$

1. Each edge in the sequence is connected to the preceding and following vertices.
2. Repetition of vertices and edges is allowed.

   The end vertices v_0 *and* v_n are the end or terminal vertices of the walk. The vertices $v_1, v_2, \dots\dots, v_{n-1}$ are called its internal vertices. The

NOTE

integer n, the number of edges in the walk is called the length of the walk. A walk. is called open when the terminal vertices are distinct. For the same end terminal vertices, it is termed as closed. note that a walk may repeat both vertices and edges.

16.4.2 Special Types of Walks

- A walk is called a **trail** if all its edges are distinct. A Trail is open or closed depends on whether its end vertices are distinct or not.
- A closed trail is called a **circuit.** A walk is called a path if all its vertices' edges are distinct A path in which only repeated vertex is the first vertex is called a **cycle** to describe such a closed path.

Walks are classified based on specific rules applied to vertices or edges.

16.5 CLASSIFICATION OF WALK

A walk in a graph is a sequence of vertices and edges where each edge connects two consecutive vertices in the sequence. Walks can be classified into different types based on repetition of vertices and edges.

Types of Walks:

1. **Open Walk:** A walk where the starting and ending vertices are different.
2. **Closed Walk:** A walk where the starting and ending vertices are the same.
3. **Trail:** A walk in which no edge is repeated.
4. **Path:** A walk in which no vertex is repeated (except in a closed path where the starting and ending vertices are the same).
5. **Circuit**: A closed trail (i.e., a closed walk with no repeated edges).
6. **Eulerian Path**: A path that visits every edge of the graph exactly once.
7. **Eulerian Circuit:** A closed Eulerian path (starts and ends at the same vertex).
8. **Hamiltonian Path:** A path that visits every vertex exactly once.
9. **Hamiltonian Circuit:** A closed Hamiltonian path (starts and ends at the same vertex).

	Repeated Edge	Repeated Vertex	Starts and Ends At same points?
Walk (open)	allowed	allowed	No
Walk (closed)	allowed	allowed	Yes
Trail	no	allowed	no
Circuit	no	allowed	yes
Path	no	no	no
Cycle	no	first and last only	yes

NOTE

Example 1: Fig No.16.5:

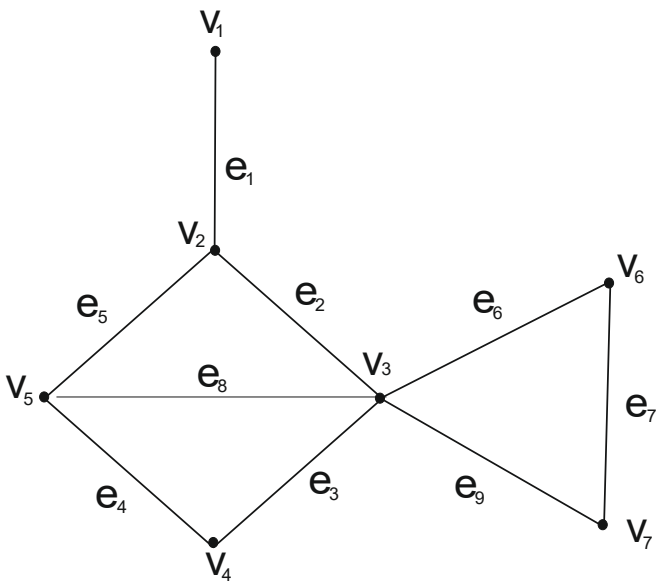


Figure No.16.5: Walk in a Graph

(i) The sequence

$v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$

Is a **walk** of length 8. it contains repeated v_2, v_3 and v_5 and repeated edge e_5.

(ii) The sequence

$v_1 - e_1 - v_2 - e_5 - v_5 - e_3 - v_3 - e_3 - v_4 - e_4 - v_5$

is a **path**. it does not contain repeated vertex v_5 but does not contain repeated edge.

(iii) The sequence

$v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4$

is a **path**. it does not contain repeated vertex and edge.

(iv) The sequence

$v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$

is a **cycle**. it does not contain repeated vertex and repeated edge except the first and last vertex.

16.5.1 Some More Examples On Walk:

Example 1:

Graph:

Vertices: A, B, C, D

Edges: $A - B, B - C, C - D, D - A, B - D$

NOTE

Walks:

1. **Open Walk**: $A \to B \to C \to D$
   - The walk starts at A and ends at D, which are different.
2. **Closed Walk**: $A \to B \to C \to D \to A$
   - The walk starts and ends at A.
3. **Trail**: $A \to B \to C \to D$
   - No edge is repeated in this walk.
4. **Path**: $A \to B \to D \to C$
   - No vertex is repeated in this walk.

Example 2: Graph:

Vertices: P, Q, R, S

Edges: $P - Q, Q - R, R - S, S - P, P - R$

Walks:

1. **Open Walk**: $P \to Q \to R$
   - Starts at P, ends at R.
2. **Closed Walk**: $P \to Q \to R \to P$
   - Starts and ends at P.
3. **Trail**: $P \to Q \to R \to S$
   - No edge is repeated.
4. **Cycle**: $P \to Q \to R \to S \to P$
   - A closed walk with no repeated vertices, except the starting and ending vertex P.

16.6 APPLICATIONS OF WALK

Walks in graphs play a crucial role in discrete mathematics, particularly in understanding connectivity, optimization, and traversal techniques. Some key applications include:

1. **Graph Traversal Algorithms:** Walks form the basis for algorithms like Depth-First Search (DFS) and Breadth-First Search (BFS), which are used in pathfinding and network analysis.
2. **Shortest Path Problems:** Walks help in finding the shortest or optimal path between two vertices, such as Dijkstra's Algorithm and Bellman-Ford Algorithm, used in routing and transportation networks.
3. **Network Connectivity:** Walks determine whether all nodes in a graph are reachable, aiding in network design, social network analysis, and circuit design.

NOTE

4. **Eulerian and Hamiltonian Paths:** Applications in scheduling, DNA sequencing, and computer chip design involve finding walks that traverse all edges (Eulerian) or visit all vertices exactly once (Hamiltonian).
5. **Markov Chains:** In probability and statistics, walks model stochastic processes where transitions occur between different states in applications like queueing theory, finance, and machine learning.



Activity 16.1

Exploring Graph Operations and Walks

Objective:

To understand different operations on graphs and classify different types of walks in a given graph.

Materials Required:

- Graph paper or whiteboard
- Markers or pens
- A set of example graphs (can be hand-drawn or provided as printed sheets)

Instructions:

1. **Graph Operations Task:**
   - Given two simple graphs G_1 and G_2, perform the following operations:
     a) Union $(G_1 \cup G_2)$ Combine all vertices and edges.
   - b) Intersection $(G_1 \cap G_2)$ Identify common vertices and edges.
   - c) Cartesian Product $(G_1 \times G_2)$ Construct a new graph by pairing vertices from both graphs.
2. **Walk Classification Task:**
   - Consider a graph with labeled vertices and edges.
   - Identify and classify the following walks:
   - a) A general walk (any sequence of vertices connected by edges)
   - b) A trail (a walk without repeating edges).
   - c) A path (a walk without repeating vertices).
   - d) A circuit (a closed trail).
   - e) A cycle (a closed path).
3. **Discussion Questions:**
   - How do different graph operations affect the connectivity of a network?

NOTE

- What real-life examples can be modeled using different types of walks?

Expected Outcome:

Students will be able to perform basic graph operations and classify walks accurately. This will enhance their understanding of graph structures and their practical applications in network analysis, transportation, and computer science.

☑ Check Your Progress 16.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a walk-in graph theory.

.....

.....

.....

2. Every Eulerian graph is also a Hamiltonian graph. (True or False).

.....

3. Which of the following statements is true about a path in a graph?

1. It allows repeated vertices.
2. It allows repeated edges.
3. It does not allow repeated vertices.
4. It does not allow edges.

4. A Hamiltonian Circuit visits every _____ exactly once before returning to the starting vertex.

16.7 SUMMARY

This unit introduced the fundamental concepts of graph operations and walks, which are essential topics in graph theory and discrete mathematics. Students learned about different operations performed on graphs, including how graphs can be combined, modified, or transformed to create new graph structures. The unit also explained the definition of graph operations and explored their practical applications in computer science, communication networks, transportation systems, and data organization. These concepts help in understanding how complex systems can be modeled and analyzed efficiently using graphs.

NOTE

The unit further focused on walks in graphs and their classifications. Students studied the definition of a walk and learned how vertices and edges are connected through sequences in a graph. Special types of walks such as open walks, closed walks, trails, paths, and circuits were discussed with suitable examples to strengthen conceptual understanding. The classification of walks helped learners differentiate between various graph traversal methods and understand their properties in detail. Additional examples demonstrated how walks are represented and analyzed in different graph structures.

Another important aspect of the unit was the application of walks in real-world situations. Students explored how graph walks are used in route planning, network routing, circuit design, social network analysis, web navigation, and optimization problems. These applications highlighted the practical importance of graph theory in modern technology and problem-solving. By the end of the unit, learners developed the ability to identify graph operations, classify different types of walks, and apply these concepts to mathematical and computational problems.

Students are encouraged to continue exploring advanced graph theory topics such as Euler paths, Hamiltonian cycles, graph algorithms, and network optimization techniques. Further study of these concepts will strengthen analytical thinking and provide valuable skills for fields such as mathematics, computer science, artificial intelligence, and operations research.

16.8 GLOSSARY

- **Graph Operation –** A mathematical technique to combine, modify, or analyze graphs.
- **Union of Graphs –** A new graph formed by taking all vertices and edges from two given graphs.
- **Intersection of Graphs –** A new graph that retains only the common vertices and edges of two graphs.
- **Complement of a Graph –** A graph where missing edges are added, and existing edges are removed.
- **Walk –** A sequence of vertices connected by edges in a graph.
- **Trail –** A walk where no edge is repeated.
- **Path –** A walk where no vertex is repeated.
- **Circuit –** A closed trail, meaning it starts and ends at the same vertex.

NOTE

- **Cycle –** A closed path that does not repeat any vertex except the starting and ending vertex.
- **Eulerian and Hamiltonian Graphs –** Special types of graphs based on path traversal properties.

16.9 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 16.1

1. Define a walk-in graph theory

Answer: A **walk**-in graph theory is a sequence of vertices and edges in a graph such that each edge's endpoints are the preceding and following vertices in the sequence. Vertices and edges may be repeated in a walk.

Example: In a graph with vertices A, B, C, a walk could be A → B → A → C.

2. Every Eulerian graph is also a Hamiltonian graph. (True or False)

Answer: False.

Not every Eulerian graph is Hamiltonian. An Eulerian graph contains an Eulerian circuit (which uses every edge exactly once), while a Hamiltonian graph contains a Hamiltonian circuit (which visits every vertex exactly once). These properties are not dependent on each other.

3. Which of the following statements is true about a path in a graph?

It does not allow repeated vertices.

Answer: A path is a walk in which no vertex is repeated, meaning it goes through each vertex at most once (except in the case of a cycle, where the first and last vertex may be the same).

4. A Hamiltonian Circuit visits every _____ exactly once before returning to the starting vertex.

Answer: A Hamiltonian Circuit visits every vertex exactly once before returning to the starting vertex.

NOTE

16.10 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1256	

16.11 REFERENCE

- https://www.hello-algo.com/en/chapter_graph/graph_operations/
- https://en.wikipedia.org/wiki/Graph_operations
- https://www.geeksforgeeks.org/union-and-intersection-operation-on-graph/
- https://cdnx.uobabylon.edu.iq/undergrad_projs/Qj7hOcjQ0apxmpS7AODIw.pdf
- https://www.geeksforgeeks.org/complement-of-graph/
- https://www.geeksforgeeks.org/walks-trails-paths-cycles-and-circuits-in-graph/
- https://www.geeksforgeeks.org/difference-between-hamiltonian-path-and-eulerian-path/
- https://www.tutorialspoint.com/euler-and-hamiltonian-paths
- https://www.tpointtech.com/eulerian-and-hamiltonian-path
- https://math.libretexts.org/Bookshelves/Combinatorics_and_Discrete_Mathematics/Applied_Discrete_Structures_(Doerr_and_Levasseur)/09%3A_Graph_Theory/9.04%3A_Traversals-_Eulerian_and_Hamiltonian_Graphs

16.12 EXERCISE

Part 1

A. Multiple choice question

1. Which graph operation involves keeping only the common vertices and edges of two graphs?

a) Union

b) Intersection

NOTE

c) Complement

d) Cartesian Product

2. A closed walk in which no edge is repeated is called a :

a) Path

b) Circuit

c) Eulerian Walk

d) Cycle

3. Which of the following is true for an Eulerian Circuit?

a) It visits every edge exactly once and starts and ends at different vertices.

b) It visits every vertex exactly once and starts and ends at the same vertex.

c) It visits every edge exactly once and starts and ends at the same vertex.

d) It visits every vertex exactly once and starts and ends at different vertices.

4. Which of the following graphs allows multiple edges between the same pair of vertices?

a) Simple Graph

b) Multigraph

c) Pseudograph

d) Directed Graph

5. A Hamiltonian Graph must contain:

a) A Eulerian Path

b) A Eulerian Circuit

c) A path that visits each vertex exactly once

d) A cycle that repeats one vertex multiple times

Answers with Explanations

1. b) Intersection

Explanation: The intersection of two graphs retains only their shared vertices and edges.

2. b) Circuit

Explanation: A circuit is a closed trail, meaning it starts and ends at the same vertex.

NOTE

3. c) It visits every edge exactly once and starts and ends at the same vertex.

Explanation: A Eulerian Circuit traverses every edge of the graph exactly once, returning to the starting vertex.

4. b) Multigraph

Explanation: A multigraph permits multiple edges (parallel edges) between two vertices.

5. c) A path that visits each vertex exactly once

Explanation: A Hamiltonian Graph contains a Hamiltonian Path or Circuit that visits each vertex only once.

B. Short answer type question

1. Define the Cartesian product of two graphs.
2. What is the difference between a walk and a trail?
3. Give an example of a real-world application of graph operations.
4. Define a closed walk in a graph.
5. What is a multigraph?

Part 2

C. Long answer type question (200 words)

1. Explain different types of graph operations with examples.
2. Discuss Eulerian and Hamiltonian paths and circuits with applications.
3. Define and classify different types of walks in graphs.
4. What are some real-world applications of walks in graph theory?
5. Explain the differences between simple graphs, multigraphs, and pseudographs with examples.

Part 3

D. Long answer type question (300 words)

1. Explain in detail the different types of graph operations with proper illustrations and real-life applications.
2. Provide an in-depth discussion of the properties and applications of Eulerian and Hamiltonian graphs.
3. Discuss the classification of walks, trails, paths, circuits, and cycles in graph theory with multiple examples.
4. How do graph operations contribute to various fields like computer science, transportation, and social networks?
5. Explain the significance of different types of walks in real-world scenarios, providing at least three applications.

UNIT 17
CONNECTIVITY, EULERIAN GRAPHS AND PLANARITY

NOTE

UNIT STRUCTURE

17.1 Introduction
17.2 Objective
17.3 Connected and Disconnected Graphs
- **17.3.1 Definition of Connected and Disconnected Graph**
- **17.3.2 Applications of Connected and Disconnected Graph**

17.4 Euler Circuit and Euler Trail
- **17.4.1 Definition of Euler Circuit and Euler Trail**
- **17.4.2 Some More Examples of Euler Circuit and Euler Trail**
- **17.4.3 Applications of Euler Circuit and Trail**

17.5 Planar and Non-Planar Graphs
- **17.5.1 Definition of Planar and Non-Planar Graph**
- **17.5.2 Some More Examples on Planar and Non-Planar Graph**
- **17.5.3 Applications of Planar and Non-Planar Graph**

17.6 Summary
17.7 Glossary
17.8 Answer to check your progress
17.9 Suggested readings – OER
17.10 Reference
17.11 Exercise
17.12 Feedback form

17.1 INTRODUCTION

In the previous unit, we explored fundamental graph concepts such as walks, trails, and different types of graph operations. These concepts helped in understanding how graphs can be analyzed and manipulated. Now, we extend this knowledge by studying connectivity in graphs, Eulerian paths, and planar graphs. These topics are essential in solving real-world problems like circuit design, transportation networks, and urban planning.

Building on this knowledge, we now offer a deeper insight into connected and disconnected graphs, Euler circuits and trails, and planar and non-planar graphs. These concepts are crucial in applications ranging from electrical circuit design to computer networks. Understanding connectivity and Eulerian properties allows us to optimize routes and network structures efficiently.

NOTE

This unit bridges abstract algebra with real-life applications by exploring graph connectivity, Eulerian circuits, and planarity. Connected graphs help in designing reliable communication networks, Euler circuits are useful in logistics and scheduling problems, and planar graphs have applications in VLSI circuit design. By studying these concepts, we gain valuable tools for solving complex real-world challenges.

17.2 OBJECTIVE

After studying this unit, you will be able to:

- Define the concept of connected and disconnected graphs and their significance in network structures.
- Define and identify Euler circuits and Euler trails in given graphs.
- Analyze real-world applications of Eulerian paths and circuits, such as solving routing and scheduling problems.
- Differentiate between planar and non-planar graphs and understand their importance in graph theory.
- Apply graph connectivity and Eulerian properties to practical problems like circuit design, transportation, and communication networks.

17.3 CONNECTED AND DISCONNECTED

17.3.1 Definition of Connected and Disconnected Graph:

1. **Connected Graph**:

   A graph is **connected** If there exists a path between every pair of vertices in the graph. This means there are no isolated vertices or groups of vertices in the graph. Thus, a graph G is connected if given any vertices v and w in G, there is a path from v to w. i.e., if any pair of vertices are reachable from one another.

2. **Disconnected Graph**:

   A graph is **disconnected** if it is not connected, i.e., there are at least two vertices such that no path exists between them. It may consist of multiple **connected components** (subgraphs where each subgraph is connected, but there is no connection between these subgraphs). A disconnected graph is the union of two or more connected subgraphs each pair of which has no vertex in common. these disjoint connected subgraphs are called the connected components of the graph.

   The graph G and a connected component of the graph is shown in Fig No.17.1 (a) and (b):

NOTE

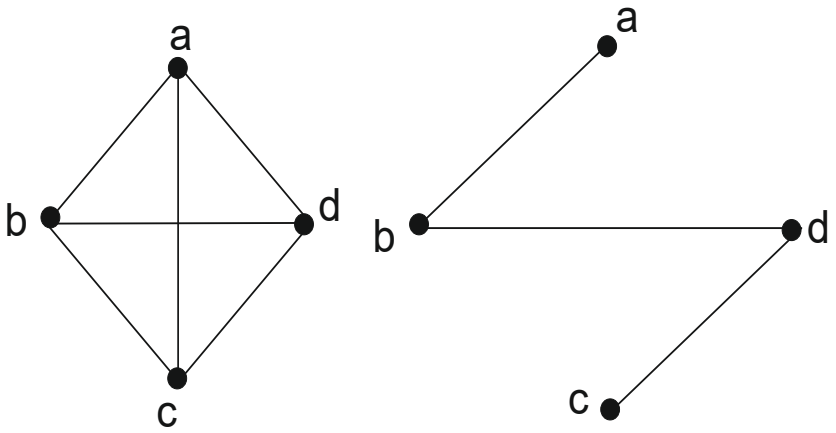


Fig No. 17.1 (*a*) *Fig No.* 17.1 (*b*)

Figure No.17.1 (a) & (b): Connected component of G

Example 1. Which of the graphs below are connected?

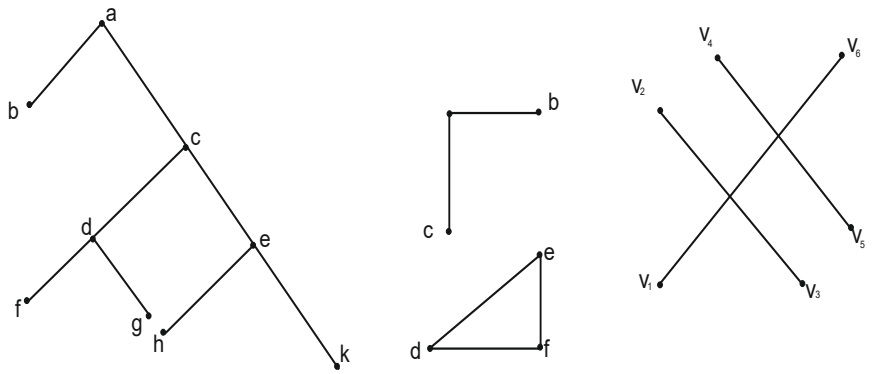


Fig No. 17.2 (*a*) *Fig No.* 17.2 (*b*) *Fig No.* 17.2 (*c*)

Figure No.17.2 (a) Connected and (b), (c) Disconnected

Solution. The graph shown in Fig No.17.2(a) are a connected graph since for every pair of distinct vertices there is a path between them.

The graph shown in Fig 17.2 (b) is disconnected graph since there is no path in the graph between vertices b and d.

The graph shown in Fig 17.2 (c) is not connected. Because while drawing a graph two edge may cross at a point which is not a vertex. The graph can be redrawn as in Fig 17.2 (d).

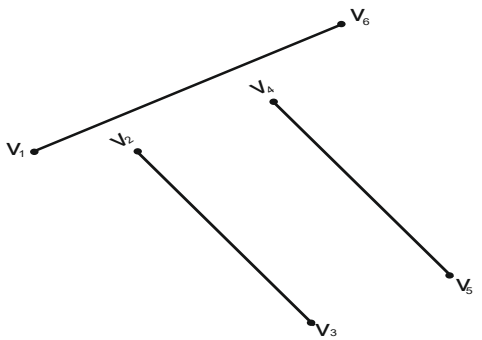


Figure* 17.2 *(d)

NOTE

Example 2: Connected Graph

Graph: Vertices: A, B, C, D

Edges: $A - B, B - C, C - D, A - D$

Solution:

1. Check all pairs of vertices:
   - $A \to B, A \to C, A \to D$: Paths exist.
   - $B \to C, B \to D$: Paths exist.
   - $C \to D$: Path exists.

Since a path exists between every pair of vertices, the graph is **connected**.

Example 3: Disconnected Graph

Graph: Vertices: P, Q, R, S, T

Edges: $P - Q, Q - R, S - T$

Solution:

1. Identify groups of connected vertices:
   - P, Q, R are connected (path exists among them).
   - S, T are connected (path exists between them).
2. Check connectivity between groups:
   - No path exists between $P, Q, R \; and \; S, T$.

Since the graph contains two distinct connected components, it is **disconnected**.

17.3.2 Applications of Connected and Disconnected Graph

1. **Social Networks**:
   - A connected graph represents a network where everyone is reachable.
   - A disconnected graph can represent separate communities.
2. **Computer Networks**:
   - Connected graphs ensure communication across the network.
   - Disconnected graphs highlight isolated systems or failures.
3. **Biology**:
   - Connected graphs model interactions in ecosystems.
   - Disconnected graphs show isolated populations or components.

17.4 EULER CIRCUIT EULER TRAIL

NOTE

17.4.1 Definition of Euler Circuit and Euler Trail:

The existence of Euler circuit and trail depends on the degree of vertices. A nonempty connected graph G is Eulerian if and only if its vertices are all even degree. A connected graph contains a Euler trail, but not a Euler circuit, it and only if it has exactly two vertices of odd degree.

1. Euler Circuit:

A **Euler circuit** (or Eulerian cycle) is a circuit (closed path) in a graph that visits every edge exactly once and returns to the starting vertex. A graph has a **Euler circuit** if and only if:

- All vertices in the graph have an even degree (an even number of edges connected to each vertex).
- The graph is connected.

2. Euler Trail:

A **Euler trail** (or Eulerian path) is a path that visits every edge of the graph exactly once but does not necessarily return to the starting vertex. A graph has a **Euler trail** if and only if:

- Exactly two vertices have an odd degree (the number of edges connected to the vertex is odd).
- The graph is connected.

If there is trail from vertex a to b in G and this trail traverses each edge in G exactly once, then the trail is called a **Euler trail**.

In Fig 17.3, $d\ c\ e\ a\ b\ c$ represents a Euler trail since it contains all the edges exactly one and vertex c is repeated but start and end vertex are not the same. it is not a Euler circuit as starting and ending at the same vertex is not possible without repeating an edge cd

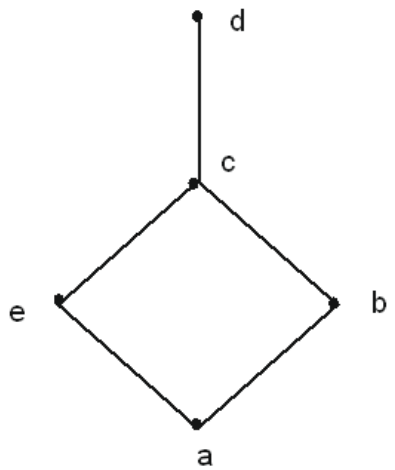


Figure No.17.3 Euler Circuits and Euler Trail

NOTE

Example 1. Let G be a graph of Fig 17.4 verify that G has a Euler circuit.

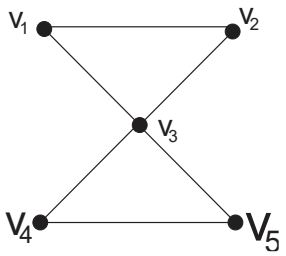


Figure No. 17.4 Euler Circuit

Solution. We observe that G is connected and all the vertices are having even degree $\deg(v_1) = deg(v_2) = \deg(v_4) = \deg(V_5) = 2$.

Thus, G has a Euler circuit.

By inspection, we find the circuit. $v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1$

Example 2. Show that the graphs in Fig. Fig No.17.5 contain no Euler circuit.

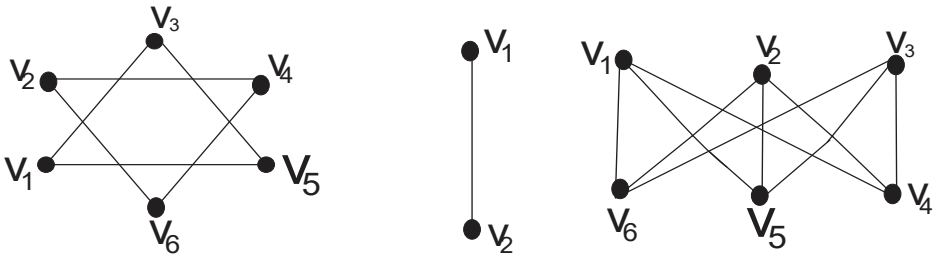


Fig No. 17. 5 (a) **Fig No. 17. 5 (b)** **Fig No 17. 5 (c)**

Figure No.17.5 Not a Euler Circuits

Solution. The graph shown in $Fig\ No.17.5(a)$ does not contain Euler circuit since it is not connected.

The graph shown in $Fig\ No.17.5\ (b)$ is connected but vertices $v_1\ and\ v_2\ are\ of$ degree 1. Hence it does not contain Euler circuit.

All the vertices of the graph shown in $Fig\ No.17.5\ (c)$ are of degree 3, hence it does not contain Euler circuit.

Example 3. Show that the graph shown in Fig 17.6 has no Eulerian circuit but has a Eulerian trail.

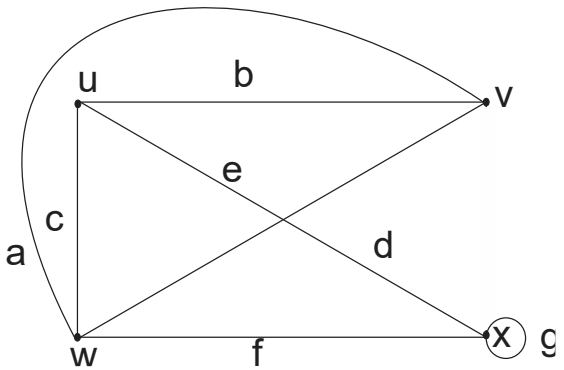


Figure No.17.6 Eulerian trail

NOTE

Solution. Here $deg\,(u) = deg\,(v) = 3\ and\ deg\,(w) = 4, deg\,(x) = 4$. since u and v here only two vertices of odd degree, the graph shown in Fig No.17.6 does not contain Eulerian circuit, but the path $b - a - c - d - g - f - e$ is a Eulerian Trail.

17.4.2 Some More Examples of Euler Circuit and Euler Trail:

Example 1: Euler Circuit

Graph: Vertices: A, B, C, D

Edges: $A - B, B - C, C - D, D - A, B - D$

Solution:

1. **Check the degree of each vertex**:
   - A has degree 2 (edges $A - B, D - A$).
   - B has degree 3 (edges $A - B, B - C, B - D$).
   - C has degree 2 (edges $B - C, C - D$).
   - D has degree 3 (edges $D - A, B - D, C - D$).

Since there are two vertices (B and D) with odd degrees, this graph **does not** have a Euler circuit, but it **may** have a Euler trail.

Example 2: Euler Trail

Graph: Vertices: P, Q, R, S

Edges: $P - Q, Q - R, R - S, S - P$

Solution:

1. **Check the degree of each vertex**:
   - P has degree 2 (edges $P - Q, S - P$).
   - Q has degree 2 (edges $P - Q, Q - R$).
   - R has degree 2 (edges $Q - R, R - S$).
   - S has degree 2 (edges $R - S, S - P$).

Since all vertices have even degrees, this graph **has a Euler circuit**, and therefore, it also has a Euler trail (since any Euler circuit is a Euler trail).

Example 3: Euler Trail (Graph with Two Odd Degree Vertices)

Graph: Vertices: A, B, C, D

Edges: $A - B, B - C, C - D$

Solution:

1. **Check the degree of each vertex**:
   - A has degree 1 (edge $A - B$).
   - B has degree 2 (edges $A - B, B - C$).

NOTE

- C has degree 2 (edges $B-C, C-D$).
- D has degree 1 (edge $C-D$).

Since exactly two vertices (A and D) have odd degrees, this graph **has a Euler trail** but **does not have a Euler circuit**.

Steps to Find a Euler Circuit or Euler Trail

1. **Check degrees**:
   - For a **Euler circuit**, all vertices must have even degrees.
   - For a **Euler trail**, exactly two vertices must have odd degrees.
2. **Check connectivity**:

   The graph must be connected for either an Euler circuit or Euler trail to exist.
3. **Start and end vertices**:
   - For a **Euler circuit**, the path starts and ends at the same vertex.
   - For a **Euler trail**, the path starts at one vertex with an odd degree and ends at the other odd-degree vertex.

17.4.3 Applications of Euler Circuits and Trails

1. **Route Planning**: Euler circuits are useful in planning routes that need to cover every edge (e.g., garbage collection, mail delivery).
2. **Circuit Design**: In electrical networks, Euler circuits can help optimize the routing of electrical current through all components.
3. **Puzzle Solving**: Euler trails are used in various puzzles and games, such as the famous "Seven Bridges of Königsberg" problem.

Understanding Euler circuits and trails is key in solving problems involving traversal, optimization, and route planning in graphs.

17.5 PLANAR AND NON-PLANAR GRAPHS

17.5.1 Definition of Planar and Non-Planar Graph

1. **Planar Graph**: A **planar graph** is a graph that can be drawn on a plane without any edges crossing. In other words, A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex. The points of intersection are called Crossovers. A graph that cannot be drawn on a plane without a crossover between its edges is called a nonplanar graph. The graphs shown in $Fig\ No.17.7\ (a)$ and $Fig\ No.17.7\ (b)$ are planer graphs.

NOTE

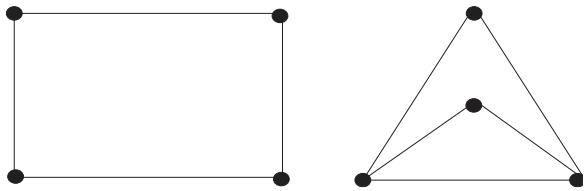


Fig No. 17. 7 (a) **Fig No. 17. 7 (b)**

Figure No.17.7 Planer Graphs

2. Non-Planar Graph: A non-planar graph is a graph that cannot be drawn on a plane without some edges crossing. These graphs cannot be embedded in the plane in such a way that no edges overlap. For example, the graph in *Fig No.*17.8 (a) is apparently nonplanar. However, the graph can be redrawn as shown in *Fig No.*17.8(b) so that edges do not cross, its planar graph, though its appearance is non-planar.

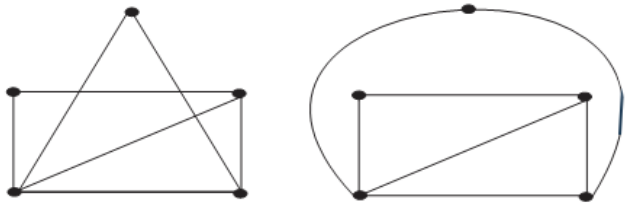


***Fig No.*17. 8 (a)** ***Fig No.*17. 8 (b)**

Figure No.17.8 Non-Planar Graph

Example 1. Two isomorphic diagrams in Fig 17.9(a) and 17.9(b) are different representation of the same graph. One of the diagrams is a planar representation: The other one is not the graph is planar.

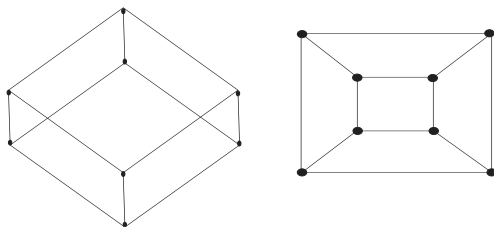


***Fig No.*17. 9(a)** ***Fig No.*17. 9(b)**

Figure No 17.9 (a) A Non-Planar Graph G., Fig No 17.9 (b) A planar Graph G

Example 2. A graph may have than one plane representation for example, the graph G has two plane representations.G_1 and G_2.

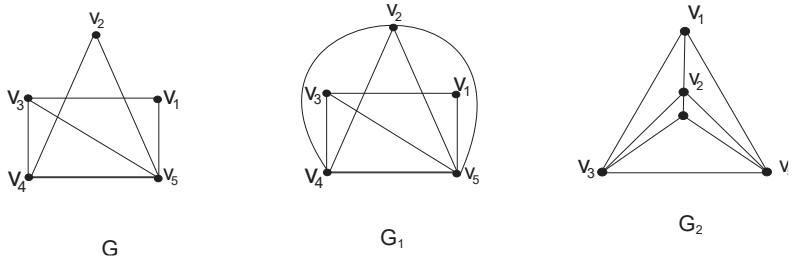


Fig No 17.10 Graph G has two planar graph representations.G_1 and G_2.

NOTE

17.5.2 Some More Examples on Planar and Non-Planar Graph:

Example 1: Planar Graph

Graph: Vertices: A, B, C, D, E

Edges: $A-B, B-C, C-D, D-A, A-E, B-E$

Solution:

1. Draw the graph:
   - Connect the vertices as described.
   - The graph can be drawn without any edges crossing.

Since the edges can be drawn without crossing each other, the graph is **planar**.

Example 2: Planar Graph

Graph: Vertices: P, Q, R, S

Edges: $P-Q, Q-R, R-S, S-P, P-R$

Solution:

1. Draw the graph:
   - Connect the vertices as described.
   - The graph can be drawn such that no edges overlap.

Since all edges can be arranged without crossing, the graph is **planar**.

Example 3: Non-Planar Graph (K_5)

Graph: The **complete graph on five vertices**, denoted K_5, is the graph where every pair of distinct vertices is connected by a unique edge.

Solution:

1. The graph K_5 consists of 5 vertices and 10 edges.

2. Attempt to draw:
   - When attempting to draw K_5, it is impossible to place the edges without some of them crossing.

3. Conclusion:

Since no arrangement of K_5 can be made without edges crossing, it is a **non-planar graph**.

Example 4: Non-Planar Graph ($K_{3,3}$)

Graph: The **complete bipartite graph** $K_{3,3}$ consists of two sets of vertices, where one set has 3 vertices and the other set also has 3 vertices. Each vertex in one set is connected to all vertices in the other set, but no vertices within the same set are connected.

Solution:

1. The graph $K_{3,3}$ consists of 6 vertices and 9 edges.

2. Attempt to draw:
   - Trying to arrange this graph on a plane without edge crossings is impossible.

NOTE

3. **Conclusion**:

   The graph $K_{3,3}$ cannot be drawn without edge crossings, so it is **non-planar**.

17.5.3 Applications of Planar and Non-Planar Graph

1. **Circuit Design**: Planar graphs are used in designing electronic circuits where paths do not overlap.
2. **Geographical Maps**: Planar graphs can model maps where different regions (vertices) are connected by paths (edges) without overlapping.
3. **Network Design**: Non-planar graphs might be used in complex network designs like those requiring multiple layers of connections.



Activity 17.1

Title: Identifying Euler Circuits, Trails, and Connectivity in Graphs

Objective:

To explore the concepts of connected and disconnected graphs, Euler circuits, and Euler trails through hands-on problem-solving.

Instructions:

1. Given the following graphs, determine whether they are connected or disconnected.
2. Identify whether each graph has a Euler circuit, Euler trail, or neither.
3. Convert a disconnected graph into a connected graph by adding the minimum number of edges required.
4. For each Eulerian graph, trace a Euler circuit or Euler trail, if possible.
5. Discuss real-life applications where Euler circuits and connectivity play a role (e.g., transportation networks, postal routes, circuit design).

☑ Check Your Progress 17.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a disconnected graph.

……
……
……

2. Every Euler circuit is a Euler trail. (True or False).

……

NOTE

3. A graph is Eulerian if:

(A) It has an Euler trail (B) It has an Euler circuit

(C) It is connected (D) It is a tree

.....

4. A graph that can be drawn without any edges crossing is called a _____ graph.

.....

17.6 SUMMARY

This unit introduced important concepts of graph theory related to connectivity, Eulerian graphs, and planar graphs. The study began with connected and disconnected graphs, where students learned how vertices in a graph may or may not be linked through paths. A connected graph allows movement between every pair of vertices, while a disconnected graph contains separate components that are not fully linked. The unit also explored practical applications of connectivity in communication networks, transportation systems, computer networking, and social network analysis. These concepts help in understanding how systems remain connected and how failures in connections can affect overall performance and communication.

The unit further explained Euler circuits and Euler trails, which are special paths used to travel through every edge of a graph exactly once. Students learned the difference between an Euler trail and an Euler circuit and studied the conditions required for their existence. Several examples were discussed to strengthen understanding of these concepts. Applications of Eulerian graphs were also examined in route planning, delivery services, road network design, garbage collection systems, and puzzle solving. These applications demonstrate how graph theory can be used to optimize movement and reduce repetition in real-life situations.

Another major topic covered was planar and non-planar graphs. Students learned that a planar graph can be drawn on a plane without edges crossing, whereas a non-planar graph cannot be represented without intersections. Various examples helped students identify and distinguish between these graph types. The applications of planar graphs were connected to circuit design, geographical mapping, engineering layouts, and network visualization. Overall, this unit developed analytical and problem-solving skills by showing how graph theory models real-world systems efficiently. Students are encouraged to further explore advanced graph theory concepts, algorithms, and practical applications in computer science, mathematics, data analysis, and engineering for deeper understanding and future academic growth.

17.7 GLOSSARY

NOTE

- **Graph**: A collection of vertices (nodes) connected by edges. It is a fundamental structure in graph theory.
- **Connected Graph**: A graph in which there is a path between every pair of vertices, ensuring no isolated components.
- **Disconnected Graph**: A graph that has two or more separate components with no path connecting them.
- **Euler Circuit**: A closed trail that visits every edge of a graph exactly once and returns to the starting vertex.
- **Euler Trail**: A trail that visits every edge of a graph exactly once but does not necessarily start and end at the same vertex.
- **Planar Graph**: A graph that can be drawn on a plane without any edges crossing.
- **Non-Planar Graph**: A graph that cannot be drawn on a plane without some edges crossing.
- **Degree of a Vertex**: The number of edges connected to a particular vertex in a graph.
- **Kuratowski's Theorem**: A theorem that helps determine whether a graph is planar by checking for the presence of subgraphs homeomorphic to K5K_5 or K3,3K_{3,3}.
- **Graph Applications**: Graphs are widely used in real-life scenarios such as computer networks, transportation systems, circuit design, and social networks.

17.8 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 17.1

1. Define a disconnected graph.

Answer: A disconnected graph is a graph in which at least two vertices are not joined by any path.

It consists of two or more components that are isolated from each other.

2. Every Euler circuit is a Euler trail. (True or False)

Answer: True.

An Euler circuit starts and ends at the same vertex, and since it uses every edge exactly once, it automatically qualifies as an Euler trail.

3. A graph is Eulerian if:

Answer: (B) It has an Euler circuit.

NOTE

A graph is Eulerian only when it has an Euler circuit, meaning all vertices have even degree and the graph is connected.

4. A graph that can be drawn without any edges crossing is called a _____ graph.

Answer: Planar graph.

A planar graph is one that can be drawn on a plane without any edges intersecting each other.

17.9 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1257	

17.10 REFERENCE

- https://testbook.com/maths/connected-graph
- https://www.tutorialspoint.com/connected-vs-disconnected-graphs
- 3.https://www.codechef.com/learn/course/graphs/GRAPHTYPES/problems/GRAPHTYPE2
- 4.https://jlmartin.ku.edu/courses/math105-F11/Lectures/chapter5-part2.pdf
- https://www.tpointtech.com/euler-graph-in-discrete-mathematicS
- https://discrete.openmathbooks.org/dmoi3/sec_planar.html
- https://transportgeography.org/contents/methods/graph-theory-definition-properties/planar-non-planar-graph/

17.11 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following graphs always has a Euler circuit?

a) A tree

b) A graph where all vertices have even degrees

NOTE

c) A graph where all vertices have odd degrees

d) A disconnected graph

2. If a graph has exactly two vertices of odd degree, then it has:

a) An Euler circuit

b) An Euler trail

c) No Euler path

d) A Hamiltonian circuit

3. A planar graph can be represented without:

a) Multiple edges

b) Loops

c) Edge crossings

d) Vertices

4. Which theorem is used to determine whether a graph is planar?

a) Euler's theorem

b) Dijkstra's theorem

c) Kuratowski's theorem

d) Kruskal's theorem

5. What is a major application of Euler circuits?

a) Social media analytics

b) Shortest path finding

c) Mail delivery routes

d) GPS mapping

Answers with Explanations:

1. b) A graph where all vertices have even degrees

Explanation: Euler circuits exist only in graphs where all vertices have even degrees and the graph is connected.

2. b) An Euler trail

Explanation: A graph with exactly two vertices of odd degree has an Euler trail but not an Euler circuit.

3. c) Edge crossings

Explanation: By definition, a planar graph can be drawn without edges crossing each other.

NOTE

4. c) Kuratowski's theorem

Explanation: Kuratowski's theorem states that a graph is non-planar if it contains a subgraph homeomorphic to K5K_5K5 or K3,3K_{3,3}K3,3.

5. c) Mail delivery routes

Explanation: Euler circuits help in designing efficient mail delivery and garbage collection routes where each road (edge) needs to be covered exactly once.

B. Short answer type question

1. Define a connected graph.
2. What is an Euler trail?
3. Give an example of a planar graph.
4. What is the degree of a vertex?
5. Name one real-life application of Euler circuits.

Part 2

C. Long answer type question (200 words)

1. Explain connected and disconnected graphs with examples.
2. Describe Euler circuits and Euler trails with their properties.
3. Explain the significance of Euler circuits in solving real-world problems.
4. Define a planar graph and explain Kuratowski's theorem.
5. Discuss the importance of graph theory in transportation and network design.

Part 3

D. Long answer type question (300 words)

1. Compare and contrast connected, disconnected, Eulerian, and Hamiltonian graphs with examples.
2. Explain the applications of Euler circuits and Euler trails in various industries.
3. Describe in detail the conditions for a graph to be Eulerian. Provide examples.
4. Explain the characteristics and applications of planar and non-planar graphs in real life.
5. Discuss how graph theory helps in solving problems related to communication networks and circuit design.

DR. C.V. RAMAN UNIVERSITY

Chhattisgarh, Bilaspur AN A/SECT GROUP UNIVERSITY

Approved by : PCI | AICTE | NCTE | BCI | Member of : AIU | Joint Committee : (UGC | DEB | AICTE) | Recognized by : UGC | A NAAC Accredited University

DR. C. V. RAMAN UNIVERSITY (C.G.)

SLM FEEDBACK FORM

PROGRAMME : _____

COURSE : _____

BLOCK : _____

UNIT : _____

Dear Student,

While studying the units of this block, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill up and send us the following questionnaire, which pertains to this block. If you find the space provided insufficient, kindly use a separate sheet.

Please mail to aisectcontent@aisect.org/ for online submission please scan QR code:

Questionnaire

1. **Name** : _____

2. **Enrolment No.** : | | | | | | | | | | | |

3. **How many hours did you need for studying the units?** : | |

4. **Please give your reactions to the following items based on your reading of the block:**

Items	Excellent	Very Good	Good	Poor	Give specific Examples (If poor)
Presentation Quality					
Language and Style					
Illustrations					
Used (diagrams, tables, etc.)					
Conceptual Clarity					
Check Your Progress Questions					
Suggested Readings - OER					

5. **Any other comments:**

_____

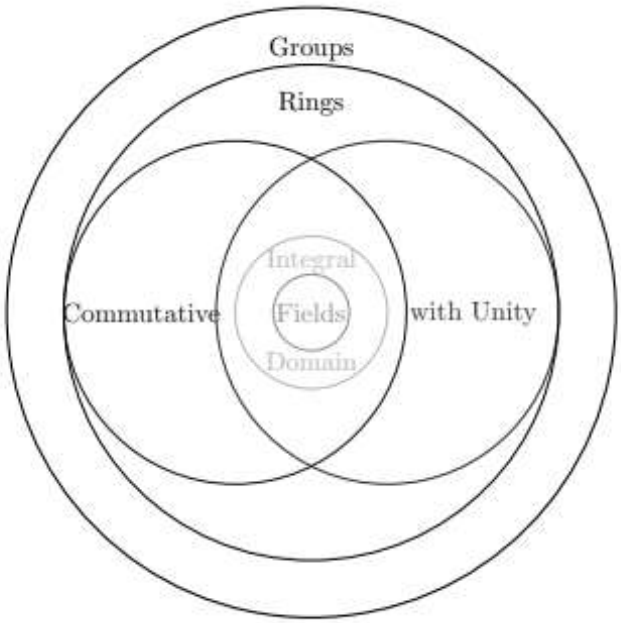
_____

_____

_____

***

BLOCK V	**GROUP**	**321-384**
Unit 18	**Group**	**323**
Unit 19	**Subgroup**	**337**
Unit 20	**Ring**	**357**
Unit 21	**Subring**	**371**



It focuses on the core ideas of abstract algebra, beginning with the structure of a group, its axioms, and fundamental properties. It then explores subgroups, helping learners understand subsets that preserve group operations and contribute to the internal architecture of algebraic structures. Moving ahead, the block introduces rings, expanding algebraic study to systems with two operations—addition and multiplication. It concludes with subrings, examining subsets of rings that behave as rings themselves. Collectively, this block builds a foundation for advanced algebra, cryptography, and mathematical modelling.

UNIT 18
GROUP

NOTE

UNIT STRUCTURE

18.1 Introduction
18.2 Objective
18.3 Group
18.3.1 Definition of a Group
18.3.2 Examples of Group
18.4 Types of a Group
18.5 Summary
18.6 Glossary
18.7 Answer to check your progress
18.8 Suggested readings – OER
18.9 Reference
18.10 Exercise

18.1 INTRODUCTION

In the previous unit, you learned about fundamental algebraic structures. Now, we introduce the concept of a group, a set with a binary operation satisfying specific properties. We will explore its definition, key examples, and different types of groups used in mathematics and applications.

Building on this knowledge, we now offer a detailed study of groups, a fundamental algebraic structure. We will define a group, provide key examples, and explore various types of groups that play a crucial role in mathematics and its applications.

This unit bridges abstract algebra with real-life applications by introducing the concept of a group. A group is a set with a binary operation that satisfies closure, associativity, identity, and invertibility. We will define groups, explore important examples, and classify different types of groups. Understanding groups helps in various fields, including cryptography, physics, and computer science.

18.2 OBJECTIVE

After studying this unit, you will be able to:

- Describe the definition and fundamental properties of a group.
- Identify and analyze examples of groups in mathematics and real life.

NOTE

- Differentiate between various types of groups, such as Abelian and non-Abelian groups.
- Apply group theory concepts to solve algebraic and computational problems.
- Recognize the significance of groups in diverse fields like cryptography, physics, and computer science.

18.3 GROUP

18.3.1 Definition of a Group

A set G and a binary operation $*$ constitutes a group, which satisfies four essential characteristics referred to as the group axioms:

A group is a pair $(G,*)$, where:

1. Closure: The outcome of the operation $a * b$ is likewise in G for any $a, b \in G$.

$$a * b \in G$$

2. Associativity: The operation satisfies for any $a, b, and\ c$ in G

$$(a * b) * c = a * (b * c)$$

3. Identity Element: For any a in G, there is an element $e \in G$ (referred to as the identity element) such that:

$$a * e = e * a = a$$

4. Inverse Element: There is an element $a^{-1} \in G$ (which is referred to as the inverse of a) such that for every $a \in G$:

$$a * a^{-1} = a^{-1} * a = e$$

The group is referred to as an abelian group if $*$ is commutative as well (i.e., $\boldsymbol{a * b = b * a}$ for any $\boldsymbol{a, b \in G}$).

18.3.2 Examples of Group

Example 1: Integers under Addition $(\mathbb{Z}, +)$

Set: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Operation: Addition $(+)$

Verification of Group Properties:

- **Closure:**

  The sum of any two integers is an integer.

  $$(e.g., 3 + (-2) = 1)$$

- **Associativity:**

  $(a + b) + c = a + (b + c)$ holds for all integers.

NOTE

- **Identity Element:**

  0 is the identity element

  since $a + 0 = a$ for all a.

- **Inverse Element:**

  The inverse of any integer a is $-a$ (e.g., the inverse of 5 is -5).

Conclusion: $(\mathbb{Z}, +)$ is a group. It is also Abelian since addition is commutative.

Example 2: Nonzero Real Numbers under Multiplication $(\mathbb{R}^*, \times)$

Set: $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ (all real numbers except zero)

Operation: Multiplication $(\times)$

Verification of Group Properties:

- **Closure:**

  The product of any two nonzero real numbers is nonzero. (e.g., $2 \times 3 = 6$)

- **Associativity:**

  Multiplication is associative: $(a \times b) \times c = a \times (b \times c)$

- **Identity Element:**

  The identity element is 1, since $a \times 1 = a$

- **Inverse Element:**

  Every nonzero real number aa has an inverse $1/a$

  $$(e.g., 5^{-1} = 1/5).$$

Conclusion: $(\mathbb{R}^*, \times)$ is a group. It is Abelian since multiplication is commutative.

Example 3: The Set of 2×2 Invertible Matrices under Matrix Multiplication

Set: $G = \{A \mid A \text{ is a } 2 \times 2 \text{ invertible matrix}\}$

Operation: Matrix multiplication

Verification of Group Properties:

- **Closure:**

  The product of two invertible matrices is also an invertible matrix.

- **Associativity:**

  Matrix multiplication is associative: $(AB)C = A(BC)$

- **Identity Element:**

  The identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies $AI = IA = A$

NOTE

- **Inverse Element:**

  Every invertible matrix A has an inverse A^{-1}

  such that $AA^{-1} = A^{-1}A = I$

Conclusion: The set of invertible 2×2 matrices form a **group**. It is **non-Abelian** because matrix multiplication is generally not commutative.

Example 4: Set of Complex Numbers of Unit Magnitude under Multiplication $(\boldsymbol{U(1)},\times)$

Set: $U(1) = \{z \in \mathbb{C} \mid\mid z \mid = 1\}$ (all complex numbers with absolute value 1)

Operation: Multiplication

Verification of Group Properties:

- **Closure:**

  The product of two unit-magnitude complex numbers also has unit magnitude.

- **Associativity:**

  Multiplication of complex numbers is associative.

- **Identity Element:**

  The identity element is 1 (since $z \times 1 = z$)

- **Inverse Element:**

  The inverse of z is $\frac{1}{z} = \bar{z}$ (the complex conjugate).

Conclusion: $U(1)$ is a group. It is Abelian because multiplication of complex numbers is commutative.

Example 5: The Symmetric Group S_3 (Permutations of Three Elements) under Composition

Set: $\boldsymbol{S_3} = \{ Identity\ (e), (12), (13), (23), (123), (132) \}$ (all possible rearrangements of 3 elements)

Operation: Function composition

Verification of Group Properties:

- **Closure:** The composition of two permutations results in another valid permutation.
- **Associativity:** Function composition is associative.
- **Identity Element:** The identity permutation ee leaves elements unchanged.
- **Inverse Element:** Every permutation has an inverse that undoes its effect.

NOTE

Conclusion: S_3 is a group. It is non-Abelian because composition of permutations is not always commutative.

18.4 TYPES OF A GROUP

A group in abstract algebra is a set that has a single binary operation that meets four essential characteristics: associativity, closure, the existence of inverses, and the existence of an identity element. Groups can be categorized into many sorts according to their characteristics and organization.

Below are the main types of groups:

18.4.1 Finite and Infinite Group

Finite Group: A group with a finite number of elements is called a finite group. The order of the group is the number of elements. The group of integers modulo n under addition ($\mathbb{Z}_n$) is one example

Infinite Group: A group with an unlimited number of elements is called an infinite group. The set of integers ($\mathbb{Z}$) that are subject to addition is one example.

Example 1: A Finite Group - Symmetric Group S_3

- **Set:** $S_3 = \{ e, (12), (13), (23), (123), (132) \}$
- **Operation:** Function composition
- **Order: 6** (finite)

Conclusion: S_3 is a finite non-Abelian group.

Example 2: An Infinite Group - Nonzero Rational Numbers $(\mathbb{Q}^*, \times)$

- **Set:** $\mathbb{Q}^* = \{q \mid q \in \mathbb{Q}, q \neq 0\}$
- **Operation:** Multiplication
- **Order:** Infinite

Conclusion: $(\mathbb{Q}^*, \times)$ is an infinite Abelian group.

18.4.1 PROBLEMS OF FINITE AND INFINITE GROUPS

Problem 1: Prove that the set $G = \{1, -1, i, -i\}$ under multiplication forms a finite group.

Solution:

Step 1: Verify Closure

- The multiplication of any two elements remains in G.
- $1 \times 1 = 1, 1 \times i = i, i \times i = -1$, etc.

Step 2: Check Associativity

- Since complex number multiplication is associative, this holds.

NOTE

Step 3: Find the Identity Element

- The identity element must satisfy $a \times e = e \times a$ for all a.
- Here, $e = 1$, since $1 \times a = a$

Step 4: Find Inverses

- Each element must have an inverse:
- $1^{-1} = 1$
- $(-1)^{-1} = -1$
- $i^{-1} = -i$
- $(-i)^{-1} = -i$

Conclusion: Gsatisfies all group properties and has only four elements, making it a finite group of order 4.

Problem 2: Prove that the set of integers $(\mathbb{Z}, +)$forms an infinite group.

Solution:

Step 1: Verify Closure

- The sum of any two integers is always an integer (e.g., $3 + (-2) = 1$).

Step 2: Check Associativity

- Addition is associative: $(a + b) + c = a + (b + c)$

Step 3: Find the Identity Element

- The identity element must satisfy $a + e = a$
- Here, $e = 0$, since $a + 0 = a$

Step 4: Find Inverses

- Every integer a has an inverse $-a$, such that $a + (-a) = 0$

Step 5: Check if the Group is Finite or Infinite

- The set $\mathbb{Z}$ has infinitely many elements.

Conclusion: $(\mathbb{Z}, +)$ is an infinite group because it has an infinite number of elements.

2. Abelian and Non-Abelian Group

Abelian Group: An Abelian group is one in which, for every a, b in the group, the operation is commutative $(a. b = b. a)$. The set of real numbers that are subject to addition $(\mathbb{R})$ is an example.

Non-Abelian Group: A non-Abelian group is one in which, for some a, b in the group, the operation $a. b \neq b. a$ is not commutative. The set of three object permutations (S_3) is an example.

NOTE

3. Cyclic Group

Cyclic Group: A cyclic group is one that may be produced by a generator, which is a single element. All the group's components are powers (or multiples) of this generator. $\mathbb{Z}_n$ under addition modulo n is an example.

4. Subgroups

Subgroup: A subgroup is a subset of a group that, when subjected to the same procedure, also forms a group. A subgroup of $\mathbb{Z}$ is made up of even integers ($2\mathbb{Z}$) is an example.

5. Normal Group

Normal Subgroup: A normal subgroup is a subgroup N of a group G such that, for any g in G, $gNg^{-1} = N$. To define quotient groups, normal subgroups are crucial.

For instance, a normal subgroup is always the center of a group is an example.

6. Simple and Composite Group

Simple Group: Simple Group is a nontrivial group with only the trivial group and the group itself as normal subgroups. such as A_5, the alternating group of order 5 is an example

Composite Group: A composite group is one that contains normal subgroups besides the trivial ones.

7. Direct Product and Semidirect Product Group

Direct Product: A group created by joining two groups in such a way that the operations are applied component-by-component is known as a "direct product." For instance, $\mathbb{Z}_2 \times \mathbb{Z}_3$ is an example.

Semidirect Product: Semidirect product is a generalization of the direct product that permits the structure of one group to influence another.

8. Permutation Group

Permutation Group: A group whose members are permutations of a set is called a permutation group. The group operation is the composition of permutations.

An illustration would be the symmetric group S_n, which is made up of every possible combination of n components is an example.

9. Matrix Group

Matrix Group: A collection of matrices subject to matrix multiplication is called a matrix group. among the subtypes are:

General Linear Group: ($\boldsymbol{GL(n, F)}$): All invertible $n \times n$ matrices over a field F are members of the General Linear Group ($GL(n, F)$).

NOTE

Special Linear Group ($SL(n,F)$): Matrices with determinant 1 in $GL(n,F)$ that belong to the Special Linear Group ($SL(n,F)$)

10. Lie Group

Lie Group: Lie Group is a group having smooth group operations that is also a differentiable manifold.

For instance, the group of rotations in three dimensions, or $SO(3)$ is an example.

- **Quotient Group**

Quotient Group: When a group G is divided by one of its normal subgroups N, a quotient group is created. The Cosets of N in G are the elements.

For instance, $\mathbb{Z}/n$, $\mathbb{Z}$ is the integer modulo n is an example.

a. Free Groups

Free Group: A group that has a basis that allows each element to be expressed uniquely as a finite product of basic elements and their inverses is known as a free group.

The free group on two generators $\{a,b\}$ is one example.



Activity 18.1

Identifying Groups

Objective: To help students understand the definition of a group by analyzing different sets and operations.

Instructions:

1. **Consider the following sets with the given operations:**
   - Set A: Integers ($\mathbb{Z}$) under addition (+)
   - Set B: Nonzero real numbers ($\mathbb{R}\{0\}$) under multiplication (×)
   - Set C: Even integers under addition (+)
   - Set D: Natural numbers under subtraction (−)
2. **For each set, check if it satisfies the four group properties:**
   - Closure (Is the operation closed in the set?)
   - Associativity (Does the operation satisfy associativity?)
   - Identity Element (Is there an identity element?)
   - Inverse Element (Does every element have an inverse?)
3. **Discuss which of these sets form a group and explain why or why not.**
4. **Present your findings in small groups and justify your answers.**

NOTE

Expected Outcome:

Students will identify which sets satisfy all group properties and understand why some sets do not form groups.

☑ Check Your Progress 18.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is the identity element in the group of integers under addition?

.....

.....

.....

2. Every Abelian group is a group, but not every group is Abelian. (True or False).

3. Which of the following is NOT a requirement for a set to be a group?

a) Closure b) Associativity

c) Commutativity d) Identity element

4. A group is called _____ if its binary operation is commutative.

18.5 SUMMARY

This unit introduced the fundamental concept of a group in mathematics and explored its importance in abstract algebra. Students learned that a group is an algebraic structure consisting of a set of elements together with a binary operation that satisfies four essential properties: closure, associativity, identity, and inverses. The unit explained how these properties work together to create a well-defined mathematical system that can describe symmetry, structure, and operations in various mathematical and practical contexts. Through the study of different examples, learners gained a clearer understanding of how groups appear in everyday mathematical operations such as addition and multiplication. Examples of finite and infinite groups were discussed to demonstrate the wide applicability of the concept. The unit also examined various types of groups, including Abelian and non-Abelian groups, helping students recognize the differences between commutative and non-commutative operations.

By the end of the unit, students should be able to define a group, identify whether a given set and operation form a group, and classify groups based on their properties. They should also understand the

NOTE

significance of identity elements and inverse elements in maintaining the structure of a group. These concepts form the foundation for advanced topics in algebra, cryptography, coding theory, computer science, and physics. Group theory is especially useful in studying symmetry in geometry, solving algebraic equations, and understanding transformations in scientific applications.

Students are encouraged to continue exploring more advanced concepts such as subgroups, cyclic groups, permutation groups, and homomorphisms to deepen their understanding of abstract algebra. Reading textbooks on modern algebra, practicing proof-based problems, and exploring real-life applications of symmetry and transformations can further strengthen conceptual clarity and analytical thinking skills.

18.6 GLOSSARY

- **Group –** A set with a binary operation satisfying closure, associativity, identity, and invertibility properties**.**
- **Binary Operation –** A rule that combines two elements of a set to produce another element within the same set.
- **Closure Property –** The result of applying the binary operation to any two elements of the set remains in the set.
- **Associative Property –** The grouping of elements does not affect the result of the operation, i.e., $(a * b) * c = a * (b * c)$.
- **Identity Element –** An element that does not change other elements under the operation, e.g., **0** for addition and **1** for multiplication.
- **Inverse Element –** For every element in the set, there exists another element that combines with it to produce the identity element.
- **Abelian Group –** A group in which the operation is commutative, meaning

  $a * b = b * a$ for all elements a and b.
- **Non-Abelian Group –** A group where the operation is not necessarily commutative, meaning $a * b$ may not equal $b * a$.
- **Cyclic Group –** A group where all elements can be generated by repeatedly applying the operation to a single element called the generator.
- **Permutation Group –** A group consisting of all possible rearrangements (permutations) of a set, used in combinatorics and algebra.

18.7 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

NOTE

Check Your Progress 18.1

1. **What is the identity element in the group of integers under addition?**

   Answer: The **identity element** in the group of integers under addition is **0**, because adding 0 to any integer does not change its value.

   Example: For any integer a, we have a + 0 = a = 0 + a.

2. **Every Abelian group is a group, but not every group is Abelian. (True or False).**

   True.

   Answer: An Abelian group is a group in which the operation is commutative (i.e., $a \cdot b = b \cdot a$ for all elements). All Abelian groups satisfy the group axioms, but some groups do not have the commutative property and are therefore not Abelian.

3. **Which of the following is NOT a requirement for a set to be a group?**

   Answer: c) Commutativity

   Commutativity is not required for a set to be a group. A group must satisfy closure, associativity, identity, and inverse, but commutativity is only needed if the group is Abelian.

4. **A group is called _____ if its binary operation is commutative.**

   Answer: A group is called Abelian if its binary operation is commutative.

18.8 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1258	

NOTE

18.9 REFERENCE

- https://math.mit.edu/research/highschool/primes/circle/documents/2022/Sophia%20&%20Jaeyi.pdf
- https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_group_theory.htm
- https://www.studysmarter.co.uk/explanations/math/pure-maths/group-mathematics/
- https://www.jmilne.org/math/CourseNotes/GT.pdf
- https://math.mit.edu/research/highschool/primes/circle/documents/2022/Sophia%20&%20Jaeyi.pdf
- https://www.geeksforgeeks.org/groups-discrete-mathematics/
- https://byjus.com/maths/finite-and-infinite-sets/
- https://www.geeksforgeeks.org/properties-of-abelian-group/
- https://www.tpointtech.com/abelian-groups-in-discrete-mathematics
- https://www.math.stonybrook.edu/~kirillov/mat552/liegroups.pdf

18.10 EXERCISE

Part 1

A. Multiple choice question

1. **Which of the following is an example of an Abelian group?**

   a) The set of integers under addition

   b) The set of 2×2 matrices under multiplication

   c) The set of nonzero real numbers under multiplication

   d) The set of natural numbers under subtraction

2. **Which of the following is NOT a requirement for a set to be a group?**

   a) Closure

   b) Associativity

   c) Commutativity

   d) Identity element

3. **What is the identity element in the group of rational numbers under multiplication?**

   a) 0

   b) 1

NOTE

c) -1

d) ∞

4. Which of the following is NOT a group?

a) Set of natural numbers under addition

b) Set of integers under addition

c) Set of real numbers under multiplication

d) Set of nonzero real numbers under multiplication

5. What is the inverse of 5 in the group of integers under addition?

a) 5

b) 0

c) -5

d) 1

Answers with Explanations:

1. a) The set of integers under addition

Explanation: A group is Abelian if its operation is commutative. Addition of integers satisfies this property, making it an Abelian group.

2. c) Commutativity

Explanation: A group does not necessarily have to be commutative (Abelian), but it must satisfy closure, associativity, identity, and invertibility.

3 b) 1

Explanation: The identity element under multiplication is the number that does not change any other number when multiplied. For rational numbers, this is 1.

4 a) Set of natural numbers under addition

Explanation: Natural numbers do not have an inverse element under addition (e.g., no negative numbers), so they do not form a group.

5 c) -5

Explanation: The inverse of an element in a group is the element that, when combined with it under the operation, gives the identity

NOTE

element. Since the identity element under addition is 0, the inverse of 5 is -5.

B. Short answer type question

1. What are the four fundamental properties of a group?
2. Give an example of a non-Abelian group.
3. What is the identity element in the group of real numbers under multiplication?
4. What is a cyclic group?
5. Who introduced the concept of a group in mathematics?

Part 2

C. Long answer type question (200 words)

1. Define a group and explain its four properties with suitable examples.
2. Differentiate between Abelian and non-Abelian groups with proper examples.
3. Explain the concept of cyclic groups with an example and proof.
4. Describe the importance of group theory in real-life applications.
5. Prove that the set of integers ($\mathbb{Z}$) under addition forms a group.

Part 3

D. Long answer type question (300 words)

1. Explain in detail the history and development of group theory, mentioning key mathematicians and contributions.
2. Discuss different types of groups, including Abelian, non-Abelian, cyclic, and permutation groups, with detailed examples.
3. Prove that the set of nonzero rational numbers ($\mathbb{Q}\{0\}$) under multiplication forms a group.
4. Describe permutation groups in detail, including their applications in combinatorics and cryptography.
5. Explain the role of groups in symmetry and physics, giving real-world examples such as molecular structures and quantum mechanics.

UNIT 19
SUBGROUP

NOTE

UNIT STRUCTURE

19.1 Introduction
19.2 Objective
19.3 Subgroup
- **19,3,1 Definition of a Subgroup**
- **19.3.2 Examples of Subgroup**
- **19.3.3 Necessary and sufficient condition of group to be a subgroup**

19.4 Coset
- **19.4.1 Definition of coset:**
- **19.4.2 Partition of a Group**
- **19.4.3 Characteristics of Coset**

19.5 Normal Subgroup
- **19.5.1 Definition of Normal Subgroup**
- **19.5.2 Examples of Normal Subgroup**
- **19.5.3 Necessary and Sufficient Condition for Normal Subgroup**
- **19.5.4 Characteristics of Normal Subgroup**

19.6 Quotient Group
- **19.6.1 Definition of Quotient Group**
- **19.6.1 Examples of Quotient Group**
- **19.6.2 Properties of Quotient Group**

19.7 Homomorphism
- **19.7.1 Definition of Group Homomorphism**
- **19.7.2 Examples of Group Homomorphism**
- **19.7.3 Properties of Homomorphism**

19.8 Summary
19.9 Glossary
19.10 Answer to check your progress
19.11 Suggested readings – OER
19.12 Reference
19.13 Exercise

19.1 INTRODUCTION

In the previous unit, you learned about groups and their fundamental properties. Now, we extend these ideas to subgroups, cosets, and normal subgroups. We also explore quotient groups and their

NOTE

properties. Additionally, we study homomorphisms and their significance in group theory.

Building on this knowledge, we now offer a deeper understanding of subgroups and their necessary and sufficient conditions. We examine cosets, their role in partitioning groups, and the importance of normal subgroups. The concept of quotient groups and their properties will be explored. Finally, we analyze homomorphisms and their properties in group structures.

This unit bridges abstract algebra with real-life applications by examining how groups break into subgroups and cosets. Normal subgroups and quotient groups play a crucial role in symmetry and cryptography. Understanding homomorphisms helps in structural comparisons between groups. These concepts are widely used in physics, coding theory, and computer science.

19.2 OBJECTIVE

After studying this unit, you will be able to:

- Define subgroups and determine the necessary and sufficient conditions for a subset of a group to be a subgroup.
- Explain cosets, their characteristics, and how they partition a group into distinct equivalence classes.
- Identify normal subgroups and understand the necessary and sufficient conditions for a subgroup to be normal.
- Define quotient groups, explore their properties, and understand their role in group theory.
- Explain group homomorphisms, their properties, and their significance in preserving group structures.

19.3 SUBGROUP

19.3.1 Definition of a Subgroup

Any subset H of a group G that is also a group under the same operation defined on G is called a subgroup.

A subset $H \subseteq G$ is formally a subgroup of G if it meets the requirements listed below:

1. **Closure:** The outcome of the group operation $a * b$ is likewise in H for any $a, b \in H. a * b \in H$
2. **Identity Element:** G is identity element e is present in H as well $e \in H$
3. **Inverses:** The inverse a^{-1} is likewise in H for any element $a \in H. a^{-1} \in H$

NOTE

If H satisfies these three properties,

we write $H \leq G$, which means H is a subgroup of G.

19.3.2 Examples of Subgroup

Example 1: Subgroup of Integers $(\mathbb{Z})$

- Group: $(\mathbb{Z}, +)$, the integer set that is being added.
- Subgroup: The collection of even numbers $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
  - Closure: The sum of two even integers is even.
  - Identity: 0 is even and is in $2\mathbb{Z}$
  - Inverses: The additive inverse of an even integer is also even.

Example 2: Subgroup of Real Numbers $(\mathbb{R}^*)$

- Group: $(\mathbb{R}^*, \cdot)$, the set of non-zero real numbers under multiplication is known as the group.
- Subgroup: The collection of positive real numbers is a subgroup $\mathbb{R}^+$
  - Closure: When two positive real numbers are multiplied, the result is positive.
  - Identity: 1 is in $\mathbb{R}^+$ and is a positive real number.
  - Inverses: A positive real number has a positive multiplicative inverse.

19.3.3 Necessary and Sufficient Condition of Group to be a Subgroup

A group G is subset H is a subgroup of G if and only if the following criteria are met: Under the group operation:

- H is closed.
- The identification element of G is contained in H.
- Every element in H has an inverse in H.

These three conditions are both necessary and sufficient for H to be a subgroup of G.

1. **Closure under the group operation:**

   The product $a.b$ (using the group operation) must also be in H for any element $a, b \in H$. This indicates that: $a \cdot b \in H, \quad \forall\, a, b \in H$.

2. **Existence of the identity element:**

   G is identification component, e must to be in H.

   In other words: $e \in H$

NOTE

3. **Existence of inverses:**

   The inverse of each element $a \in H$, a^{-1}must likewise be in H.

   In other words, $a^{-1} \in H, \ \forall a \in H$

Example 1: The Additive Group of Integers $\mathbb{Z}$ is subgroup

Solution: Let $H = 2\mathbb{Z}$ (the set of all even numbers) and $G = \mathbb{Z}$ (the set of all integers) be added using the standard method.

To check if $\boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$:

1. **Closure under addition:** We must determine whether the sum of any two even numbers is likewise an even number.

   If $b \in H$, then for certain integers m and n, $a = 2m$ and $b = 2n$.

   Then, an even integer, $a + b = 2m + 2n = 2(m + n)$, is obtained.

   H is closed under addition as a result.

2. **Contains the identity element**: Zero is the identification element in $\mathbb{Z}$ that is being added. Given that $0 = 2 \times 0$, $0 \in H$

   Thus, the identity is included in H.

3. **Contains the inverses element**: We need to determine whether - $a \in H$ for every $a \in H$, such as $a = 2m$.

   $a = 2m$, which is likewise an even number, is the inverse of a.

   As a result, every element in H has an inverse.

   Given that all three requirements are met, $H = 2\mathbb{Z}$ is a subgroup of $G = \mathbb{Z}$

19.4 COSET

19.4.1 Definition of Coset

A coset is a way to arrange items in a group according to a subgroup. In particular, the left Coset of H by an element $g \in G$ is defined as follows given a group G and a subgroup H of G:

There are two types of cosets:

- Left coset: $gH = \{gh: h \in H\}$
- Right coset: $Hg = \{hg: h \in H\}$

19.4.2 Partition of A Group

In group theory, a **partition of a group** refers to a way of dividing the group into distinct, non-overlapping subsets (called *block*s) such that:

1. **The subsets encompass the complete group:** Each group element needs to be present in precisely one subset.
2. **Every subset of the group is a non-empty subset.**
3. **The subsets are disjoint:** No two subsets share any element.

NOTE

Example 1: Group Partition (Division) Using Cosets

Consider the group $G = \mathbb{Z}_6$ (the integers modulo 6) and the subgroup $H = \{0,3\}$

Group G: $G = \{0,1,2,3,4,5\}$

Subgroup H: $H = \{0,3\}$

We can partition the group G into cosets of H.

A set of the form $g + H$ where $g \in G$ is called a coset of H.

Now let us locate the cosets:

- $0 + H = \{0,3\}$
- $1 + H = \{1,4\}$
- $2 + H = \{2,5\}$

Thus, the partition of G based on the cosets of H is: $\{\{0,3\},\{1,4\},\{2,5\}\}$

These cosets cover the entire group, and each element of G appears in exactly one coset, so this is a valid partition.

Example 2: Partition of a Group by Conjugacy Classes

Take the group of all permutations of three elements, $G = S_3$, which is the symmetric group of degree 3.

- **Group** G: $G = \{e, (12), (13), (23), (123), (132)\}$ where e is the identity element.

The conjugacy classes in a group are sets of elements that are conjugate to each other.

Two elements g and h of a group are conjugate

if there is some x in the group such that $xgx^{-1} = h$.

Let us find the conjugacy classes of S_3:

- The identity element e is only conjugate to itself, so $\{e\}$ is one conjugacy class.
- The transpositions $(12), (13), (23)$ are conjugate to each other,

so $\{(12), (13), (23)\}$ is another conjugacy class.

- The 3-cycles (123) and (132) are conjugate to each other,

  so $\{(123), (132)\}$ is another conjugacy class.

Thus, the partition of S_3 into conjugacy classes is:

$$\{\{e\}, \{(12), (13), (23)\}, \{(123), (132)\}\}$$

This is a valid partition of G since, once more, every element of G occurs in exactly one conjugacy class, and these classes contain the entire group.

NOTE

19.4.3 Characteristics of Coset

Cosets are crucial ideas in group theory that are used to examine the relationship between a subgroup and the wider group. A coset is a set created by multiplying every member of a subgroup H by a fixed element $g \in G$ in a group G.

Cosets partition the group: The cosets of a subgroup H divide the entire group G into disjoint sets, where each coset has the same size as H.

Coset size: All cosets of a subgroup have the same number of elements as the subgroup itself.

Coset are disjoint or identical: If two cosets g_1H and g_2H (or Hg_1 and Hg_2) are not identical, they do not share any elements. In other words, cosets are either disjoint or the same.

1. **Cosets form a partition of the group:** Group G is partitioned by the set of all cosets of a subgroup H in G.
2. **Every element of G belongs to exactly one coset.**
3. **All Cosets have the same size:** All left cosets and all right cosets of the same subgroup have the same number of elements, which is equal to the order (size) of the subgroup.
4. **Coset operation:** If two cosets g_1H and g_2H are identical, then g_1 and g_2 differ by an element of H, i.e., $g_1 = g_2h$ for some $h \in H$.

19.5 NORMAL SUBGROUP

19.5.1 Definition of Normal Subgroup

If a subgroup H of a group G meets the following criteria, it is referred to as a normal subgroup (commonly $H \lhd G$):

Since $gH = Hg$,for every g in G.

This indicates that for each element g in group G, the left and right cosets of H are the same.

Alternatively, the condition can be denoted as:

$$ghg^{-1} \in H, \qquad \forall g \in G, h \in H$$

where ghg^{-1} is called the **conjugate** of h by g.

19.5.2 Examples of Normal Subgroup

Example 1: Normal Subgroup in $\mathbb{Z}_6$(Integers Modulo 6)

Let $G = \mathbb{Z}_6$ be the integers' additive group modulo 6. Take the subgroup $H = \{0,3\}$. as an example.

Step 1: Verify Normality

To check if $H = \{0,3\}$ is normal in G, we observe that $\mathbb{Z}_6$ is an abelian group.

NOTE

Since the group operation in abelian groups is commutative, all the subgroups are normal. We obtain

$$g + h = h + g$$

for all $g \in G$ and $h \in H$.

This suggests that for any element of G, the left and right cosets are identical.

Step 2: Conclusion

Given that $G = \mathbb{Z}_6$is abelian, a normal subgroup of G is $H = \{0,3\}$.

Example 2: Normal Subgroup in S_3 (Symmetric Group on 3 Elements)

Let $H = \{e, (12)\}$, where e is the identity permutation and (12) is the transposition that switches 1 and 2, and let G=S_3, the symmetric group of all permutations of the set {1,2,3}.

Step 1: Verify Normality

We need to check if $H = \{e, (12)\}$, is normal in $G = S_3$.

We check if conjugating any element of H by any element of G results in an element still in H.

- Conjugate e by any element $g \in G$:

  $geg^{-1} = e$ (since the identity is unchanged under conjugation)

  Thus, $e \in H$.

- Conjugate (12) by each element of G:
  - Conjugate by (13):

    $(13)(12)(13)^{-1} = (132)$

This is **not** in H, so (12) is not normal in S_3

Step 2: Conclusion

Since conjugating (12) by (13) does not result in an element of H,

$H = \{e, (12)\}$ not a normal subgroup in S_3

19.5.3 Necessary and Sufficient Condition for Normal Subgroup

Necessary and sufficient condition for a subgroup H of a group G to be normal.

$gH = Hg,$ for all $g \in G$

On the other hand, a subgroup H is normal in G if and only if the conjugate of h by g belongs to H for every g in G and every h in H.

Formally, this is expressed as follows: $ghg^{-1} \in H \quad for\ all \quad g \in G, h \in H.$

Consequently,

NOTE

- A **normal subgroup** H of a group G satisfies the condition that for all elements $g \in G$, the left and right cosets of H are equal, i.e., $gH = Hg$.
- **Abelian groups** have all their subgroups as normal subgroups because the group operation is commutative, making left and right cosets identical for all elements.
- In general, to verify that a subgroup is normal, we check if conjugating any element of the subgroup by any element of the group results in an element still in the subgroup.

19.5.4 Characteristics of Normal Subgroup

1. **Coset Equality:** The left cosets and right cosets of a normal subgroup are the same for every element g in the group. This means that $gH = Hg$ for all $g \in G$.
2. **Conjugates Remain in Subgroup:** Any element of H that is conjugated by any element of G produces an element that remains in H if H is a normal subgroup of G. Specifically, $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.
3. **Group Homomorphisms:** The set of cosets of H in G, or the quotient group G/H, is a group. Due to the well-defined nature of the Coset multiplication, this is only feasible when H is normal.
4. **Subgroup of the Quotient Group:** The group operation on G/H is well defined, and a normal subgroup of G is a subgroup of the quotient group G/H.
5. **Trivial and Whole Group:** The trivial subgroup $\{e\}$ and the entire group G are always normal subgroups of G.

19.6 QUOTIENT GROUP

19.6 1 Definition of Quotient Group

When a group G is divided by one of its normal subgroups N, a quotient group also known as a factor group is created. The cosets of N in G are the elements of the quotient group, and the multiplication of these cosets defines the group operation.

The quotient group G/N is formed by partitioning the group G into cosets of a normal subgroup N, and the operation on cosets is defined as $(gN) \cdot (hN) = (gh)N$.

Condition for Normal Subgroup: For the quotient group to be clearly defined, N needs to be a normal subgroup of G.

i.e., $gN = Ng$ for all $g \in G$.

NOTE

- **Example 1:** $G = \mathbb{Z}_6$, $N = \{0,3\}$, the quotient group G/N is isomorphic to $\mathbb{Z}_3$.
- **Example 2:** $G = S_3$, $N = A_3$, the quotient group S_3/A_3 is isomorphic to $\mathbb{Z}_2$.

19.6.2 Examples of Quotient Group

Example 1: Quotient Group in $\mathbb{Z}_6$

Let $N = \{0,3\}$ be a normal subgroup of G, and let $G = \mathbb{Z}_6$, , the group of integers modulo 6 under addition. The quotient group G/N will be formed.

Step 1: Identify the Cosets of N

The cosets of N in G are the sets of the form $g + N$, where $g \in G$.

We calculate the cosets:

- $0 + N = \{0 + 0, 0 + 3\} = \{0,3\}$
- $1 + N = \{1 + 0, 1 + 3\} = \{1,4\}$
- $2 + N = \{2 + 0, 2 + 3\} = \{2,5\}$

Thus, the cosets of N in G are: $\{0,3\}$, $\{1,4\}$, $\{2,5\}$

Step 2: Define the Quotient Group Operation

The quotient group G/NG/N consists of the cosets $\{0,3\}, \{1,4\}, \{2,5\}$.

The operation on these cosets is defined by:

$$(gN) \cdot (hN) = (gh)N$$

Example:

- $(0 + N) \cdot (1 + N) = (0 + 1) + N = 1 + N = \{1,4\}$
- $(1 + N) \cdot (2 + N) = (1 + 2) + N = 3 + N = \{0,3\}$

Thus, the quotient group G/N is $\{\{0,3\}, \{1,4\}, \{2,5\}\}$, and it behaves like the cyclic group $\mathbb{Z}_3$ with three elements.

Step 3: Conclusion

The quotient group G/N is isomorphic to $\mathbb{Z}_3$, the cyclic group of order 3.

Example 2: Quotient Group in S_3(Symmetric Group on 3 Elements)

Consider the symmetric group S_3, the group of all permutations of $\{1,2,3\}$, and

let $N = A_3$ be the alternating group, consisting of all even permutations.

The group A_3 is a normal subgroup of S_3.

Step 1: Elements of S_3 and A_3

- $S_3 = \{e, (12), (13), (23), (123), (132)\}$
- $A_3 = \{e, (123), (132)\}$, which is a normal subgroup of S_3.

NOTE

Step 2: Identify the Cosets of A_3 in S_3

We form the Cosets of A_3in S_3:

- $eA_3 = A_3 = \{e, (123), (132)\}$
- $(12)A_3 = \{(12) \cdot e, (12) \cdot (123), (12) \cdot (132)\} = \{(12), (13), (23)\}$

Thus, the cosets of A_3in S_3 are:

$A_3 = \{e, (123), (132)\}, (12)\ A_3 = \{(12), (13), (23)\}$

Step 3: Define the Quotient Group Operation

The quotient group S_3/A_3 consists of the cosets A_3 and $(12)A_3$.

The operation on these cosets is defined by: $(gA_3) \cdot (hA_3) = (gh)A_3$

For example:

- $A_3 \cdot A_3 = A_3$
- $A_3 \cdot (12)A_3 = (12)A_3$

Step 4: Conclusion

The quotient group S_3/A_3 has two elements, A_3 and $(12)A_3$, and is isomorphic to $\mathbb{Z}_2$the cyclic group of order 2.

In fact, S_3/A_3 is a simple group of order 2, which is the group of even and odd permutations.

19.6.3 Properties of Quotient Groups:

1. A normal subgroup of G must have N for the operation on cosets to be well-defined. This implies that the representatives chosen from each coset have no bearing on the coset product.
2. Order of Quotient Group: The order of the quotient group G/N is $|G/N| = |G| / |N|$ if $|G|$ is the order (size) of G and $|N|$ is the order of the normal subgroup N.
3. Each element of G maps to its coset in G/N, forming a natural homomorphism between G and G/N. This map has a kernel of exactly N and is surjective.

19.7 HOMOMORPHISM

19.7 1 Definition of Group Homomorphism

A group homomorphism is a function $f: G \to H$ between two groups $(G,*)$ and $(H,\cdot)$ that preserves the group operation. This means:

$$f(a * b) = f(a) \cdot f(b) \qquad for\ all \quad a, b \in G.$$

Here:

- $*$ is the operation in G,
- $\cdot$ is the operation in H,

NOTE

- f maps the identity element of G to the identity element of H,
- $f(a^{-1}) = \big(f(a)\big)^{-1} \qquad for\ all\ \ a \in G.$

19.7 2 Examples of Group Homomorphism

Example 1: Homomorphism from Integers to Integers Modulo n

Let $G = (\mathbb{Z}, +)$ (integers under addition) and $H = (\mathbb{Z}_n, +)$ (integers modulo n under addition).

Solution: Define $f: \mathbb{Z} \to \mathbb{Z}_n$ as: $f(a) = a \bmod n$

Check if f is a homomorphism:

- Let $a, b \in \mathbb{Z}$.
- $f(a + b) = (a + b) \bmod n$.
- $f(a) + f(b) = (a \bmod n) + (b \bmod n) = (a + b) \bmod n$.

Since $f(a + b) = f(a) + f(b)$, f is a group homomorphism.

Conclusion: This is a valid homomorphism, as it preserves the group operation.

Example 2: Homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \cdot)$

Let $G = (\mathbb{R}, +)$ (real numbers under addition) and $H = (\mathbb{R}^+, \cdot)$ (positive real numbers under multiplication).

Solution: Define $f: \mathbb{R} \to \mathbb{R}^+$ as: $f(x) = e^x$

Check if f is a homomorphism:

- Let $x, y \in \mathbb{R}$.
- $f(x + y) = e^{(x+y)} = e^x \cdot e^y$
- $f(x) \cdot f(y) = e^x \cdot e^y$

Since $f(x + y) = f(x) \cdot f(y)$, f is a group homomorphism.

Conclusion: This is a valid homomorphism, as it preserves the group operation.

19.7 3 Properties of Homomorphism

Property 1: Preserves Additive Identity

A ring homomorphism $f: R \to S$ maps the additive identity of R to the additive identity of S: $f(0_R) = 0_S$.

Example 1: Identity Homomorphism

Let $R = \mathbb{Z}$, the ring of integers. The identity map $f: \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = x$, is a ring homomorphism.

- Here, $0_R = 0$ (additive identity of $\mathbb{Z}$) and $0_S = 0$.
- Clearly, $f(0_R) = 0$, satisfying the property.

NOTE

Example 2: Modulo Reduction Map

Let $R = \mathbb{Z}$ and $S = \mathbb{Z}_6$ (integers modulo 6). The map $f: \mathbb{Z} \to \mathbb{Z}_6$, defined by $f(x) = x \ mod \ 6$, is a ring homomorphism.

- Here, $0_R = 0$ (in $\mathbb{Z}$).and $0_S = 0$ (in $\mathbb{Z}_6$).
- $f(0) = 0 \ mod \ 6 = 0$, satisfying the property.

Property 2: Preserves Negatives

For any $a \in R$, a ring homomorphism f maps the additive inverse $-a$ in R to the additive inverse $-f(a)$ in S:$f(-a) = -f(a)$.

Example 1: Identity Homomorphism

Let $f: \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x$.

- If $a = 5$, then $-a = -5$.
- $f(-a) = f(-5) = -5$ and $-f(a) = -(f(5)) = -5$.
- Thus, $f(-a) = -f(a)$, satisfying the property.

Example 2: Polynomial Evaluation Map

Let $f: \mathbb{Z}[x] \to \mathbb{Z}$, defined by $f(p(x)) = p(0)$ (evaluation at $x = 0$).

- For $p(x) = 2x^2 - 3x + 1$, the additive inverse is $-p(x) = -2x^2 + 3x - 1$.
- $f(-p(x)) = (-2x^2 + 3x - 1)\ |_{x=0} = -1$.
- $-f(p(x)) = -(2x^2 - 3x + 1)\ |_{x=0} = -1$.
- Thus, $f\big(-p(x)\big) = -f\big(p(x)\big)$, satisfying the property.

Property 3: Kernel and Image

1. **Kernel:** The kernel of f, indicated as $\ker(f)$, is the collection of every component in R that are mapped to 0_S: $ker(f) = \{a \in R: f(a) = 0_S\}$.
2. **Image:** The image of f, denoted as $Im(f)$, is the set of all values $f(a)$ for $a \in R$: $Im(f) = \{f(a): a \in R\}$.

Example: Map $f: \mathbb{Z} \to \mathbb{Z}_6$

- $f(x) = x \ mod \ 6$
  - **Kernel:** The kernel is $ker(f) = \{x \in \mathbb{Z}: x \ mod \ 6 = 0\} = \{0, \pm 6, \pm 12, \dots\} = 6\mathbb{Z}$
  - **Image:** The image is $Im(f) = \{0,1,2,3,4,5\}$, which is a subring $\mathbb{Z}_6$.

Verification:

- For $x = 12, f(12) = 12 \ mod \ 6 = 0$, so $12 \in ker(f)$.
- For $x = 8, f(8) = 8 \ mod \ 6 = 2$, so $2 \in Im(f)$.

NOTE

Activity 19.1

Identifying Subgroups and Normal Subgroups

Objective:

To understand the concept of subgroups, cosets, and normal subgroups through hands-on verification of necessary and sufficient conditions.

Instructions:

1. **Given Group:** Consider the group $G = (\mathbb{Z}, +)$ (the set of integers under addition).
2. **Subgroup Verification:**
   - Take the subset $H = 2\,\mathbb{Z}$ (set of even integers).
   - Verify whether H satisfies the subgroup criteria:
     - Closure under addition
     - Existence of identity
     - Existence of inverses
3. **Coset Formation:**
   - Find the left and right cosets of H in G.
   - Check if left and right cosets are the same.
4. **Normal Subgroup Verification:**
   - Check if H is a normal subgroup of G by verifying if $gH = Hg$ for all $g \in G$.
5. **Quotient Group Construction:**
   - Construct the quotient group G/H and describe its elements.
6. **Discussion:**
   - Compare the results and discuss the significance of normal subgroups in forming quotient groups.

Expected Outcome:

By completing this activity, you will develop a practical understanding of subgroups, cosets, and normal subgroups, reinforcing their theoretical properties.

NOTE

☑ Check Your Progress 19.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a subgroup.?

.....

.....

.....

2. Every subgroup of a group is also a normal subgroup.

(True or False).

.....

3. If H is a normal subgroup of G, which of the following is always true?

a) H is a cyclic group

b) Left and right cosets of H are the same

c) H contains the identity element only

d) Every subgroup of G is normal

4. The kernel of a homomorphism is always a _____ of the given group.

19.8 SUMMARY

This unit introduced the fundamental concepts of subgroup theory in abstract algebra and explained how smaller algebraic structures can exist within larger groups. The study began with the definition of a subgroup, including various examples that demonstrated how subsets of a group can themselves satisfy all group properties. Important tests and necessary and sufficient conditions for determining whether a subset forms a subgroup were also discussed, helping students understand the logical structure behind subgroup verification.

The unit further explored the concept of cosets, including their definitions, properties, and the way they partition a group into disjoint subsets. Students learned how left and right cosets are formed and how cosets play a significant role in understanding the structure of groups. The characteristics of cosets also provided insight into equivalence relations and group decomposition.

Another major topic covered was the normal subgroup. The definition, examples, and identifying conditions for normal subgroups were explained in detail. Students learned that normal subgroups are essential because they allow the construction of quotient groups. The

NOTE

characteristics and algebraic importance of normal subgroups highlighted their central role in advanced group theory.

The unit also introduced quotient groups, their definitions, examples, and important properties. Students understood how quotient groups simplify complex group structures and provide a deeper understanding of symmetry and algebraic operations. In addition, the concept of group homomorphism was studied, including its definition, examples, and important properties. Homomorphisms demonstrated how one group can be mapped into another while preserving algebraic structure, making them useful tools in mathematics and related fields.

Overall, this unit developed a strong foundation in subgroup-related concepts that are widely used in higher mathematics, cryptography, coding theory, physics, and computer science. Students are encouraged to further explore advanced algebraic structures and solve more examples to strengthen their conceptual understanding and analytical skills.

19.9 GLOSSARY

- **Subgroup:** A subset H of a group G that itself forms a group under the same operation as G.
- **Necessary and Sufficient Condition for a Subgroup:** A non-empty subset H of G is a subgroup if for all $a, b \in H$, the element ab^{-1} is also in H.
- **Coset:** A set of elements obtained by multiplying a fixed element g from G with all elements of a subgroup H, forming left cosets (gH) and right cosets (Hg).
- **Partition of a Group:** The division of a group into distinct cosets of a subgroup, ensuring no overlap between different cosets.
- **Characteristics of Coset:** Cosets have the same number of elements as the subgroup, and if a subgroup is normal, left, and right cosets are identical.
- **Normal Subgroup**: A subgroup N of G is normal if for every element $g \in G$
- $gNg^{-1} = N$, meaning it remains unchanged under conjugation.
- **Quotient Group:** The set of cosets G/N forms a new group when N is a normal subgroup, with operation defined by $(aN)(bN) = (ab)N$.
- **Properties of Quotient Group:** Quotient groups simplify group structures, play a key role in classification, and are used in the First Isomorphism Theorem.

NOTE

- **Homomorphism:** A function $\phi: G \to G'$ that preserves the group operation, meaning $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.
- **Kernel of a Homomorphism:** The set of elements in G that map to the identity element in G'; it is always a normal subgroup of G.

19.10 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 19.1

1. Define a subgroup.

Answer: A subgroup is a subset H of a group G that is itself a group under the same operation as G.

To be a subgroup, H must satisfy:

1. Closure: If $a, b \in H$, then $ab \in H$
2. Identity: The identity element of G is in H
3. Inverses: If $a \in H$, then $a^{-1} \in H$

We denote this by $H \leq G$, meaning H is a subgroup of G.

2. Every subgroup of a group is also a normal subgroup. (True or False):

Answer: False.

Not every subgroup is normal.

A subgroup H is normal in G if $gH = Hg$ for all $g \in G$.

Only in abelian groups, every subgroup is normal.

3. If H is a normal subgroup of G, which of the following is always true?

Answer: b) Left and right cosets of H are the same

Explanation:

This is the definition of a normal subgroup:

$gH = Hg$ for all $g \in G$

Other options are not necessarily true for all normal subgroups.

4. The kernel of a homomorphism is always a _____ of the given group.

Answer: Normal subgroup

The kernel of a group homomorphism $\phi: G \to G'$, defined as

$ker(\phi) = \{g \in G \mid \phi(g) = e'\}$

is always a normal subgroup of G.

NOTE

19.11 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1259	

19.12 REFERENCE

- https://www.geeksforgeeks.org/subgroup-and-order-of-group-mathematics/
- https://www.emathzone.com/tutorials/group-theory/necessary-and-sufficient-condition-for-a-subgroup.html
- https://www.geeksforgeeks.org/cosets-in-mathematics/
- https://byjus.com/maths/normal-subgroup/
- https://www.geeksforgeeks.org/quotient-group-in-group-theory/
- https://testbook.com/maths/quotient-group
- https://www.socratica.com/pages/kernel-of-group-homomorphisms

19.13 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following is always true for a subgroup H of a group G?

a) H must contain all elements of G

b) H must be normal in G

c) H must contain the identity element of G

d) H must have infinite elements

2. If H is a normal subgroup of G, which of the following is true?

a) Left and right cosets of H are identical

b) H must be a cyclic group

c) Every element of G is in H

d) H contains only the identity element

3. Which of the following statements is false?

a) The intersection of two subgroups is also a subgroup

NOTE

b) Every subgroup of a cyclic group is cyclic

c) Every subgroup is a normal subgroup

d) Every normal subgroup is a subgroup

4. Which of the following is a homomorphism property?

a) $\phi(ab) = \phi(a) + \phi(b)$

b) $\phi(ab) = \phi(a)\phi(b)$

c) $\phi(a) = a^2$

d) $\phi(a) = a^{-1}$

5. Which of the following is the kernel of a homomorphism $\phi: G \to G'$?

a) The set of all elements in G

b) The set of elements in G that map to the identity element of G'

c) The set of all elements in G'

d) The set of cosets of G

Answers with Explanations:

1. (c) H must contain the identity element of G

Explanation: A subgroup must contain the identity element of the group and be closed under the group operation and inverses.

2. (a) Left and right cosets of H are identical

Explanation: A subgroup is normal if $gH = Hg$ for all $g \in G$.

3. (c) Every subgroup is a normal subgroup

Explanation: Not every subgroup is normal. A subgroup is normal only if left and right cosets are equal.

4. (b) $\phi(ab) = \phi(a)\phi(b)$

Explanation: A group homomorphism preserves the group operation, meaning $\phi(ab) = \phi(a)\phi(b)$

5. (b) The set of elements in G that map to the identity element of G'

Explanation: The kernel of a homomorphism consists of elements that map to the identity of the co-domain and forms a normal subgroup.

B. Short answer type question

1. Define a subgroup.
2. What is a coset?

NOTE

3. State the necessary and sufficient condition for a subset to be a subgroup.
4. What is a normal subgroup?
5. Define a quotient group.

Part 2

C. Long answer type question (200 words)

1. Prove that the intersection of two subgroups of a group is also a subgroup.
2. Explain how cosets partition a group with an example.
3. State and prove the necessary and sufficient condition for a subgroup to be normal.
4. Define homomorphism and prove that the kernel of a homomorphism is a normal subgroup.
5. Prove that the quotient group G/N forms a group if N is a normal subgroup of G.

Part 3

D. Long answer type question (300 words)

1. Discuss the properties of cosets and their role in Lagrange's theorem.
2. Explain the concept of normal subgroups and derive the necessary and sufficient condition for normality.
3. Define and explain quotient groups with an example. Prove that the order of the quotient group is equal to the index of the normal subgroup.
4. State and prove the First Isomorphism Theorem for groups. Provide an example illustrating the theorem.
5. Discuss the significance of homomorphism in group theory and prove that the image of a homomorphism is also a group.

UNIT 20
RING

NOTE

UNIT STRUCTURE

20.1 Introduction
20.2 Objective
20.3 Ring
20.3.1 Definition of Ring
20.3.2 Some special types of Rings
20.4 Elementary Properties of Ring
20.5 Homomorphism of Ring
20.6 Summary
20.7 Glossary
20.8 Answer to check your progress
20.9 Suggested readings OER
20.10 Reference
20.11 Exercise

20.1 INTRODUCTION

In the previous unit, you studied groups and their properties, which form the foundation of abstract algebra. Now, we extend these ideas to rings, which have two operations: addition and multiplication. Rings are fundamental in algebra and have applications in number theory and linear algebra. This unit explores the definition, special types, properties, and homomorphism of rings.

Building on this knowledge, we now offer a deeper exploration of rings, an essential algebraic structure that extends groups by incorporating both addition and multiplication. This unit covers the definition of a ring, its special types, elementary properties, and homomorphisms, which describe structural mappings between rings. Understanding these concepts is crucial for applications in algebra, number theory, and advanced mathematical structures.

This unit bridges abstract algebra with real-life applications by introducing the concept of rings, which extend groups by incorporating both addition and multiplication. Rings play a crucial role in number theory, cryptography, and linear algebra. We will explore the definition of a ring, its special types, elementary properties, and homomorphisms, which describe structural relationships between rings. These concepts provide a foundation for further studies in modern algebra and advanced mathematics.

NOTE

20.2 OBJECTIVE

After studying this unit, you will be able to:

- Explain the fundamental structure of a ring and its properties.
- Recognize and differentiate between commutative rings, integral domains, and fields.
- Utilize key properties such as distributivity, associativity, and the existence of additive identity and inverses.
- Analyse and apply the concept of homomorphisms to establish structural relationships between rings.
- Use ring theory in real-world applications, such as coding theory, cryptography, and algebraic geometry.

20.3 RING

20.3.1 Definition of Ring

A ring is called a commutative ring if multiplication is commutative

($a \cdot b = b \cdot a$ for all $a, b \in R$).

A ring is said to have a multiplicative identity (or a unity) if there exists an element $1 \in R$ such that $a \cdot 1 = a$ for all $a \in R$.

A ring is a set R equipped with two operations:

a) Addition (+): Makes R an abelian group.

b) Multiplication (·): Is associative and satisfies distributive laws over addition.

1. **Additive Properties:**

   R is an abelian group under addition:

   Closure: For all $a, b \in R, a + b \in R$.

   Associativity: $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$.

   Identity: There exists an element $0 \in R$ such that $a + 0 = a$ for all $a \in R$.

   Inverses: For every $a \in R$, there exists an element -a ∈ R such that $a + (-a) = 0$.

   Commutativity: $a + b = b + a$ for all $a, b \in R$.

2. **Multiplicative Properties:**

   Closure: For all a, b ∈ R, a · b ∈ R.

   Associativity: (a · b) · c = a · (b · c) for all a, b, c ∈ R.

3. **Distributive Properties:**

   Left distributivity: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$.

   Right distributivity: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a, b, c \in R$.

   Example 1: Ring of Integers Modulo n ($\mathbb{Z}_6$)

NOTE

Let $R = \mathbb{Z}_6$, the set of integers modulo 6, with addition and multiplication modulo 6.

The elements are $\{0, 1, 2, 3, 4, 5\}$.

Addition Table (mod 6):

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Multiplication Table (mod 6):

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Solution Examples:

Compute $(3 + 5)\ mod\ 6$:

$3 + 5 = 8, and\ 8\ mod\ 6 = 2.$

Answer is 2.

Compute $(3 \cdot 4)\ mod\ 6$:

$3 \cdot 4 = 12, and\ 12\ mod\ 6 = 0.$

Answer is 0.

Example 2: Ring of Polynomials $(\mathbb{R}[x])$

Let $R = \mathbb{R}[x]$, the set of all polynomials with real coefficients.

Given Polynomials:

$P(x) = 2x^2 + 3x + 1$

$Q(x) = x^2 - x + 2$

NOTE

Solution:

1. **Addition of Polynomials:**

$P(x) + Q(x) = (2x^2 + 3x + 1) + (x^2 - x + 2)$

Combine like terms:

$P(x) + Q(x) = (2x^2 + x^2) + (3x - x) + (1 + 2)$

$= 3x^2 + 2x + 3.$

2. **Multiplication of Polynomials:**

$$P(x) \cdot Q(x) = (2x^2 + 3x + 1) \cdot (x^2 - x + 2)$$

Expand using the distributive property:

$$= 2x^2(x^2 - x + 2) + 3x(x^2 - x + 2) + 1(x^2 - x + 2)$$

$$= 2x^4 - 2x^3 + 4x^2 + 3x^3 - 3x^2 + 6x + x^2 - x + 2$$

Combine like terms:

$$= 2x^4 + x^3 + 2x^2 + 5x + 2.$$

20.3.2 Some Special Types of Rings

1. Commutative Ring

A ring is commutative if multiplication is commutative, i.e., $ab = ba$ for all $a, b \in R$.

Example 1:

The ring of integers $\mathbb{Z}$ under addition and multiplication is a commutative ring because:

- Addition and multiplication are associative and commutative.
- 0 is the additive identity.
- Multiplication distributes over addition.

Example 2:

The ring $\mathbb{Z}_6$ (integers modulo 6) is a commutative ring. Check commutativity of multiplication:

- $2 \cdot 3 \equiv 6 \equiv 0 \ (mod\ 6)$,
- $3 \cdot 2 \equiv 6 \equiv 0 \ (mod\ 6)$, showing $ab = ba\ modulo\ 6$.

2. Ring with Unity (Unitary Ring)

A ring with unity has a multiplicative identity element 1

such that $1 \cdot a = a \cdot 1 = a \qquad for\ all\ a \in R.$

Example 1: The ring $\mathbb{Z}$ is a ring with unity where 1 is the multiplicative identity.

Example 2: The ring $M_2(\mathbb{R})$ (the set of all 2 × 2 matrices with real entries) is a ring with unity.

The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies IA = A for any matrix A.

NOTE

3. Division Ring (Skew Field)

A division ring is a ring where every non-zero element has a multiplicative inverse, but multiplication may not be commutative.

Example 1: The set of quaternions $\mathbb{H}$ (used in $\boldsymbol{3D}$ rotations) is a division ring.

$q = 1 + i + j + k$ (where $i^2 = j^2 = k^2 = ijk = -1$) has an inverse q^{-1}

such that $q \cdot q^{-1} = 1$.

Example 2:

The field of real numbers $\mathbb{R}$ is a division ring and a field.

For any $a \neq 0, a^{-1} = 1/a$ exists.

4. Integral Domain

An integral domain is a commutative ring with unity

where $ab = 0$ implies $a = 0 \; or \; b = 0$ (no zero divisors).

Example 1: The ring of integers $\mathbb{Z}$ is an integral domain because if

$$\mathbf{ab = 0, either\ a = 0\ or\ b = 0}$$

Example 2: The ring of polynomials $\mathbb{Z}[x]$ (polynomials with integer coefficients) is an integral domain. If $f(x)g(x) = 0$, then $f(x) = 0 \; or \; g(x) = 0$.

Example3: Verification (Commutative Ring $\mathbb{Z}_6$):

Show $ab = ba$: $4 \cdot 5 \equiv 20 \equiv 2 \; (mod\ 6) \; and \; 5 \cdot 4 \equiv 20 \equiv 2 \; (mod\ 6)$.

Example 4: Verification (Integral Domain $\mathbb{Z}$):

Check zero divisors: For integers $a, b \in \mathbb{Z}, ab = 0 \; implies \; either \; a = 0 \; or \; b = 0$, proving no zero divisors.

20.4 ELEMENTARY PROPERTIES OF RING

The elementary properties of rings are fundamental characteristics that all rings satisfy. These properties are derived directly from the axioms of a ring.

A ring $(R, +, \cdot)$ is defined as a set R with two operations $+ \; (addition) \; and \; \cdot \; (multiplication)$, satisfying the following properties:

Properties of Rings

1. **Additive Closure:** If $a, b \in R, then \; a + b \in R$.
2. **Additive Associativity:** $(a + b) + c = a + (b + c) \; for \; all \; a, b, c \in R$.

NOTE

3. **Additive Identity:** There exists an element $0 \in R$ such that $a + 0 = a \, for \, all \, a \in R$.
4. **Additive Inverse:** For every $a \in R$, there exists $-a \in R$ such that $a + (-a) = 0$.
5. **Additive Commutativity:** $a + b = b + a \, for \, all \, a, b \in R$
6. **Multiplicative Closure:** If $a, b \in R, then \, a \cdot b \in R$.
7. **Multiplicative Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c) \, for \, all \, a, b, c \in R$.
8. **Distributive Property:** $For \, all \, a, b, c \in R$:
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
   - $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Example 1: The Ring of Integers ($\mathbb{Z}$)

Let R = $\mathbb{Z}$, the set of integers, with the usual addition + and multiplication ·.

Verify that $\mathbb{Z}$ satisfies the ring properties.

1. **Additive Closure:** If $a, b \in \mathbb{Z}, then \, a + b \in \mathbb{Z}$.
   Example: $3 + 5 = 8 \in \mathbb{Z}$.
2. **Additive Associativity:** $(a + b) + c = a + (b + c)$.
   Example: Let $a = 2, b = 3, c = 4$.
   Then $(2 + 3) + 4 = 5 + 4 = 9 \, and \, 2 + (3 + 4) = 2 + 7 = 9$.
3. **Additive Identity:** The additive identity is 0 because $a + 0 = a$.
   Example: $7 + 0 = 7$.
4. **Additive Inverse:** For any $a \in \mathbb{Z}$, the additive inverse is $-a$.
   Example: $5 + (-5) = 0$.
5. **Additive Commutativity:** $a + b = b + a$.
   Example: $4 + 6 = 10 \, and \, 6 + 4 = 10$.
6. **Multiplicative Closure:** If $a, b \in \mathbb{Z}, then \, a \cdot b \in \mathbb{Z}$.
   Example: $3 \cdot 4 = 12 \in \mathbb{Z}$.
7. **Multiplicative Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   Example: $(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24, and \, 2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$.
8. **Distributive Property:** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

NOTE

Example: $2 \cdot (3 + 4) = 2 \cdot 7 = 14, and\ (2 \cdot 3) + (2 \cdot 4) = 6 + 8 = 14$.

Thus, $\mathbb{Z}$ is a ring.

Example 2: The Set of $\mathbf{2 \times 2}$ Matrices ($\boldsymbol{M_2(\mathbb{R})}$)

Let R = $M_2(\mathbb{R})$, the set of all 2 × 2 matrices with real entries, under matrix addition and multiplication. ***Verify that*** $\boldsymbol{M_2(\mathbb{R})}$ ***satisfies the ring properties***.

Solution: Let $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

1. **Additive Closure:** Matrix addition is closed in $M_2(\mathbb{R})$.

   Example: $A + B = \begin{bmatrix}1+0 & 2+1\\ 3+1 & 4+0\end{bmatrix} = \begin{bmatrix}1 & 3\\ 4 & 4\end{bmatrix} \in M_2(\mathbb{R})$.

2. **Additive Identity:** The additive identity is the zero matrix $O = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

   Example: A + O = A.

3. **Additive Inverse:** The additive inverse of A is −A, where −A == $\begin{bmatrix}-1 & -2\\ -3 & -4\end{bmatrix}$

   Example: A + (−A) = O.

4. **Multiplicative Closure:** Matrix multiplication is closed in $M_2(\mathbb{R})$.

   Example: $A \cdot B = \begin{bmatrix}1\cdot 0 + 2\cdot 1 & 1\cdot 1 + 2\cdot 0\\ 3\cdot 0 + 4\cdot 1 & 3\cdot 1 + 4\cdot 0\end{bmatrix} = \begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix} \in M_2(\mathbb{R})$.

5. **Distributive Property:** Matrix multiplication is distributive over addition.

   Example: $A \cdot (B + O) = A \cdot B$.

   Thus, $M_2(\mathbb{R})$ is a ring.

20.5 HOMOMORPHISM OF RING

A ring homomorphism is a function $f: R \to S$ between two rings R and S that preserves the ring operations:

1. **Addition Preservation:** $f(a + b) = f(a) + f(b)$, for all $a, b \in R$.
2. **Multiplication Preservation:** $f(a \cdot b) = f(a) \cdot f(b)$, for all $a, b \in R$.
3. $f(I_R) = I_S$, where I_R and I_S are the multiplicative identities of R and S (if the rings have unity).

Example 1: The Identity Map

Let $R = \mathbb{Z}$, the ring of integers. The identity map $f: \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = x$, is a ring homomorphism because:

NOTE

1. $f(a+b) = a+b = f(a)+f(b)$
2. $f(a \cdot b) = a \cdot b = f(a) \cdot f(b)$
3. $f(1) = 1$.

Example 2: Modulo Reduction Map

Let $R = \mathbb{Z}$ and $S = \mathbb{Z}_n$ (integers modulo n). The map $f: \mathbb{Z} \to \mathbb{Z}_n$ defined by $f(x) = x \ mod \ n$, is a ring homomorphism because:

1. $f(a+b) = (a+b) \ mod \ n = (a \ mod \ n + b \ mod \ n) = f(a) + f(b)$
2. $f(a \cdot b) = (a \cdot b) \ mod \ n = (a \ mod \ n \cdot b \ mod \ n) = f(a) \cdot f(b)$
3. $f(1) = 1 \ mod \ n = 1$.

Example 3: Prove that the map $f: \mathbb{Z}[x] \to \mathbb{Z}$ defined by $f(p(x)) = p(0)$ (evaluation at zero), is a ring homomorphism.

Solution:

1. **Addition Preservation:**

   Let $p(x), q(x) \in \mathbb{Z}[x]$.

   Then: $f\big(p(x)+q(x)\big) = \big(p(x)+q(x)\big)$ at $x = 0$

   $= p(0) + q(0) = f(p(x)) + f(q(x))$.

2. **Multiplication Preservation:**

   Let $p(x), q(x) \in \mathbb{Z}[x]$

   Then: $f\big(p(x) \cdot q(x)\big) = \big(p(x) \cdot q(x)\big)$ at $x = 0$

   $= p(0) \cdot q(0) = f(p(x)) \cdot f(q(x)).f(p(x)$

3. $f(1) = 1$, where 1 is the constant polynomial 1.

Thus, f is a ring homomorphism.



Activity 20.1

Identifying and Classifying Rings

Objective: To help students understand the definition of a ring and classify different types of rings based on their properties.

Instructions:

1. **Group Work:** Divide students into small groups and provide each group with a set of algebraic structures (e.g., $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, M_2(\mathbb{R})$, polynomial rings, etc.).
2. **Classification Task:** Each group must determine whether the given structure is a ring and classify it as:

NOTE

- Commutative or non-commutative
- With or without identity
- Integral domain or field

3. **Justification:** Groups must justify their classification by verifying ring axioms and properties.
4. **Presentation:** Each group presents their findings to the class, explaining their reasoning.

Expected Outcome:

Students will develop a deeper understanding of different types of rings and their properties by applying theoretical knowledge to concrete examples.

☑ Check Your Progress 20.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. Define a ring.?

.....

.....

.....

2. Every ring is a field. (True or False).

3. Which of the following is an example of a commutative ring with identity?

a) The set of 2×2 matrices under addition and multiplication

b) The set of integers $\mathbb{Z}$ under usual addition and multiplication

c) The set of all polynomials over real numbers without a constant term

d) The set of odd integers under usual addition and multiplication

4. A ring in which every nonzero element has a multiplicative inverse is called a _____.

20.6 SUMMARY

The unit on Rings introduced one of the fundamental algebraic structures used in abstract algebra and advanced mathematics. It began with the definition of a ring, where students learned that a ring is a set equipped with two binary operations, namely addition and multiplication, satisfying specific algebraic properties. The unit further

NOTE

explored several special types of rings such as commutative rings, rings with identity, integral domains, and division rings, helping students understand how different algebraic systems are classified based on their operational properties. Elementary properties of rings were also discussed, including closure, associativity, distributive laws, additive identity, additive inverse, and multiplication-related properties. These concepts provided a strong foundation for understanding the internal structure and behavior of rings. Another major topic covered in the unit was ring homomorphism, which explained how mappings between rings preserve algebraic operations and help establish relationships between different ring structures. Students learned the importance of kernels, images, and the role of homomorphisms in simplifying complex algebraic systems. By the end of the unit, students should be able to define and identify different types of rings, verify ring properties, and apply homomorphism concepts to solve algebraic problems. The study of rings is highly relevant in various fields such as cryptography, coding theory, computer science, quantum physics, and higher mathematics because ring structures are widely used in modern algebraic applications and computational systems. Understanding rings also builds a strong conceptual base for further topics like fields, vector spaces, and advanced algebraic theories. Students are encouraged to continue exploring abstract algebra through problem-solving, theorem verification, and practical examples to deepen their understanding. Referring to algebra textbooks, lecture notes, and mathematical exercises can further strengthen conceptual clarity and analytical skills in ring theory.

20.7 GLOSSARY

- **Ring:** An algebraic structure consisting of a set with two operations (addition and multiplication) satisfying specific properties, such as associativity and distributive laws.
- **Commutative Ring:** A ring in which multiplication is commutative, i.e., $ab = ba$ for all elements a, b in the ring.
- **Ring with Unity (Identity):** A ring that has a multiplicative identity element (denoted as 1) such that $a \cdot 1 = a$ for all a in the ring.
- **Zero Divisor:** A nonzero element a in a ring such that there exists a nonzero bb where $ab = 0$. Rings without zero divisors are called integral domains.
- **Integral Domain:** A commutative ring with an identity element that has no zero divisors, ensuring that the product of two nonzero elements is always nonzero.

NOTE

- **Field:** A special type of commutative ring in which every nonzero element has a multiplicative inverse, meaning division (except by zero) is possible.
- **Subring:** A subset of a ring that forms a ring under the same operations as the original ring, satisfying closure properties.
- **Ideal:** A special subset of a ring that absorbs multiplication from ring elements and plays a crucial role in defining quotient rings.
- **Ring Homomorphism:** A function between two rings that preserves both addition and multiplication, ensuring structural consistency.
- **Kernel of a Ring Homomorphism:** The set of elements mapped to zero under a ring homomorphism, used to define quotient rings and analyze ring structures.

20.8 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 20.1

1. Define a ring.

Answer: A ring is a set R equipped with two binary operations: addition (+) and multiplication (×), such that:

1. R is an abelian group under addition (i.e., closed, associative, has identity 0, and every element has an additive inverse).
2. R is closed under multiplication and multiplication is associative.
3. Distributive laws hold:

$$-\ a(b+c) = ab + ac$$

$$-\ (a+b)c = ac + bc$$

for all $a, b, c \in R$.

2. Every ring is a field. (True or False):

Answer: False.

A field is a special kind of ring in which:

- Every nonzero element has a multiplicative inverse.
- Multiplication is commutative.

Not all rings satisfy these conditions, so not every ring is a field.

3. Which of the following is an example of a commutative ring with identity?

Answer: b) The set of integers Z\mathbb{Z} under usual addition and multiplication

NOTE

Explanation:

- $\mathbb{Z}$ is a commutative ring (multiplication is commutative) and has an identity (1).
- 2×2 matrices do not form a commutative ring.
- Polynomials without constant term do not contain the identity element (1).
- Odd integers are not closed under addition (e.g., $3 + 5 = 8$, not odd).

4. A ring in which every nonzero element has a multiplicative inverse is called a _____.

Answer: Field

A field is a commutative ring with identity in which every nonzero element has a multiplicative inverse.

20.9 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1260	

20.10 REFERENCE

- https://math.libretexts.org/Courses/Mount_Royal_University/Abstract_Algebra_I/Chapter_5%3A_Rings_and_fileds/5.1%3A_Introduction_to_Rings
- https://en.wikipedia.org/wiki/Ring_(mathematics)
- https://en.wikipedia.org/wiki/Commutative_ring
- https://mathworld.wolfram.com/ZeroDivisor.html
- https://www.britannica.com/science/integral-domain
- https://www.geeksforgeeks.org/group-rings-and-fields-in-group-theory/
- https://en.wikipedia.org/wiki/Ideal_(ring_theory)
- https://en.wikipedia.org/wiki/Ring_homomorphism
- https://arbital.com/p/kernel_of_ring_homomorphism/

NOTE

- https://uomustansiriyah.edu.iq/media/lectures/9/9_2021_03_29!01_55_48_PM.pdf

20.11 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following is an example of a commutative ring?

a) The set of all 2×2 matrices under addition and multiplication
b) The set of integers $\mathbb{Z}$ under usual addition and multiplication
c) The set of upper triangular matrices
d) The set of polynomials with integer coefficients

2. Which property is not necessarily true for all rings?

a) Associativity of multiplication
b) Commutativity of addition
c) Existence of multiplicative identity
d) Existence of additive inverse

3. A ring in which every nonzero element has a multiplicative inverse is called a:

a) Integral domain
b) Commutative ring
c) Field
d) Subring

4. The kernel of a ring homomorphism is always:

a) A subring
b) An ideal
c) A field
d) A commutative ring

5. Which of the following is true for every ring?

a) It has a multiplicative identity
b) It has no zero divisors
c) It is associative under multiplication
d) It is commutative under multiplication

Answers with Explanations:

1. (b) The set of integers $\mathbb{Z}$

Explanation: The set of integers satisfies the ring axioms and is commutative under multiplication.

2. (c) Existence of multiplicative identity

Explanation: Not all rings have a multiplicative identity. Some rings, like the set of even integers, do not have a multiplicative identity.

NOTE

3 (c) Field

Explanation: A field is a ring in which every nonzero element has a multiplicative inverse.

4 (b) An ideal

Explanation: The kernel of a ring homomorphism forms an ideal of the ring, as it satisfies the necessary absorption property under ring multiplication.

5 (c) It is associative under multiplication

Explanation: By definition, multiplication in a ring must be associative, but it may not be commutative or have a multiplicative identity.

B. Short answer type question

1. Define a ring.
2. Give an example of a commutative ring.
3. What is the additive identity in a ring?
4. What is a zero divisor in a ring?
5. State one property of ring homomorphism.

Part 2

C. Long answer type question (200 words)

1. Explain the definition of a ring with suitable examples.
2. Describe the different types of rings with examples.
3. Prove that every ring is an abelian group under addition.
4. What are the elementary properties of rings? Explain in detail.
5. Define ring homomorphism and explain its properties with examples.

Part 3

D. Long answer type question (300 words)

1. Discuss in detail the concept of rings, their types, and their real-life applications.
2. Explain the elementary properties of rings with proofs and examples.
3. Define ring homomorphism and prove that the kernel of a homomorphism is an ideal.
4. What is the significance of commutative rings and integral domains? Explain with examples.
5. Discuss in detail the concept of ideals and their role in ring theory.

UNIT 21
SUBRING

NOTE

UNIT STRUCTURE

21.1 Introduction

21.2 Objective

21.3 Subring

21.3.1 Definition of Subring

21.3.2 Results of Subring of a ring

21.4 Ideal

21.4.1 Definition of Ideal

21.4.2 Properties of ideal

21.5 Summary

21.6 Glossary

21.7 Answer to check your progress

21.8 Suggested readings OER

21.9 Reference

21.10 Exercise

21.11 Feedback form

21.1 INTRODUCTION

In the previous unit, you learned about the fundamental concepts of rings, including their definitions, properties, and homomorphisms. We explored various types of rings and their structures. The concept of ring operations and their algebraic significance was discussed. This knowledge forms the basis for understanding subrings, ideals, and their properties.

Building on this knowledge, we now offer a deeper study of subrings, which are subsets of rings that maintain the ring structure. We introduce the concept of ideals, which play a crucial role in ring theory and algebraic structures. The properties and significance of ideals in the formation of quotient rings are explored. Understanding these concepts helps in solving advanced algebraic problems.

This unit bridges abstract algebra with real-life applications by showing how subrings and ideals are used in cryptography, coding theory, and computational algebra. Ideals provide a foundation for constructing quotient rings, which have applications in solving polynomial equations. The study of subrings helps in understanding algebraic structures used in modern mathematics. These concepts play a key role in areas like number theory and computer science.

NOTE

21.2 OBJECTIVE

After studying this unit, you will be able to:

- Identify conditions under which a subset of a ring forms a subring.
- Explore important results related to subrings
- Define and analyze ideals and distinguish them from subrings in terms of their algebraic properties.
- Examine the properties of ideals and their role in forming quotient rings.
- Apply the concepts of subrings and ideals in abstract algebra and real-world applications such as cryptography and coding theory.

21.3 SUB RING

21.3.1 Definition of Subring

A subring of a ring is a subset of a ring that is itself a ring under the same addition and multiplication operations. Specifically, for a subset S of a ring R to be a subring, S must satisfy the following conditions:

1. S is closed under addition ($a + b \in S$ for all $a, b \in S$).
2. S is closed under multiplication ($a.b \in S$ for all $a, b \in S$).
3. S contains the additive identity (0).
4. S is closed under additive inverses ($-a \in S$ for all $a \in S$).

Example 1: The Ring of Integers $\mathbb{Z}$ and Its Subring $2\mathbb{Z}$}

Let $R = \mathbb{Z}$, the set of all integers under standard addition and multiplication.

The set $S = 2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$, the set of all even integers, is a subring of $\mathbb{Z}$.

Verification:

1. **Closed under addition:** If $a, b \in 2\mathbb{Z}$, then $a = 2m$ and $b = 2n$ for integers m and n. Thus, $a + b = 2m + 2n = 2(m + n) \in 2\mathbb{Z}$.
2. **Closed under multiplication:**

   If $a, b \in 2\mathbb{Z}$, then $ab = (2m)(2n) = 4mn = 2(2mn) \in 2\mathbb{Z}$
3. **Contains 0:** The additive identity $0 \in 2\mathbb{Z}$.
4. **Closed under additive inverses:** If $a \in 2\mathbb{Z}$, then $-a = -2m \in 2\mathbb{Z}$.

   Hence, $2\mathbb{Z}$ is a subring of $\mathbb{Z}$

Example 2: The Ring of Real Numbers $\mathbb{R}$ and Its Subring $\mathbb{Z}$

Let $R = \mathbb{R}$, the set of real numbers under standard addition and multiplication.

The set $S = \mathbb{Z}$, the set of integers, is a subring of $\mathbb{R}$.

Verification:

1. **Closed under addition:** If $a, b \in \mathbb{Z}$ then $a + b \in \mathbb{Z}$.
2. **Closed under multiplication:** If $a, b \in \mathbb{Z}$, then $ab \in \mathbb{Z}$.
3. **Contains 0:** The additive identity $0 \in \mathbb{Z}$.
4. **Closed under additive inverses:** If $a \in \mathbb{Z}$, then $-a \in \mathbb{Z}$.

Thus, $\mathbb{Z}$ is a subring of $\mathbb{R}$.

21.3.2 Results of Subring Of A Ring:

Below are the key **results** about subrings of a ring along with examples:

Result 1: Intersection of Subring is a Subring

If S_1 and S_2 are subrings of a ring R, then their intersection $S_1 \cap S_2$ is also a subring of R.

Example 1:

Let $R = \mathbb{Z}$ (the ring of integers).

- $S_1 = 2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$ (even integers).
- $S_2 = 3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$ (integers divisible by 3).

The intersection $S_1 \cap S_2 = 6\mathbb{Z}$ (integers divisible by 6) is a subring of $\mathbb{Z}$.

Verification:

- Closed under addition: If $a, b \in 6\mathbb{Z}$, then $a + b \in 6\mathbb{Z}$.
- Closed under multiplication: If $a, b \in 6\mathbb{Z}$, then $ab \in 6\mathbb{Z}$.
- Contains 0: $0 \in 6\mathbb{Z}$.
- Closed under additive inverses: If $a \in 6\mathbb{Z}$, then $-a \in 6\mathbb{Z}$.

Thus, $6\mathbb{Z}$ is a subring.

Result 2: Subring Contain Additive Identity

Every subring S of a ring R contains the additive identity 0 of R.

Example 2:

Let $R = \mathbb{R}$ (the ring of real numbers), and let $S = \mathbb{Z}$ (the integers).

- The additive identity of $\mathbb{R}$ is 0.
- Since $0 \in \mathbb{Z}$, the subring $\mathbb{Z}$ contains the additive identity.

Verification:

- $\mathbb{Z}$ is closed under addition, multiplication, and additive inverses.
- $0 \in \mathbb{Z}$, satisfying the requirement for a subring.

NOTE

Thus, $\mathbb{Z}$ is a subring of $\mathbb{R}$.

Result 3: Smallest Subring Containing a Subset

For any subset $A \subseteq R$, there exists a smallest subring S of R containing A. This is called the **subring generated by A**.

Example 3:

Let $R = \mathbb{Q}$ (the field of rational numbers), and let $A = \{1/2\}$.

The smallest subring containing A is the set of all rational numbers of the form

$$\mathbb{Z}\left[\frac{1}{2}\right] = \{m/2^n : m \in \mathbb{Z}, n \in \mathbb{N}\}$$

Verification:

- $\mathbb{Z}\left[\frac{1}{2}\right]$ is closed under addition and multiplication, contains 0, and is closed under additive inverses.
- $\mathbb{Z}\left[\frac{1}{2}\right]$ is the smallest subring containing $1/2$.

Thus, $\mathbb{Z}\left[\frac{1}{2}\right]$ is a subring of $\mathbb{Q}$.

21.4 IDEAL

21.4.1 DEFINITION OF IDEAL

In abstract algebra, **ideals** are fundamental subsets of rings that generalize certain properties of divisibility in integers. They help in constructing quotient rings and studying ring homomorphisms.

21.4.2 PROPERTIES OF IDEAL

The **properties of ideals** refer to the key conditions that an ideal must satisfy to be classified as such. Below are the primary properties of ideals, along with some additional related concepts.

1. **Additive Subgroup Property:** An ideal I of a ring R is an additive subgroup of R. This means that:
   - **Closure under addition:** If $a, b \in I$ then $a + b \in I$.
   - **Additive identity:** The additive identity $0 \in R$ must be in I (i.e., $0 \in I$).
   - **Additive inverses:** If $a \in I$, then the additive inverse $-a \in I$.

   This property guarantees that the ideal behaves like a subgroup in terms of the ring's addition operation.
2. **Absorption Property (Multiplicative Closure):** An ideal I is closed under multiplication by any element from the ring. This means that:
   - **Left absorption:** For any $r \in R$ and $a \in I$, the product $r \cdot a \in I$.

NOTE

- **Right absorption:** For any $r \in R$ and $a \in I$, the product $a \cdot r \in I$.

This ensures that multiplying elements of I by any element from R results in an element that still belongs to the ideal.

- For two-sided ideals, both left and right multiplication must hold.

3. Types of Ideals:

- **Two-Sided Ideal (Ideal):** A subset $I \subseteq R$ is a two-sided ideal (simply called an ideal) if for all $r \in R$ and $a \in I$, both $r \cdot a \in I$ and $a \cdot r \in I$.

This is the most general form of an ideal.

- **Left Ideal:** A subset $I \subseteq R$ is a left ideal if for all $r \in R$ and $a \in I, r \cdot a \in I$.

Notice that here, the absorption property applies only on the left side.

- **Right Ideal:** A subset $I \subseteq R$ is a right ideal if for all $r \in R$ and $a \in I$,

$a \cdot r \in I$. The absorption property applies only on the right side.

4. The Zero Ideal:

The zero ideal in any ring R is the ideal {0}, which contains only the additive identity (zero).It is trivially an ideal because it satisfies both the additive subgroup property (as 0+0=0) and the absorption property ₹

(since $r \cdot 0 = 0$ for any $r \in R$).

5. The Whole Ring as an Ideal:

The whole ring R itself is always an ideal of R, called the unit ideal or the improper ideal. This is because for any element r∈R and any a∈R,both r·a∈R and a·r∈R,so the absorption property holds.

6. Principal Ideals:

A principal ideal is an ideal generated by a single element of R. The ideal generated by an element $a \in R$ is denoted as (a) and consists of all multiples of a:$(a) = \{r \cdot a \mid r \in R\}$.

- If R is a commutative ring and a is a nonzero element, then the ideal (a) is a principal ideal, and it is the smallest ideal containing a.
- Every ideal in a **principal ideal domain (PID)** is a principal ideal.

7. Intersection of Ideals:

If $I_1, I_2, \ldots\ldots I_n$, are ideals of a ring R, then their intersection is also an ideal: $I_1 \cap I_2 \cap \ldots\ldots \cap I_n$

NOTE

This intersection contains all the elements that are in every ideal I_i. Since ideals are closed under addition and absorption, the intersection will also satisfy these properties.

8. **Sum of Ideals:** The sum of two ideals I_1 and I_2 in a ring R is also an ideal, and is defined as: $I_1 + I_2 = \{a + b \mid a \in I_1, b \in I_2\}$.

   This sum is the smallest ideal containing both I_1 and I_2.

9. **Quotient Ring (Factor Ring):** If I is an ideal of a ring R, the quotient ring (also called the factor ring) is the set of cosets of I in R, denoted by R/I. The cosets are of the form $r + I$, where $r \in R$, and the operations are defined in a way that makes R/I a ring.

10. **Maximal Ideals:** An ideal M in a ring R is called maximal if there is no other ideal I of R such that $M \subsetneq I \subsetneq R$. In other words, there is no ideal of R strictly between M and R. Maximal ideals play a key role in the study of ring homomorphisms and quotient rings.

11. **Prime Ideals:** An ideal P in a ring R is called a prime ideal if for all $a, b \in R$,

Whenever $ab \in P$, then either $a \in P$ or $b \in P$. Prime ideals generalize the concept of prime numbers in number theory and are important in the study of integral domains and algebraic geometry.



Activity 21.1

Identifying Subrings and Ideals

Objective:

To help students understand the difference between subrings and ideals by analyzing specific subsets of a given ring.

Materials Needed:

- Pen and paper
- Access to a whiteboard (if in a classroom setting)

Instructions:

1. Consider the ring $\mathbb{Z}$ (set of integers) under addition and multiplication.
2. Define the following subsets of $\mathbb{Z}$:
   - $2\mathbb{Z}$ (: The set of all even integers
   - $\mathbb{N}$: The set of natural numbers
   - $\mathbb{Z} \geq 0$: The set of non-negative integers
3. **Task 1:** Determine whether each subset is a subring by checking if it satisfies:

- Closure under addition and multiplication
- Containment of additive identity (0)
- Closure under additive inverses

4. **Task 2:** Determine whether each subset is an ideal by checking if for all $r \in \mathbb{Z}$ and $a \in$ subset, ra and ar remain in the subset.
5. Discuss the results with a group or write a conclusion explaining which sets are subrings and which are ideals, highlighting the difference between them.

Expected Outcome:

- $2\mathbb{Z}$ is both a subring and an ideal of $\mathbb{Z}$.
- $\mathbb{N}$ is not a subring (no additive inverses) and not an ideal (not closed under multiplication with negative integers).
- $\mathbb{Z} \geq 0$ is not an ideal (not closed under multiplication by negative integers) but may satisfy some subring properties.

Discussion Questions:

1. Why is $2\mathbb{Z}$ an ideal but $\mathbb{N}$ is not?
2. Can you give an example of a subring that is not an ideal?
3. How do ideals help in constructing quotient rings?

OUTCOME: This activity encourages analytical thinking and helps students internalize the properties of subrings and ideals.

☑ Check Your Progress 21.1

Notes: a) Space is given below for your answer.

b) Check your answer with the one given at the end of this unit.

1. What is a subring?

.....

.....

.....

2. Every subring of a ring is an ideal. (True or False).

3. Which of the following is always an ideal of a ring?

a) The set of all even integers in $\mathbb{Z}$

b) The set of positive integers in $\mathbb{Z}$

c) The set of prime numbers in $\mathbb{Z}$

d) The set of odd integers in $\mathbb{Z}$

.....

NOTE

4. An ideal is a special type of _____ that is closed under multiplication by elements from the ring.

21.5 SUMMARY

This unit introduced the fundamental concepts of Subrings and Ideals in ring theory, which are essential topics in abstract algebra. The unit began with the study of subrings, explaining how a subset of a ring can itself form a ring under the same operations. Students learned the definition of a subring and the important conditions required for a subset to qualify as a subring, such as closure under subtraction and multiplication.

The unit also covered important results related to subrings, helping learners understand how algebraic structures behave within larger rings and how these structures preserve ring properties. The discussion then moved to ideals, another major concept in ring theory that plays a vital role in advanced algebra. Students explored the definition of ideals and learned how ideals differ from ordinary subrings because they absorb multiplication from elements of the ring. The properties of ideals were also studied in detail, including their role in forming quotient rings and simplifying algebraic structures. By the end of the unit, students gained a strong understanding of how subrings and ideals help mathematicians analyze and organize algebraic systems.

These concepts are highly relevant in higher mathematics, computer science, cryptography, coding theory, and many branches of modern algebra. Understanding ideals is especially important for studying homomorphisms, factor rings, and advanced algebraic structures. The unit also develops logical thinking, proof-writing skills, and mathematical reasoning abilities. Students are encouraged to continue exploring related topics such as quotient rings, ring homomorphisms, and polynomial rings to deepen their understanding of abstract algebra. Regular practice of examples and theorem-based problems will further strengthen conceptual clarity and analytical skills in this area of mathematics.

21.6 GLOSSARY

- **Ring:** A set equipped with two operations (addition and multiplication) that satisfy specific properties.
- **Subring:** A subset of a ring that is itself a ring under the same operations.
- **Ideal:** A subring that is closed under multiplication by elements from the ring.
- **Left Ideal:** A subset I of R where $rI \subseteq I$ for all $r \in R$.

- **Right Ideal:** A subset I of R where $Ir \subseteq I$ for all $r \in R$.
- **Two-Sided Ideal:** A subset I that is both a left ideal and a right ideal.
- **Quotient Ring:** A ring formed by partitioning a ring using an ideal.
- **Ring Homomorphism:** A function between two rings that preserves their operations.
- **Kernel of a Ring Homomorphism:** The set of elements mapped to zero, which always forms an ideal.
- **Principal Ideal:** An ideal generated by a single element.

21.7 ANSWER TO CHECK YOUR PROGRESS - POSSIBLE ONE

Check Your Progress 21.1

1. What is a subring?

Answer: A subring is a subset of a ring that is itself a ring under the same operations of addition and multiplication.

To be a subring, the subset must:

- Be non-empty
- Be closed under subtraction and multiplication
- Contain the additive identity (0)

2. Every subring of a ring is an ideal. (True or False):

Answer: False.

While every ideal is a subring, the converse is not always true.

A subring may not satisfy the condition of being closed under multiplication with *any* element from the ring (required for an ideal).

3. Which of the following is always an ideal of a ring?

Answer: a) The set of all even integers in $\mathbb{Z}$

Explanation:

- The even integers form an ideal in Z\mathbb{Z}, because:
  - They are closed under subtraction.
  - Multiplying an even integer by any integer still gives an even integer.
- Other options:
  - Positive integers: not closed under subtraction.
  - Prime numbers: not closed under addition or subtraction.

NOTE

- Odd integers: not closed under addition or multiplication.

4. An ideal is a special type of _____ that is closed under multiplication by elements from the ring.

Answer: Subring

An ideal is a special type of subring that, in addition, satisfies the absorbing property: If $a \in I$ (ideal) and $r \in R$ (ring), then both $ra \in I$ and $ar \in I$.

21.8 SUGGESTED READINGS – OER

Sr. No.	Title	URL	QR Code
1	Open Educational Resources -1	http://books.aisectonline.com/User/UnitUrls?unitId=1261	

21.9 REFERENCE

- https://math.libretexts.org/Courses/Mount_Royal_University/Abstract_Algebra_I/Chapter_5%3A_Rings_and_fileds/5.1%3A_Introduction_to_Rings
- https://en.wikipedia.org/wiki/Subring
- https://en.wikipedia.org/wiki/Quotient_ring
- https://www.britannica.com/science/integral-domain
- https://www.geeksforgeeks.org/group-rings-and-fields-in-group-theory/
- https://en.wikipedia.org/wiki/Ideal_(ring_theory)
- https://en.wikipedia.org/wiki/Ring_homomorphism
- https://arbital.com/p/kernel_of_ring_homomorphism/
- https://arbital.com/p/kernel_of_ring_homomorphism/
- https://mathworld.wolfram.com/PrincipalIdeal.html

21.10 EXERCISE

Part 1

A. Multiple choice question

1. Which of the following is always true for a subring S of a ring R?

a) S must contain the multiplicative identity of R

NOTE

b) S must be closed under addition and multiplication

c) S must be an ideal of R

d) S must have a multiplicative inverse for every element

2. Which of the following is an ideal in $\mathbb{Z}$?

a) $\mathbb{N}$(natural numbers)

b) $2\mathbb{Z}$ (set of even integers)

c) $\mathbb{Z}_{>0}$ (positive integers)

d) $\mathbb{Q}$ (rational numbers)

3. An ideal I of a ring R is called a maximal ideal if:

a) I is the only ideal of R

b) I contains all elements of R

c) The only ideals containing I are I and R itself

d) I is the intersection of all ideals of R

4. Which of the following statements is true about ideals?

a) Every ideal is a subring

b) Every subring is an ideal

c) Every field has exactly one non-trivial ideal

d) The set of all odd integers forms an ideal in $\mathbb{Z}$

5. If I is an ideal of R, which of the following operations is well-defined in the quotient ring R/I?

a) Addition

b) Multiplication

c) Both addition and multiplication

d) Neither addition nor multiplication

Answers with Explanations:

1. b) S must be closed under addition and multiplication

Explanation: A subring needs to be closed under addition and multiplication but does not necessarily contain the identity or have multiplicative inverses.

2. b) $2\mathbb{Z}$ (set of even integers)

Explanation: An ideal in $\mathbb{Z}$ must be closed under multiplication by any integer. The set of even integers $2\mathbb{Z}$ satisfies this condition.

3. c) Every left coset of N is equal to its corresponding right coset.

Explanation: A maximal ideal is one that is not contained in any larger proper ideal of the ring.

NOTE

4. c)Every field has exactly one non-trivial ideal

Explanation: Every field has only two ideals: the trivial ideal {0} and the entire field itself.

5. c) Both addition and multiplication

Explanation: In a quotient ring R/I , both addition and multiplication are defined in a way that respects the ideal I.

B. Short answer type question

1. Define a subring.
2. What is an ideal in a ring?
3. Give an example of a subring of $\mathbb{Z}$.
4. State one necessary condition for a subset to be an ideal.
5. What is the difference between a subring and an ideal?

Part 2

C. Long answer type question (200 words)

1. Prove that the set of even integers forms a subring of $\mathbb{Z}$.
2. Explain the necessary and sufficient conditions for a subset of a ring to be a subring.
3. Show that the set of all multiples of a fixed integer n forms an ideal in $\mathbb{Z}$
4. Prove that every ideal of a field is either {0} or the entire field.
5. Explain the relationship between ideals and quotient rings with an example.

Part 3

D. Long answer type question (300 words)

1. Prove that the intersection of two ideals of a ring is also an ideal. Give an example to illustrate your answer.
2. Define left ideal, right ideal, and two-sided ideal. Give examples for each in the ring of 2×2 matrices.
3. Prove that in a commutative ring with unity, every maximal ideal is prime.
4. Explain the importance of the concept of ideals in ring theory and their role in constructing quotient rings.
5. State and prove the First Isomorphism Theorem for rings, illustrating with an example.

DR. C.V. RAMAN UNIVERSITY

Chhattisgarh, Bilaspur AN A/SECT GROUP UNIVERSITY

Approved by : PCI | AICTE | NCTE | BCI | Member of : AIU | Joint Committee : (UGC | DEB | AICTE) | Recognized by : MSC | A NAAC Accredited University

DR. C. V. RAMAN UNIVERSITY (C.G.)

SLM FEEDBACK FORM

PROGRAMME : _____

COURSE : _____

BLOCK : _____

UNIT : _____

Dear Student,

While studying the units of this block, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill up and send us the following questionnaire, which pertains to this block. If you find the space provided insufficient, kindly use a separate sheet.

Please mail to aisectcontent@aisect.org/ for online submission please scan QR code:

Questionnaire

1. **Name** : _____

2. **Enrolment No.** : | | | | | | | | | | | |

3. **How many hours did you need for studying the units?** : | |

4. **Please give your reactions to the following items based on your reading of the block:**

Items	Excellent	Very Good	Good	Poor	Give specific Examples (If poor)
Presentation Quality					
Language and Style					
Illustrations					
Used (diagrams, tables, etc.)					
Conceptual Clarity					
Check Your Progress Questions					
Suggested Readings - OER					

5. **Any other comments:**

_____

_____

_____

_____

***